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On modules over group rings of locally soluble groups for a ring of p-adic integers

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Abstract. The author studies the $\mathbf{Z_{p^\infty}}G$-module A such that $\mathbf{Z_{p^\infty}}$ is a ring of p-adic integers, a group G is locally soluble, the quotient module $A/C_A(G)$ is not Artinian $\mathbf{Z_{p^\infty}}$-module, and the system of all subgroups $H \leq G$ for which the quotient modules $A/C_A(H)$ are not Artinian $\mathbf{Z_{p^\infty}}$-modules satisfies the minimal condition on subgroups. It is proved that the group G under consideration is soluble and some its properties are obtained.

1. Introduction

At present a number of works of scientists is devoted to the investigation of infinite dimensional linear groups with the large system of sugroups which are similar to finite dimensional groups. The one of basic notions which is applied here is the central dimension of subgroup, i.e. the codimension of centralizer of this subgroup in the vector space. The linear groups of finite central dimension turned out very similar to the usual finite dimensional linear groups. The one of directions in the theory of infinite dimensional linear groups is engaged in the study of linear groups with some essential restrictions on the family of subgroups of infinite central dimension. The natural extension of the theory of linear groups is the theory of modules over group rings. In this work the one of analogues of the notion of central dimension is considered. In the theory of modules there exists a number of generalizations of finite dimensional vector space. These are modules with finite composition series, finitely generated modules, Noetherian modules, Artinian modules.

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The broad class of modules over group rings is Artinian modules over group rings. Remind that a module is called Artinian if partially ordered set of all submodules of this module satisfies the minimal condition. It should be noted that many problems of Algebra require the investigation of some specific Artinian modules. The Artinian modules over group rings with different restrictions on groups were studied in [1]. Naturally, it is arised the question on investigation of modules over group rings which are not Artinian but which are similar to Artinian modules in some sense.

Let A be a $\mathbf{D}G$-module where $\mathbf{D}$ is Dedekind domain, G is a group. If $H \leq G$ then the quotient module $A/C_A(H)$ is called the cocentralizer of H in module A. The subject of investigation of this paper is a $\mathbf{D}G$-module A where $\mathbf{D} = \mathbf{Z_{p^\infty}}$ is a ring of p-adic integers.

Let A be $\mathbf{Z_{p^\infty}}G$-module such that the cocentralizer of the group G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module and let $L_{nad}(G)$ be a system of all subgroups of G such that its cocentralizers in module A are not Artinian $\mathbf{Z_{p^\infty}}$-modules. The system of $L_{nad}(G)$ is the partially ordered set relative to usual inclusion of subgroups. If the system $L_{nad}(G)$ satisfies the minimal condition on subgroups then we shall say that the group G satisfies the condition $min - nad$.

Later on in the work it is considered $\mathbf{Z_{p^\infty}}G$-module A such that $C_G(A) = 1$. Let A be a $\mathbf{Z_{p^\infty}}G$-module such that the cocentralizer of group G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module and the group G is locally soluble and satisfies the condition $min - nad$. In the work it is proved that in this case the group G is soluble, and some its properties are obtained.

The basic results of the work are the following theorems 1.1, 1.2 and 1.3.

Theorem 1.1. *Let A be $\mathbf{Z_{p^\infty}}G$-module, G be a locally soluble group which satisfies the condition $min - nad$. If the cocentralizer of group G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module then the group G is soluble.*

Theorem 1.2. *Let A be $\mathbf{Z_{p^\infty}}G$-module, G be a locally soluble group which satisfies the condition $min - nad$. If the cocentralizer of group G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module then the group G contains the normal nilpotent subgroup H such that the quotient group G/H is Chernikov group.*

Theorem 1.3. *Let A be $\mathbf{Z_{p^\infty}}G$-module, G be a locally soluble group. Suppose that the cocentralizer of group G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. If the cocentralizer of each proper subgroup of group G in module A is Artinian $\mathbf{Z_{p^\infty}}$-module then $G \simeq C_{q^\infty}$ for some prime q.*

