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RESEARCH ARTICLE

Some properties of nilpotent groups

Anthony M. Gaglione, Seymour Lipschutz, Dennis Spellman

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Dedicated to Professor Leonid A. Kurdachenko on his 60^{th} birthday

Abstract. Property S, a finiteness property which can hold in infinite groups, was introduced by Stallings and others and shown to hold in free groups. In [2] it was shown to hold in nilpotent groups as a consequence of a technical result of Mal'cev. In that paper this technical result was dubbed property R. Hence, more generally, any property R group satisfies property S. In [7] it was shown that property R implies the following (labeled there weak property R) for a group G:

> If G_0 is any subgroup in G and G_0^* is any homomorphic image of G_0, then the set of torsion elements in G_0^* forms a locally finite subgroup.

It was left as an open question in [7] whether weak property R is equivalent to property R. In this paper we give an explicit counterexample thereby proving that weak property R is strictly weaker than property R.

1. An alphabet soup of properties

In this paper, we use the following notation. If G is a group and $S \subseteq G$, then $\langle S\rangle$ denotes the subgroup of G generated by the elements of S. Also $\langle ...;\ ...\rangle$ indicates a description of a group in terms of generators and relations.

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Definition 1.1. *Subgroups A and B of a group G are* ***commensurable*** *provided has $A \cap B$ finite index in each of A and B.*

Definition 1.2. *The group G satisfies* ***property S*** *provided, whenever A and B are finitely generated commensurable subgroups, $A \cap B$ has finite index in $\langle A, B\rangle$.*

Below are examples of groups which violate property S. Let $n > 1$ be an integer.

Example 1.3. The cyclically pinched one-relator group $G = \langle a, b;\ a^n = b^n\rangle$. Let $a^n = z = b^n$ so that z is central in the torsion free group G. Let $A = \langle a\rangle$ and $B = \langle b\rangle$. Then $A \cap B = \langle z\rangle$ has finite index in each of A and B but $\langle A, B\rangle / \langle z\rangle$ is the free product $\langle a; a^n = 1\rangle * \langle b; b^n = 1\rangle$ which is infinite.

Example 1.4. Suppose is n odd and sufficiently large (e.g., $n \geq 667$ will do) so that the rank 2 free group $F_2(\mathcal{B}_n)$ in the Burnside variety of exponent n is infinite. Suppose $\{a, b\}$ freely generates $F_2(\mathcal{B}_n)$ relative to $\mathcal{B}_n$. Let $A = \langle a\rangle$ and $B = \langle b\rangle$. Then $A \cap B = \{1\}$ has finite index in each of A and B. But $\langle A, B\rangle / \{1\} \cong G/\{1\} \cong G$ is infinite.

Example 1.5. Let $G = \langle a, b;\ a^n = b^n = 1\rangle$ be the free product $\langle a; a^n = 1\rangle * \langle b; b^n = 1\rangle$. Let $A = \langle a\rangle$ and $B = \langle b\rangle$. Then $A \cap B = \{1\}$ has finite index in each of A and B but $\langle A, B\rangle / \{1\} \cong G/\{1\} \cong G$ is infinite.

Theorem 1.6. *Free groups satisfy property S.*

For a proof see, for example, [2]. Note that, since there are examples of groups which violate property S, property S is not, in general, preserved in homomorphic images.

Now we go on an orgy of giving definitions of properties. A group G satisfies property

(1) FC provided every element has only finitely many conjugates.

(2) S_1 provided, whenever A and B are finitely generated commensurable subgroups, $A \cap B$ has finite index in $\langle N_A(A \cap B), N_B(A \cap B)\rangle$. Here N indicates normalizer.

(3) S_2 provided, whenever A and B are finitely generated commensurable subgroups, $(A \cap B)^{\langle A,B\rangle}$ has finite index in $\langle A, B\rangle$. Here $(A \cap B)^{\langle A,B\rangle}$ is the normal closure of $(A \cap B)$ in $\langle A, B\rangle$.