2. Preliminary results

We mention some elementary facts on $\mathbf{Z_{p^\infty}}G$-modules. Recall that if $K \leq H \leq G$ and the cocentralizer of subgroup H in module A is Artinian $\mathbf{Z_{p^\infty}}$-module then the cocentralizer of subgroup K in module A is Artinian $\mathbf{Z_{p^\infty}}$-module also. If $U, V \leq G$ such that their cocentralizers in module A are Artinian $\mathbf{Z_{p^\infty}}$-modules then the quotient module $A/(C_A(U) \cap C_A(V))$ is Artinian $\mathbf{Z_{p^\infty}}$-module also. Therefore the cocentralizer of subgroup $< U, V >$ in module A is Artinian $\mathbf{Z_{p^\infty}}$-module.

Suppose now that the group G satisfies the condition $min - nad$. If $H_1 > H_2 > H_3 > ...$ is an infinite strictly descending chain of subgroups of G, then there is a natural number n such that the the cocentralizer of subgroup H_n in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Moreover, if N is a normal subgroup of G and the cocentralizer of subgroup N in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module then the quotient group G/N satisfies the minimal condition on subgroups.

Lemma 2.1 [2]. *Let A be $\mathbf{Z_{p^\infty}}G$-module and suppose that G satisfies the condition $min - nad$. Let X, H be subgroups of G and Λ be an index set such that:*

(i) $X = Dr_{\lambda \in \Lambda} X_\lambda$, where $1 \neq X_\lambda$ is an H-invariant subgroup of X, for each $\lambda \in \Lambda$.

(ii) $H \cap X \leq Dr_{\lambda \in \Gamma} X_\lambda$ for some subset Γ of Λ.

If $\Omega = \Lambda \setminus \Gamma$ is infinite, then the cocentralizer of subgroup H in module A is Artinian $\mathbf{Z_{p^\infty}}$-module.

Lemma 2.2. *Let A be $\mathbf{Z_{p^\infty}}G$-module and suppose that G satisfies the condition $min - nad$. Let H, K be subgroups of G such that K is a normal subgroup of H and suppose that there exists an index set Λ and subgroups H_λ of G such that $K \leq H_\lambda$ for all $\lambda \in \Lambda$. Suppose that $H/K = Dr_{\lambda \in \Lambda} H_\lambda / K$ and that Λ is infinite. Then the cocentralizer of subgroup H in module A is Artinian $\mathbf{Z_{p^\infty}}$-module.*

Proof. Suppose that Λ is infinite. Let Γ and Ω be infinite disjoint subsets of Λ such that $\Lambda = \Gamma \cup \Omega$. Let $U/K = Dr_{\lambda \in \Gamma} H_\lambda / K$, let $V/K = Dr_{\lambda \in \Omega} H_\lambda / K$, and let $\Gamma_1 \supseteq \Gamma_2 \supseteq ...$ be a strictly descending chain of subsets of Γ. Then we obtain an infinite descending chain

$$< U, H_\lambda | \lambda \in \Gamma_1 > \; >< U, H_\lambda | \lambda \in \Gamma_2 > \; > ... \; .$$

of subgroups. It follows from the condition $min - nad$ that the cocentralizer of subgroup U in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Likewise, the cocentralizer of subgroup V in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Since $H = UV$, it follows that the cocentralizer of subgroup H in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. The lemma is proved. □

Lemma 2.3. *Let A be $\mathbf{Z_{p^\infty}}G$-module and suppose that G satisfies the condition $min-nad$. If the element $g \in G$ has infinite order, then the cocentralizer of subgroup $< g >$ in module A is Artinian $\mathbf{Z_{p^\infty}}$-module.*

Proof. Let p, q be distinct primes greater than 3 and let $u = g^p$, $v = g^q$. Then there is an infinite descending chain $< u >>< u^2 >>< u^4 >> \ldots$ Therefore for some natural number k, the cocentralizer of subgroup $< u^{2^k} >$ in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Simirlarly, there exists a natural number l such that the cocentralizer of subgroup $< v^{3^l} >$ in module A is Artinian $\mathbf{Z_{p^\infty}}$-module also. Therefore the cocentralizer of subgroup $< g >=< u^{2^k} >< v^{3^l} >$ in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. The lemma is proved. □

The following result gives an important information about the derived quotient group.