(4) τ_0 provided the set of torsion elements forms a subgroup $\tau(G)$.

(5) τ provided the set of torsion elements forms a locally finite subgroup $\tau(G)$.

(6) U provided roots, when they exist, are unique.

Clearly property S implies each of properties S_1 and S_2 and property τ implies property τ_0. Moreover, every property U group is torsion free and every torsion free group satisfies property τ.

Theorem 1.7 (B.H. Neumann [3]). *Let G be an FC-group. Then G satisfies property τ_0. Moreover, $G/\tau(G)$ is abelian and torsion free.*

Corollary 1. *A torsion free FC-group is abelian.*

Note that Example 1.3 violates property U, Example 1.4 violates property τ and Example 1.5 violates property τ_0.

Theorem 1.8. *Property S implies property τ.*

Theorem 1.9. *Suppose G is torsion free and satisfies either property S_1 or property S_2. Then G satisfies property U.*

Proof of Theorem 1.8. To show that $\tau(G)$ forms a subgroup it will suffice to show that it is closed. Let $(a, b) \in \tau(G)^2$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$. Then $A \cap B$ has finite index in each of the finite groups A and B.Hence $A \cap B$ has finite index in $\langle A, B \rangle$.Thus, $|\langle A, B \rangle| = [\langle A, B \rangle : A \cap B]\,|A \cap B| < \infty$. Since $ab \in \langle A, B \rangle$ it has finite order. Thus, $\tau(G)$ is closed and forms a subgroup. We use induction on n to show that, if $(g_1, ..., g_n) \in \tau(G)^n$ then $\langle g_1, ..., g_n \rangle$ is finite. Clearly the result holds for $n = 1$. Now assume $n > 1$ and the result holds for $n - 1$. Then each of $A = \langle g_1, ..., g_{n-1} \rangle$ and $B = \langle g_n \rangle$ is finite so $A \cap B$ has finite index in both A and B. Then $A \cap B$ has finite index in $\langle A, B \rangle = \langle g_1, ..., g_n \rangle$. Hence, $|\langle A, B \rangle| = [\langle A, B \rangle : A \cap B]\,|A \cap B| < \infty$. Thus, by induction, we are finished. □

Remark 1.10. Essentially the same argument shows that each of the properties S_1 and S_2 implies property τ_0.

Proof of Theorem 1.9. If $a^n = 1 = b^n$ then $a = 1 = b$ since G is torsion free. So suppose $n > 1$ and $a^n = b^n \neq 1$. We may assume $G = \langle a, b \rangle$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$. Then $A \cap B$ is central in G since every element of $A \cap B$ commutes with both a and b. Thus, $N_A(A \cap B) = A$, $N_B(A \cap B) = B$, and $(A \cap B)^{\langle A,B \rangle} = A \cap B$. So, if either property S_1 or S_2 is satisfied, then $A \cap B$ has finite index in G. Now let $g \in G$ be arbitrary. Since $A \cap B$ is central in G, we must have $A \cap B \leq N_G(g)$. This forces $[G : N_G(g)] < \infty$. Since g was arbitrary, G is an FC-group.

It now follows from Corollary 1 that G is abelian. But in the torsion free abelian group $G, a^n = b^n \Rightarrow \left(ab^{-1}\right)^n = 1 \Rightarrow a = b$. □

2. Some standard and some coined universal algebraic terminology applied to groups

In this section we shall strive to be somewhat more precise about our terminology. We shall reserve the word *class* to mean any nonempty class of groups closed under isomorphism. We shall reserve the word *property* to mean any property consistent with the group axioms and preserved under group isomorphism. We shall find it convenient to commit the abuse of identifying classes and properties both in our verbiage and in our notation. By a *direct product*, or a *direct power* or a *subdirect product* we shall mean, in the case of infinitely many factors, the unrestricted version. That is, we do not insist that at most finitely many coordinates of an element must be distinct from the identity. In the case of finitely many factors we shall speak of *finite direct products*, etc. In what follows all terminology and notation that is more or less standard will be assumed familiar to the reader. None the less we have included an appendix containing a glossary of terms which the reader should feel free to consult at any point.