Lemma 2.4. *Let A be $\mathbf{Z_{p^\infty}}G$-module. Suppose that G satisfies the condition $min-nad$ and the cocentralizer of G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Then the quotient group G/G' is Chernikov group.*

Proof. Suppose for a contradiction that G/G' is not Chernikov group. Let S be a set of all subgroups $H \leq G$ such that H/H' is not Chernikov group and the cocentralizer of subgroup H in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Since $G \in S$ then $S \neq \emptyset$. Since S satisfies the minimal condition it has a minimal element D. If U, V are proper subgroups of D such that $D = UV$ and $U \cap V = D'$, then at least for one of these subgroups, U say, the cocentralizer of U in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. The choice of D implies that U/U' is Chernikov group. Hence, $U/D' \simeq (U/U')/(D'/U')$ is Chernikov group too. Since the cocentralizer of subgroup U in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module, it follows that the abelian group D/U is also Chernikov group. Hence D/D' is Chernikov group. Contrary to the choice of D. Therefore the quotient group D/D' is indecomposable. Hence D/D' is isomorphic to the subgroup of C_{q^∞} for some prime q. Contradiction. The lemma is proved. □

Let A be $\mathbf{Z_{p^\infty}}G$-module and G satisfies the condition $min-nad$. Let $AD(G)$ be the set of all elements $x \in G$ such that the cocentralizer of group $< x >$ in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Since $C_A(x^g) = C_A(x)g$ for all $x, g \in G$ then $AD(G)$ is a normal subgroup of G.

Lemma 2.5. *Let A be $\mathbf{Z_{p^\infty}}G$-module. Suppose that G satisfies the condition $min-nad$ and the cocentralizer of G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Then G either is periodic or $G = AD(G)$.*

Proof. We suppose to the contrary that G is neither periodic nor $G \neq AD(G)$. Let S be the set of all subgroups $H \leq G$ such that H is non-periodic group and $H \neq AD(H)$. Then S is non-empty. If $H \neq AD(H)$, then there is an element $h \in H$ such that the quotient module $A/C_A(h)$ is not Artinian $\mathbf{Z_{p^\infty}}$-module. Hence $S \subseteq L_{nad}(G)$ and S therefore has the minimal condition. Let D be a minimal element of S, let $L = AD(D)$. Note that $L \neq 1$, since D is non-periodic. Let $L \leq W \leq D$ and $W \neq D$. By lemma 2.3 L contains all elements of infinite order of non-periodic subgroup D then W is non-periodic subgroup. Therefore $W = AD(W)$ so $W \leq L$. Hence D/L has order q for some prime q. Let $x \in D \setminus L$. If a is an element of infinite order, then the minimal choice of D implies that $< x, a >= D$. Since $|D : L|$ is finite and D is a finitely generated subgroup then by theorem 1.41 [3] L is also finitely generated subgroup. Since $L = AD(L)$, the quotient module $A/C_A(L)$ is Artinian $\mathbf{Z_{p^\infty}}$-module. Since L is a normal subgroup of D, then $C = C_A(L)$ is $\mathbf{Z_{p^\infty}}G$-submodule of the module A. If $R = C_D(A/C)$, then R is a normal subgroup of the group D. Since the quotient module A/C is Artinian $\mathbf{Z_{p^\infty}}$-module, and by the theorem 7.13 [4] $A/C = A_1/C \oplus A_2/C \oplus ... \oplus A_n/C, i = 1, 2, ...n$, where each direct summand is either Prüfer $\mathbf{Z_{p^\infty}}$-module or finitely generated $\mathbf{Z_{p^\infty}}$-module. In the case $\mathbf{D} = \mathbf{Z_{p^\infty}}$ for maximal ideal P of D the additive group of D/P has the order p. By corollary 1.28 [4] D/P^k and P/P^{k+1} are isomorphic as D-modules for any $k = 1, 2, ..., n, ...$. In particular, the additive group of D/P^k is a cyclic group of order p^k. Let $D/P^k =< a_k >$. We can define the mapping $\pi_k^{k+1} : D/P^k \longrightarrow D/P^{k+1}$ such that $\pi_k^{k+1}(a_k) = pa_{k+1}$. Therefore we can consider the injective limit of the family of D-modules $D/P^k, k = 1, 2, ..., n, ...$. From the choice of a_1 it followes that $pa_1 = 0$. By the definition of Prüfer bfZ_{p^∞}-module (see chapter 5 [4]) this module is the injective limit of the family of D-modules $D/P^k, k = 1, 2, ..., n, ...$. In the case $\mathbf{D} = \mathbf{Z_{p^\infty}}$ the additive group of Prüfer $\mathbf{Z_{p^\infty}}$-module is quasicyclic p-group. Since each ideal of $\mathbf{Z_{p^\infty}}$ has in this ring finite index (see chapter 8 [5]), then a finitely generated $\mathbf{Z_{p^\infty}}$-module is finite. Therefore the additive group of the quotient module A/C is Chernikov group, and its divisible part is p-group. By theorem 60.1.1 [6] the quotient group D/R is isomorphic to some subgroup of $GL(r, \mathbf{Z_{p^\infty}})$. Let U be a normal subgroup of D of finite index. Then U is not periodic and so $< U, x >$ is not periodic and $< U, x >\neq AD(< U, x >)$. The minimal choice of D implies that $D =< U, x >$. Since U is a normal subgroup of D then the quotient group D/U is a cyclic group. Therefore D/U is an abelian quotient group. If E is the finite residual of D, it follows that D/E is abelian. Since $E \leq R$, then the quotient group D/R is also abelian. Therefore $D/(R \cap L)$ is abelian. $R \cap L$ is a subgroup of stabilizer of the series of length 2, and therefore it is abelian. So that D is a finitely generated metabelian group. By a theorem of P.Hall (theorem 9.51 [3]) D is residually finite group. As above, D is therefore abelian. Since $D = U < x >$ for every subgroup U of finite index, it follows that D is infinite cyclic. By lemma 2.3 $D = AD(D)$. The contradiction. The lemma is proved. □

3. Locally soluble groups with min-nad

Let G be a locally soluble group and the quotient module $A/C_A(G)$ be Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module. As in the proof of lemma 2.5, the quotient group $G/C_G(A/C_A(G))$ is isomorphic to the locally soluble subgroup of $GL(r, \mathbf{Z}_{\mathbf{p}^\infty})$. Since $\mathbf{Z}_{\mathbf{p}^\infty}$ is an integral ring then it can be imbeded in the field F. Therefore the quotient group $G/C_G(A/C_A(G))$ is isomorphic to some locally soluble subgroup of the linear group $GL(r, F)$. Hence by corollary 3.8 [7] the quotient group $G/C_G(A/C_A(G))$ is soluble. Since $C_G(A/C_A(G))$ is abelian group then G is a soluble group. Therefore it is necessary to concentrate attenton on the study of locally soluble groups G with $min - nad$ for which the quotient module $A/C_A(G)$ is not Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module.

Lemma 3.1. *Let A be $\mathbf{Z}_{\mathbf{p}^\infty}G$-module, G is a periodic locally soluble group which satisfies the condition $min - nad$, and the cocentralizer of G in module A is not Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module. Then G either satisfies the minimal condition on subgroups or $G = AD(G)$.*

Proof. We suppose to the contrary that G neither satisfies the minimal condition for subgroups nor $G \neq AD(G)$. Let S be a set of subgroups $H \leq G$ such that H does not satisfies the minimal condition for subgroups and not $H \neq AD(H)$. Then $S \neq \emptyset$ and satisfies the minimal condition. Let D be a minimal element of S and $L = AD(D)$. There is an infinite strictly descending chain of subgroups of D:

$$H_1 > H_2 > H_3 > ....$$

Since D has $min - nad$, there is a natural number d such that the cocentralizer of subgroup H_d in module A is Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module. Clearly $H_d \leq L$ and hence L does not satisfy the minimal condition. It follows that if $x \in D \setminus L$, then $< x, L >= D$, by the minimal choice of D. Therefore D/L has prime order q, for some prime q. Replacing x by a suitable power if necessary, we may assume that x has order q^r for some natural number r. Since D is not Chernikov group, then by D.I.Zaitzev theorem [8], D contains an $< x >$-invariant abelian subgroup