By a *class operator* we mean a function which accepts as inputs arbitrary classes and whose image on a class is again a class. We shall adopt boldface for class operators. A class operator $\mathbf{F}$ is a *closure operator* provided, for arbitrary classes $\mathcal{X}$ and $\mathcal{Y}$,

(1) $\mathcal{X} \subseteq \mathbf{F}\mathcal{X}$

(2) $\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathbf{F}\mathcal{X} \subseteq \mathbf{F}\mathcal{Y}$

(3) $\mathbf{F}(\mathbf{F}\mathcal{X}) = \mathbf{F}\mathcal{X}$.

The class operators below are (with the sole exception of our usage of $\mathbf{H}$ where he uses $\mathbf{Q}$) as defined in [4]. With the possible exception of $\mathbf{L}$ they are all closure operators.

- $\mathbf{H}\mathcal{X}$ is the class of all homomorphic images of groups in X.
- $\mathbf{S}\mathcal{X}$ is the class of all subgroups of groups in $\mathcal{X}$.
- $\mathbf{P}\mathcal{X}$ is the class of all groups isomorphic to a direct product of a family of groups in $\mathcal{X}$.
- $\mathbf{R}\mathcal{X}$ is the class of all groups isomorphic to a subdirect product of a family of groups in $\mathcal{X}$.

- $\mathbf{L}\mathcal{X}$ is the class of all groups isomorphic to a direct union of groups in $\mathcal{X}$.

A class is *hereditary* if it is closed under taking subgroups. Let us say that a class is *fg-hereditary* provided it is closed under taking finitely generated subgroups. We have the rather obvious

Lemma 2.1. *If $\mathcal{X}$ is fg-hereditary then $\boldsymbol{L}\mathcal{X}$ is the class of all groups whose finitely generated subgroups lie in $\mathcal{X}$.*

We then immediately have that $\mathbf{L}$ behaves like a closure operator when restricted to fg-hereditary classes. That is

Corollary 2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary fg-hereditary classes. Then*

(1) $\mathcal{X} \subseteq \mathbf{L}\mathcal{X}$

(2) $\mathcal{X} \subseteq \mathcal{Y} \Rightarrow \mathbf{L}\mathcal{X} \subseteq \mathbf{L}\mathcal{Y}$

(3) $\mathbf{L}(\mathbf{L}\mathcal{X}) = \mathbf{L}\mathcal{X}$.

With the possible exceptions of $R_0(G,X)$ and$R_0(G)$ (to be defined in the next section) all of the properties considered in this paper will be fg-hereditary. A group in $\mathbf{L}\mathcal{X}$ is said to be locally $\mathcal{X}$ while a group in $\boldsymbol{R}\mathcal{X}$ is said to be residually $\mathcal{X}$. Clearly G residually $\mathcal{X}$ is equivalent to the following. For every $g \in G\backslash\{1\}$ there is a group $H_g \in \mathcal{X}$ and an epimorphism $\varphi_g : G \to H_g$ such that $\varphi_g(g) \neq 1$. In the case when $\mathcal{X}$ is hereditary the insistence that the maps φ_g be surjective may be safely relaxed.

A class $\mathcal{V}$ closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ is a *variety* of groups. The isomorphism class $\mathcal{E}$ of the one element group is the *trivial variety.* All other varieties $\mathcal{V} \neq \mathcal{E}$ are nontrivial. A classical result of Garrett Birkhoff [Bi] asserts that a class $\mathcal{V}$ is a variety if and only if it is the model class of a set of laws. The variety $\mathcal{A}$ of abelian groups is the model class of the law $[x,y] = 1$ where $[x,y]$ is the commutator $x^{-1}y^{-1}xy$. If $c \geq 0$ is an integer, then the variety $\mathcal{N}_c$ of groups nilpotent of class at most c is the model class of the higher commutator law $[x_1, x_2, \ldots, x_c, x_{c+1}] = 1$ where $[x_1, \ldots, x_n]$ is defined inductively by $[x_1] = x_1, [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ and $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$ if $n \geq 3$. Note that we identify $\mathcal{N}_0$ with $\mathcal{E}$.

If $\mathcal{U}$ and $\mathcal{V}$ are varieties then their *product* $\mathcal{U}\mathcal{V}$ is the class of all extensions G of a group $K \in \mathcal{U}$ by a group $H \in \mathcal{V}$. The product of two varieties is again a variety. Multiplication of varieties is associative. In particular powers of varieties are well-defined. Thus, if $d \geq 0$ is an integer, then $\mathcal{A}^d$ is the variety of all groups solvable of length at most d and where, by convention, $\mathcal{A}^0 = \mathcal{E}$ and $\mathcal{A}^1 = \mathcal{A}$.

For each positive integer n the *Burnside variety* $\mathcal{B}_n$ is the variety determined by the law $x^n = 1$. The union $\mathcal{N}$ of the chain $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots$ is the class of nilpotent groups and the union $\mathcal{S}$ of the chain $\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \mathcal{A}^2 \subseteq \cdots$ is the class of solvable groups. Here is a good point to advertise

Theorem 2.2 (Fine, Gaglione and Spellman [5])**.** *The class $\mathcal{X}$ is a union of varieties if and only if it is closed under taking subgroups, homomorphic images and direct powers; furthermore, $\mathcal{X}$ is a direct union of varieties if and only if additionally it is closed under taking finite direct products.*

Every nontrivial variety admits free objects of all ranks. If $\mathcal{V}$ is any nontrivial variety and r is any cardinal then $F_r(\mathcal{V})$ shall denote a fixed but arbitrary group free of rank r in $\mathcal{V}$. For fixed r and $\mathcal{V} \neq \mathcal{E}$ this is unique up to isomorphism.

We now coin some terminology for our purposes.

Definition 2.3. *Let $\mathcal{X}$ be a property. A G group satisfies*

$$\textbf{\textit{strong}}\ \mathcal{X}\ \textbf{\textit{or}}\ (\mathbf{HS})^{-1}\mathcal{X}$$

*provided whenever G_0 is a subgroup in G and G_o^* is a homomorphic image of G_0 it is the case that G_0^* satisfies $\mathcal{X}$. $\mathcal{X}$ is a **strong property** provided $(\boldsymbol{HS})^{-1}\mathcal{X} = \mathcal{X}$.*

Thus property S, which is not preserved in homomorphic images, is not strong. On the other hand, in view of Theorem 2.2, each of nilpotence and solvability is strong. (Admittedly the strength of nilpotence and solvability is rather obvious without explicit appeal to Theorem 2.2.) We make the transparent observations

(1) strong $\mathcal{X} \Rightarrow \mathcal{X}$ and

(2) If $\mathcal{X} \Rightarrow \mathcal{Y}$, then strong $\mathcal{X} \Longrightarrow$ strong $\mathcal{Y}$.

3. Nilpotent groups and more

Our reference for nilpotent groups shall be [1]. It is well-known that nilpotent groups satisfy property τ and that torsion free nilpotent groups satisfy property U. Probably less well-known is the fact, proven in [2], that nilpotent groups satisfy property S (from which the aforementioned properties τ and U, in the torsion free case, follow). This is a consequence of a technical result of Mal'cev.