$B = Dr_{n\in\mathbf{N}} < b_n >$ and we may assume that b_n has prime order, for each $n \in \mathbf{N}$. Let $1 \neq c_1 \in B$ and $C_1 =< c_1 >^{<x>}$. Then C_1 is finite and there is a subgroup E_1 such that $B = C_1 \times E_1$. Let $U_1 = core_{<x>}E_1$. Then U_1 has finite index in B. If $1 \neq c_2 \in U_1$ and $C_2 =< c_2 >^{<x>}$, then C_2 is a finite $< x >$-invariant subgroup and $< C_1, C_2 >= C_1 \times C_2$. Continuing in this manner, we can construct a family $\{C_n | n \in \mathbf{N}\}$ of finite $< x >$-invariant subgroups of B such that $\{C_n | n \in \mathbf{N}\}= Dr_{n\in\mathbf{N}} C_n$. Lemma 2.1 implies that $x \in L$. The contradiction. The lemma is proved. □

From lemmas 2.5 and 3.1 validity of the theorem is followed.

Theorem 3.2. *Let A be $\mathbf{Z_{p^\infty}}G$-module, G is a locally soluble group which satisfies the condition $min-nad$, and the cocentralizer of G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Then either G satisfies the minimal condition or $G = AD(G)$.*

Lemma 3.3. *Let A be $\mathbf{Z_{p^\infty}}G$-module, G is a locally soluble group which satisfies the condition $min - nad$, and the cocentralizer of G in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Then either G is soluble or G has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq ... \leq W_n \leq ... \leq W_\omega = \cup_{n\in\mathbf{N}} W_n \leq G$, such that the cocentralizer of each subgroup W_n in module A is Artinian $\mathbf{Z_{p^\infty}}$-module and W_{n+1}/W_n is abelian for $n \geq 0$. Moreover, in this case G/S_ω is soluble.*

Proof. At first we show that G is hyperabelian. It is sufficiently to show that every non-trivial image of G contains a non-trivial normal abelian subgroup.

Let H be a proper normal subgroup of G. Suppose first that the cocentralizer of H in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Then the quotient group G/H has the minimal condition on subgroups. Therefore it is Chernikov group and has a non-trivial abelian normal subgroup. Now we suppose that the cocentralizer of H in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Let $S = \{M_\sigma/H | \sigma \in \Sigma\}$ be a family of all non-trivial normal subgroups of G/H. Suppose at first that for each $\sigma \in \Sigma$ the cocentralizer of M_σ in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. We show that in this case G/H has the minimal condition on normal subgroups. Let $\{M_\delta/H\}$ be a non-empty subset of S. For any δ the cocentralizer of M_δ in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. By the condition $min-nad$ the set $\{M_\delta\}$ has the minimal element M. Hence M/H is the minimal element of subset $\{M_\delta/H\}$. Therefore G/H has the minimal condition on normal subgroups. Hence G/H is hyperabelian, and G/H has a non-trivial abelian normal subgroup. In the case where for some $\gamma \in \Sigma$ the cocentralizer of M_γ is Artinian $\mathbf{Z_{p^\infty}}$-module, the subgroup M_γ is soluble. Hence M_γ/H is non-trivial normal soluble subgroup of G/H. Therefore G/H contains non-trivial normal abelian subgroup and so G is hyperabelian.

Let $1 = H_0 \leq H_1 \leq ... \leq H_\alpha \leq ... \leq G$ be a normal ascending series with abelian factors and let α be the least ordinal such that cocentralizer of H_α in module A is not Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module. Then, as above, H_β is soluble for all $\beta < \alpha$. Moreover, the quotient group G/H_α has the minimal condition on subgroups and so it is Chernikov group.