Lemma 3.1 (Mal'cev [1]). *Let X be a finite set of generators for a nilpotent group G and let H be a subgroup in G. Then H has finite index in G if and only if for every $x \in X$ there is a positive integer $n(x)$ such that $x^{n(x)} \in H$.*

Corollary 3. *Nilpotent groups satisfy property S.*

Proof. Suppose A and B are finitely generated commensurable subgroups of a nilpotent group G. Say $A = \langle a_1, ..., a_p \rangle$ and $B = \langle b_1, ..., b_q \rangle$. Suppose $a^{m_i} \in A \cap B, i = 1, ..., p$ and $b_j^{n_j} \in A \cap B,\ j = 1, ..., q$ where the m_i and n_j are positive integers. Since $\{a_1, ..., a_p, b_1, ..., b_q\}$ generates $\langle A, B \rangle$ we conclude from Lemma 3.1 that $[\langle A, B \rangle : A \cap B] < \infty$. □

Note that nilpotent groups actually satisfy strong property S since nilpotence is strong. Now following the development (but not the notation) in [2] let us say that a group with a finite generating set X satisfies $\mathrm{R}_0(G, X)$ provided whenever H is a subgroup in G it is the case that H has finite index in G if and only if for each $x \in X$ there is a positive integer $n(x)$ such that $x^{n(x)} \in H$. If G is any finitely generated group we say that G satisfies $\mathrm{R}_0(G)$ provided it satisfies $\mathrm{R}_0(G, X)$ for every finite set X of generators. Finally, G satisfies R_0 provided it is finitely generated and every finitely generated subgroup $H \leq G$ satisfies $\mathrm{R}_0(H)$. By its very definition property R_0 is fg-hereditary. *Property R* was defined in [2] to be our property R_0 and in [7] to be our property $\mathbf{L}\mathrm{R}_0$. Note that these formulations coincide on the class of finitely generated groups. Thus, we choose the latter. Explicitly —

Definition 3.2. *G satisfies **property R** provided whenever X is a finite subset of G and H is a subgroup in $\langle X \rangle$ it is the case that H has finite index in $\langle X \rangle$ if and only if for every $x \in X$ there is a positive integer $n(x)$ such that $x^{n(x)} \in H$.*

Examples of non-nilpotent property R groups were given in [2] and [7]. Clearly the proof of Corollary 3 yields

Theorem 3.3. *Property R implies property S.*

In fact, since it was proven in [7] that property R is preserved in homomorphic images, property R is strong and hence property R implies strong property S. Now since property S implies property τ strong property S implies strong property τ. The chain of implications gives that property R implies strong property τ. In fact it was proven in [7] that strong property τ is equivalent to a weakened version of property R.

Definition 3.4 ([7]). *The G group satisfies* ***weak property R*** *provided whenever X is a finite subset of G and K is a subgroup of $\langle X\rangle$ which is normal in $\langle X\rangle$ it is the case that K has finite index in $\langle X\rangle$ if and only if for each $x \in X$ there is a positive integer $n(x)$ such that $x^{n(x)} \in K$.*

Observe that essentially the proof of Corollary 3 establishes that a weak property R group satisfies each of the properties S_1 and S_2. Hence (from either S_1 or S_2) it follows that a torsion free weak property R group is a U-group.

Theorem 3.5 ([7]). *Weak property R is equivalent to strong property τ.*

In [7] the authors posed the question of whether or not weak property R implies property R. In the next section we present an explicit counterexample establishing that weak property R is strictly weaker than property R.

4. The counterexample

An easy induction on the solvability length establishes the classic

Lemma 4.1 (Phillip Hall). *A finitely generated solvable group of finite exponent is finite.*

Now suppose G is a solvable group whose commutator subgroup G' has finite exponent e. Suppose g and h are torsion elements in G. Then certainly there are positive integers m and n such that $g^m \equiv h^n \equiv 1(\text{mod } G')$. Let m_0 and n_0 be the least such positive integers and let L be the least common multiple of m_0 and n_0. Then $(gh)^L \equiv g^L h^L \equiv 1(\text{mod } G')$. It follows that $(gh)^{Le} = ((gh)^L)^e = 1$ since G' has exponent e. Thus, the set $\tau(G)$ of torsion elements in G forms a subgroup. Now let $(g_1, ..., g_k) \in \tau(G)^k$ be a finite tuple of torsion elements. Then, for all $i = 1, ..., k$ there is a positive integer n_i such that $g_i^{n_i} \equiv 1(\text{mod } G')$. Let m_i be the least such positive integer and let L be the least common multiple of $m_1, ..., m_k$. Now suppose is $x \in \langle g_1, ..., g_k\rangle$ is arbitrary. We may assume $x \equiv g_1^{\xi_1} \cdots g_k^{\xi_k}(\text{mod } G')$. Then $x^L \equiv g_1^{\xi_1 L} \cdots g_k^{\xi_k L} \equiv 1(\text{mod } G')$. Consequently $x^{Le} = 1$. It follows that the finitely generated solvable group $\langle g_1, ..., g_k\rangle$ has finite exponent. Hence, by Lemma 4.1, it is finite. Therefore $\tau(G)$ is locally finite and hence G satisfies property τ. But it is easy to see that solvable with derived group of finite exponent is a strong property. Hence, G satisfies strong property τ. Equivalently, G satisfies weak property R. We have proven

Theorem 4.2. *A solvable group whose commutator subgroup has finite exponent satisfies weak property R.*

Now consider the product variety $\mathcal{B}_2\mathcal{A}$. First we concentrate on the left hand factor $\mathcal{B}_2$. That is the variety of all groups satisfying the law $x^2 = 1$. But these are precisely the elementary abelian 2-groups. That is, vector spaces over the two element field. Hence, $\mathcal{B}_2 \subseteq \mathcal{A}$ and $\mathcal{B}_2\mathcal{A} \subseteq \mathcal{A}^2$ so every group in $\mathcal{B}_2\mathcal{A}$ is metabelian. Furthermore, if $G \in \mathcal{B}_2\mathcal{A}$, there is a short exact sequence

$$1 \to K \to G \to H \to 1$$

where $K \in \mathcal{B}_2$ and H is abelian. Then $G' \leq K$ must satisfy the law $x^2 = 1$. Thus, every group in $\mathcal{B}_2\mathcal{A}$ is a metabelian group whose commutator subgroup has exponent dividing 2. In view of Theorem 4.2 every group $\mathcal{B}_2\mathcal{A}$ satisfies weak property R. We shall prove that the group $G = F_2(\mathcal{B}_2\mathcal{A})$ free of rank 2 in $\mathcal{B}_2\mathcal{A}$ violates property R.

Consider the free product $X * Y$ of abelian groups X and Y. By problems 23, 24 and 34 pages 196 and 197 of [8], the commutator subgroup of $X * Y$ is freely generated by the commutators $[x, y]$ as x and y vary independently over $X\backslash\{1\}$ and $Y\backslash\{1\}$ respectively. Thus, the derived group $\widehat{F}\,'$ of $\widehat{F} = \langle \widehat{a}; \rangle * \left\langle \widehat{b}; \right\rangle = \left\langle \widehat{a}, \widehat{b}; \right\rangle$ is freely generated by the commutators $[\widehat{a}^m, \widehat{b}^n]$ as (m, n) varies over $(\mathbb{Z}\backslash\{0\})^2$. Now suppose that $\{a, b\}$ is a basis relative to $\mathcal{B}_2\mathcal{A}$ for $G = F_2(\mathcal{B}_2\mathcal{A})$. Under the epimorphism $\widehat{F} \to G$ determined by $\widehat{a} \mapsto a, \widehat{b} \mapsto b$, $\widehat{F}\,'$ maps onto G'. It follows that, as (m, n) varies over $(\mathbb{Z}\backslash\{0\})^2$, the commutators $[a^m, b^n]$ form a basis for the multiplicatively written elementary abelian 2-group G'. Hence, apart from the order of the factors in G' every element of G is uniquely expressible in the form