Suppose first that α is not a limit ordinal. Therefore H_α is soluble and it follows that G is soluble. Suppose now that α is a limit ordinal and that G is not soluble. For each positive integer d there exists an ordinal β_d such that $\beta_d < \alpha$, H_{β_d} has derived length at least d. Moreover, we may assume that $\beta_i < \beta_{i+1}$ for all positive integers i. For each positive integer i, let $T_i = H_{\beta_i}$. So the group G has an ascending series of normal subgroups $1 = T_0 \leq T_1 \leq ... \leq$. Now $T_\omega = \cup_{n\in\mathbf{N}} T_n$ is not soluble and so $T_\omega = H_\alpha$. A series $1 = W_0 \leq W_1 \leq ... \leq W_n \leq ... \leq W_\omega = \cup_{n\in\mathbf{N}} W_n \leq G$ with the properties referred to in the theorem can now be obtained by refining the series $1 = T_0 \leq T_1 \leq ... \leq T_\omega \leq G$. The lemma is proved. □

Lemma 3.4. *Let A be $\mathbf{Z}_{\mathbf{p}^\infty}G$-module, G satisfies the condition $min-nad$, the cocentralizer of G in module A is not Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module, and $G = AD(G)$. Then $G/G^{\mathfrak{F}}$ is finite.*

Proof. Let us suppose for a contradiction that $G/G^{\mathfrak{F}}$ is infinite. Then G has an infinite descending series of normal subgroups $G \geq N_1 \geq N_2 \geq ...$, such that the quotient groups G/N_i are finite for each i. It follows that, for some k, G/N_k is finite and the cocentralizer of N_k in module A is Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module. Since $G = AD(G)$, it can be choose the subgroup H such that $H = AD(H)$, and $G = HN_k$. Therefore the cocentralizer of G in module A is Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module. The contradiction. The lemma is proved. □

Lemma 3.5. *Let A be $\mathbf{Z}_{\mathbf{p}^\infty}G$-module, G satisfies the condition $min-nad$, and the cocentralizer of G in module A is not Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module. Suppose that G has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq ... \leq W_n \leq ... \leq \cup_{n\geq 1} W_n = G$ such that the cocentralizer of each subgroup W_n in module A is Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module and each W_{n+1}/W_n is abelian. Then G is soluble.*

Proof. Since $A/C_A(S_k)$ is Artinian $\mathbf{Z}_{\mathbf{p}^\infty}$-module, as in the proof of lemma 2.5 we conclude that the additive group of the module $A/C_A(S_k)$ is

Chernikov group. Therefore there is a finite series of $\mathbf{Z_{p^\infty}}G$-submodules $A = A_0 \geq A_1 \geq ... \geq A_{n(k)} = C_A(S_k)$, each factor of which is either finite $\mathbf{Z_{p^\infty}}G$-module or quasifinite $\mathbf{Z_{p^\infty}}G$-module. Since the cocentralizer of S_{k+1} in module A is Artinian $\mathbf{Z_{p^\infty}}$-module the above series is extended to a series of $\mathbf{Z_{p^\infty}}G$-submodules $A = A_0 \geq A_1 \geq ... \geq A_{n(k)} \geq ... \geq A_{n(k+1)} = C_A(S_{k+1})$, each factor of which is either finite $\mathbf{Z_{p^\infty}}G$-module or quasifinite $\mathbf{Z_{p^\infty}}G$-module. In this way we obtain an infinite descending chain of $\mathbf{Z_{p^\infty}}G$-submodules $A = A_0 \geq A_1 \geq A_2 \geq ... \geq A_\omega = C_A(G)$, each factor of which is either finite $\mathbf{Z_{p^\infty}}G$-module or quasifinite $\mathbf{Z_{p^\infty}}G$-module.