$$a^\alpha b^\beta \prod_{(m,n)\in(\mathbb{Z}\backslash\{0\})^2} [a^m, b^n]^{\gamma(m,n)}$$

where α and β are integers, $\gamma(m, n) \in \{0, 1\}$and all but finitely many $\gamma(m, n)$ are zero. (See also Corollary 21.13 of [6] and its proof.) Moreover, since G' is abelian, the order of the factors in $\prod_{(m,n)\in(\mathbb{Z}\backslash\{0\})^2} [a^m, b^n]^{\gamma(m,n)}$ is immaterial. Now, since G is free in $\mathcal{B}_2\mathcal{A}$ on $\{a, b\}$ the assignment $a \mapsto a^2, b \mapsto b^2$ extends uniquely to an epimorphism σ from G onto the subgroup $\langle a^2, b^2 \rangle$. Applying σ to $a^\alpha b^\beta \prod_{(m,n)\in(\mathbb{Z}\backslash\{0\})^2} [a^m, b^n]^{\gamma(m,n)}$ we get

$$a^{2\alpha} b^{2\beta} \prod_{(m,n)\in(\mathbb{Z}\backslash\{0\})^2} [a^{2m}, b^{2n}]^{\gamma(m,n)}.$$

By uniqueness, σ has trivial kernel and hence is an isomorphism. It follows that every element of $\langle a^2, b^2 \rangle$ is uniquely expressible in the form

$$a^{2\alpha} b^{2\beta} \prod_{(m,n)\in(\mathbb{Z}\setminus\{0\})^2} [a^{2m}, b^{2n}]^{\gamma(m,n)}.$$

More importantly for our purposes, it follows that

$$a^{\alpha} b^{\beta} \prod_{(m,n)\in(\mathbb{Z}\setminus\{0\})^2} [a^{m}, b^{n}]^{\gamma(m,n)}$$

lies in $\langle a^2, b^2 \rangle$ if and only if both α and β are even and, additionally, $\gamma(m,n) = 0$ whenever at least one of m or n is odd. Now let $k_1 \neq k_2$ be integers. Recalling that $[x,y]^{-1} = [x,y]$ for all commutators $[x,y] \in G'$ since G' is an elementary abelian 2-group,we have that $[a, b^{2k_1+1}][a, b^{2k_2+1}]^{-1} = [a, b^{2k_1+1}][a, b^{2k_2+1}]$ does not lie in $H = \langle a^2, b^2 \rangle$. Thus, $H\ [a, b^{2k_1+1}] \neq H[a, b^{2k_2+1}]$. Consequently $H = \langle a^2, b^2 \rangle$ has infinite index in G. Hence, G violates property R.

5. Questions

We conclude with two questions each of which we conjecture has a negative answer.
Question 1: *Does weak property R imply property S?*
Question 2: *Does strong property S imply property R?*

6. Appendix (Glossary)

1. *Direct union of groups*
   A group G is the *direct union* of a family $\mathcal{H}$ of subgroups provided
   (1) $G = \cup\, \mathcal{H}$ and
   (2) For all $H_0, H_1 \in \mathcal{H}$ there is $H \in \mathcal{H}$ such that $\langle H_0, H_1 \rangle \leq H$.
2. *Direct union of classes*
   The class $\mathcal{Y}$ is the *direct union* of the family $\mathcal{F}$ of subclasses provided
   (1) $\mathcal{Y} = \cup\, \mathcal{F}$ and
   (2) For all $\mathcal{X}_0, \mathcal{X}_1 \in \mathcal{F}$ there is $\mathcal{X} \in \mathcal{F}$ such that $\mathcal{X}_0 \cup \mathcal{X}_1 \subseteq \mathcal{X}$.
3. *Subdirect product of a family of groups*
   Let I be a nonempty set and let $(G_i)_{i\in I}$ be a family of (not necessarily distinct) groups indexed by I. Let $P = \prod_{i\in I} G_i$ be the direct product of the indexed family. That is, P consists of all choice functions $g : I \to \bigcup_{i\in I} G_i, i \mapsto g_i \in G_i$ for all $i \in I$ with group operations defined componentwise. For each fixed $i \in I$ let be $p_i : P \to G_i$ projection onto the i-th coordinate, $p_i(g) = g_i$. A subgroup $G \leq P$ is a *subdirect product*

of the family $(G_i)_{i\in I}$ provided, for all $i \in I$, the restriction $p_i\mid_G$maps G onto G_i.

4. *Relatively free groups and verbal subgroups*

Let $\mathcal{V}$ be a nontrivial variety of groups. Suppose X is a set of generators for a group Φ in $\mathcal{V}$. Then Φ is free in $\mathcal{V}$ on X provided every assignment of values $X \to G$ from X into a group G in $\mathcal{V}$ extends (necessarily uniquely) to a homomorphism $\Phi \to G$. It follows from abstract nonsense that, if r is any cardinal, then $F_r(\mathcal{V})$ is unique up to isomorphism whenever it exists. Existence is deduced by applying the concept of verbal subgroup defined below.

Let G be any group – not necessarily in $\mathcal{V}$. Let $\boldsymbol{V}(G)$ be the family of subgroups K normal in G such that G/K lies in $\mathcal{V}$. Clearly $G \in \boldsymbol{V}(G)$ so $\boldsymbol{V}(G)$ is nonempty. Now we get a homomorphism defined by $\varphi : G \to \prod_{K\in\boldsymbol{V}(G)} (G/K)$ defined by $\varphi(g)(K) = Kg$. Since $\mathcal{V}$ is a variety it is closed under direct products and subgroups so the image $\varphi[G]$ lies in $\mathcal{V}$. Hence, $\mathrm{Ker}(\varphi)$ lies in $\boldsymbol{V}(G)$. But $\mathrm{Ker}(\varphi) = \cap\boldsymbol{V}(G)$ is then the unique minimum element of $\boldsymbol{V}(G)$. By definition $\cap\boldsymbol{V}(G)$ is the *verbal subgroup* $V(G)$ corresponding to the variety $\mathcal{V}$.

In practice, if $w(\mathbf{x}) = 1$ is a set of laws (on tuples $\mathbf{x}$ of variables) determining $\mathcal{V}$, then $V(G)$ is the subgroup of generated by the elements $w(\mathbf{g})$ as $w(\mathbf{x}) = 1$ varies over the laws and $\mathbf{g}$ varies over tuples from G. If r is a cardinal and F is absolutely free (i.e. free in the variety of all groups) on $\{a_{\xi+1} : 0 \le \xi < r\}$, then one can prove that $F/V(F)$ is free in $\mathcal{V}$ on $\{V(F)a_{\xi+1} : 0 \le \xi < r\}$.

Remark 6.1. One can show that $\mathcal{V} = \mathcal{B}_2\mathcal{A}$ is determined by the law

$$([x, y][z, w])^2 = 1.$$

Thus, $F_2(\mathcal{B}_2\mathcal{A})$ is $\widehat{F}/V(\widehat{F})$ where $\widehat{F} = \left\langle \widehat{a}, \widehat{b}; \right\rangle$and V is the verbal subgroup operator corresponding to the law $([x, y][z, w])^2 = 1$.

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Contact information

A. M. Gaglione
Department of Mathematics
U.S. Naval Academy
Annapolis, MD 21402 USA
E-Mail: amg@usna.edu

S. Lipschutz
Department of Mathematics
Temple University
Philadephia, PA 19122 USA
E-Mail: seymour@temple.edu

D. Spellman
Department of Mathematics
Temple University
Philadephia, PA 19122 USA
E-Mail: spellman@temple.edu

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