Let $H = \cap_{j\geq 0} C_G(A_j/A_{j+1})$. By lemma 16.19 [1] for each j the quotient group $G/C_G(A_j/A_{j+1})$ is abelian-by-finite. Since G/H embeds in the Cartesian product of the groups $G/C_G(A_j/A_{j+1})$, it follows that G/H is abelian-by-(residually finite). Moreover, G is a union of subgroups such that their cocentralizers in module A are Artinian $\mathbf{Z_{p^\infty}}$-modules. Therefore $G = AD(G)$. By lemma 3.4 the quotient group G/H is abelian-by-finite. Let K/H be a normal abelian subgroup of G/H such that G/K is finite. Since $G = AD(G)$ the cocentralizer of K in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Let the cocentralizer of H in module A be Artinian $\mathbf{Z_{p^\infty}}$-module. Since the subgroup H is locally soluble, then the quotient group $H/C_H(A/C_A(G))$ is isomorphic to a locally soluble subgroup of $GL(r, \mathbf{Z_{p^\infty}})$. Since $\mathbf{Z_{p^\infty}}$ is an integral ring then it can be imbeded in the field F. Therefore the quotient group $H/C_H(A/C_A(H))$ is isomorphic to some locally soluble subgroup of linear group $GL(r, F)$. By corollary 3.8 [7] $H/C_H(A/C_A(H))$ is a soluble group. Since the quotient group $C_H(A/C_A(H))$ is a subgroup of stabilizer of the series of length 2, and therefore it is abelian. Hence H is a soluble group.

Thus, we may suppose that the cocentralizer of H in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. We show that H is soluble. Let $L_j = C_H(A/A_j)$, $j = 1, 2, ...$. If $H \neq L_j$ for some j then there exists the number t for which the quotient group H/L_t is infinite. Therefore there exists the number $k \geq j$, $k \geq t$, for which among the factors of the series $A/A_k = A_0/A_k \geq A_1/A_k \geq A_2/A_k \geq ... \geq A_j/A_k \geq ... \geq A_k/A_k$ there are infinite. By the results of chapter 8 [9] H has a nilpotent non-periodic image. Then there exists the normal subgroup H_1 of H for which the quotient group H/H_1 is a nilpotent non-periodic group. It follows that there is a normal subgroup H_2, for which the quotient group H/H_2 is an abelian torsion-free group, which contradicts lemma 2.4. Then $H = L_j$ for each j, $j = 1, 2, ...$. Finally, suppose that for each j, $j = 1, 2, ...$, the quotient group H/L_j is finite. Suppose that there is a number j, for which the cocentralizer of L_j in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Let j be the minimal with this property. Therefore the cocentralizer of L_{j-1}

in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Since L_{j-1}/L_j is finite and $G = AD(G)$, then the cocentralizer of L_{j-1} in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Contradiction. Therefore the cocentralizer of each subgroup L_j in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. Since H has the condition $min - nad$, there is the number m for which $L_j = L_m$ for each $j \geq m$. Therefore the subgroup L_m is soluble. Since H/L_m is finite then H is a soluble group also. The lemma is proved. □

From these results it follows the validity of theorem 1.1.

Proof of theorem 1.2. By theorem 1.1 G is a soluble group. To prove the theorem it is sufficient to consider the case when G is not Chernikov group.

Let $G = D_0 \geq D_1 \geq D_2 \geq ... \geq D_n = 1$ be the derived series of G. By lemma there is the number m such that the cocentralizer of D_m in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module while the cocentralizer of D_{m+1} in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. By lemma 2.4 the quotient groups D_i/D_{j+1}, $i = 0, 1, ..., m$, are Chernikov groups. Let $U = D_{m+1}$. Then G/U is Chernikov group. Let $C = C_A(U)$. Then C is $\mathbf{Z_{p^\infty}}G$-submodule of module A. Since the cocentralizer of U in module A is Artinian $\mathbf{Z_{p^\infty}}$-module then A/C is Artinian $\mathbf{Z_{p^\infty}}$-module. Therefore there exists the series of submodules

$$0 = C_0 \leq C = C_1 \leq C_2 \leq ... \leq C_t = A,$$

such that each factor C_{i+1}/C_i, $i = 1, ..., t-1$, is either finite $\mathbf{Z_{p^\infty}}G$-module or quasifinite $\mathbf{Z_{p^\infty}}G$-module. Then by lemma 16.19 [1] the quotient groups $G/C_G(C_{i+1}/C_i)$, $i = 1, ..., t-1$, are abelian-by-finite. Since G/U is Chernikov group and $U \leq C_G(C_1)$, then $G/C_G(C_1)$ is Chernikov group also. Then the quotient group $G/C_G(C_1)$ is abelian-by-finite. Let $H = C_G(C_1) \cap C_G(C_2/C_1) \cap ... \cap C_G(C_t/C_{t-1})$. It should be noted that G/H is an abelian-by-finite group. Let V/H be a normal abelian subgroup of G/H such that G/V is finite. By theorem 3.2 the cocentralizer of V in module A is not Artinian $\mathbf{Z_{p^\infty}}$-module. By lemma 2.4 V/H is Chernikov group. Therefore G/H is Chernikov group also. The subgroup H acts trivially on each factor of the series $0 = C_0 \leq C = C_1 \leq C_2 \leq ... \leq C_t = A$. Therefore H is a nilpotent group. The theorem is proved. □

Proof of theorem 1.3. By the conditions of the theorem G is not finitely generated group. We prove that G has not proper subgroups of finite index. Let $N \leq G$ and index $|G : N|$ is finite. Then the finitely generated subgroup M can be chosen for which $G = MN$. M и N are proper subgroups of G then their cocentralizers in module A are Artinian $\mathbf{Z_{p^\infty}}$-modules. Therefore the cocentralizer of G in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Contradiction. From theorem 1.2 the group G contains the normal nilpotent subgroup H such that the quotient group G/H is Chernikov group. If the quotient group G/H is finite then G is nilpotent-by-finite. Since G is not finitely generated then the subgroup H is not finitely generated also. Hence G/H' is infinitely generated abelian-by-finite group. Then $G/H' = (G_1/H')(G_2/H')$, where G_1 and G_2 are proper subgroups of G. Contradiction. Therefore the quotient group G/H is a divisible periodic abelian group. If G/H is not isomorphic to C_{q^∞} for some prime q then $G = G_1G_2$, where G_1 and G_2 are proper subgroups of G. Contradiction. Therefore G/H is isomorphic to C_{q^∞} for some prime q. Hence there exists the ascending series of normal subgroups of G

$$E \leq H \leq G_1 \leq G_2 \leq ... \leq G_n \leq ...,$$

such that the quotient groups G_1/H, $G_{i+1}/G_i, i = 1, 2, ...$ are finite of prime order q and $G/H = \bigcup_{n\geq 1}(G_n/H)$. In view of the construction of additive group of Artinian $\mathbf{Z_{p^\infty}}$-module, for each $n = 1, 2, ...,$ the module A has the series of G-invariant submodules

$$0 \leq C_A(G_n) \leq A_1 \leq A_2 \leq ... \leq A_k \leq ...,$$

such that each factor $A_k/C_A(G_n), k = 1, 2, ...,$ is finite. Hence the quotient groups $G/C_G(A_k/C_A(G_n)$ are finite for each $k = 1, 2, ...$ Previously it was proved that G has not proper subgroups of finite index. Hence, $G = C_G(A_k/C_A(G_n)$ for each $k = 1, 2, ...,$ and therefore $[A_k, G] \leq C_A(G_n)$ for each $k = 1, 2, ...$ Then $[A, G] \leq C_A(G_n)$. Since this inclusion is fulfilled for each $n = 1, 2, ...,$ then $[A, G] \leq \bigcap_{n\geq 1} C_A(G_n) = C_A(G)$. Hence G acts trivially on each factor of the series $0 \leq C_A(G) \leq A$. Therefore G is an abelian group. Let G be not isomorphic to C_{q^∞} for some prime q. The G is the product of two its proper subgroups. Therefore the cocentralizer of G in module A is Artinian $\mathbf{Z_{p^\infty}}$-module. Contradiction. Hence $G \simeq C_{q^\infty}$ for some prime q.

The theorem is proved. □

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