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On the cotypeset of torsion-free abelian groups

Fatemeh Karimi

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Abstract. In this paper the cotypeset of some torsion-free abelian groups of finite rank is studied. In particular, we determine the cotypeset of some rank two groups using the elements of their typesets.

Introduction

One of the important and known tools in the theory of torsion-free abelian groups is type and the typeset of a group. This set which is determined from the beginning of the the study the torsion-free groups, has allocated many papers which are about the identifying this set for torsion-free groups or applying it to determine the properties of these groups and the rings over them. Problems in this area are very diverse; for example, [3] is devoted to a determination of the representation type of indecomposables in the categories of almost completely decomposable groups, or in [6], the author is tried to construct indecomposable group with an special critical typeset, and some articles as well as [4], which are discussed about the representation of some categories of torsion-free abelian groups, are some of the works, which are done related to type. Moreover, [2], that provides perspectives on classification of almost completely decomposable groups and deals with the rank, regulator quotient and near-isomorphism types, is one of the major sources in [11], which is dealing with indecomposable $(1,2)-$groups with regulator quotient of

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exponent $\leqslant 3$ and shows that there are precisely four near-isomorphism types of indecomposable groups. After much theorizing has been done about the type and continued more or less to the present, another concepts named "cotype" and "cotypeset" associated to the torsion-free groups. In fact, The study of cotypeset of torsion-free abelian groups begins mainly by Schultz [12]. This concept has been a focus of study between the years 1977 to 1987, and some of the works in this area are Arnold and Vinsonhaler [5], Metelli [9] and Mutzbauer[10]. In the past two decades there are only a few researches about this subject, such as Lafleur [8] in 1994. From that time better identification of cotypeset for different groups and its relation with type is considered. Moreover, always such a question is raised that: could we have some results for cotype similar to ones about the type? For example, similar results of [3], [4], [11] or [2] could be stated for cotype instead of types?

In this paper, we deal with the cotypeset of some torsion-free abelian groups of finite rank and show that the cotypeset of any completely decomposable group is closed under mutually union of its rank one direct summand's types. Moreover, we have some results about the relation between the elements of cotypeset and typeset and determine the cotypeset of some rank two groups using their typesets.

Finally, some of the other unsolved problems in this area are as follows:

(1) Identifying the cotypeset for a completely decomposable group of rank greater than 2.

(2) If $A = A_1 \oplus A_2$ is a group of rank three, with $r(A_2) = 2$, then can we obtain CT(A), (the cotypeset of A) using the cotypesets of A_1 and A_2?

(3) Is there any relation between the cardinality of the cotypeset of a torsion-free abelian group and the existance of a non-zero ring on a group?

1. Notation and Preliminaries

All groups considered in this paper are torsion-free and abelian, with addition as the group operation. Terminology and notation will mostly follow from [7]. By the typeset of a torsion-free group A we mean the partially ordered set of types, i.e.,

$$\mathrm{T(A)} = \{\mathrm{t(x)} \mid 0 \neq \mathrm{x} \in \mathrm{A}\},$$

and for two types $t_1 = [(m_i)_{i\in\mathbb{N}}]$ and $t_2 = [(k_i)_{i\in\mathbb{N}}]$ we define:

$$\inf\{t_1, t_2\} = [(\min\{m_i, k_i\})_{i\in\mathbb{N}}], \quad \sup\{t_1, t_2\} = [(\max\{m_i, k_i\})_{i\in\mathbb{N}}].$$

Moreover, if $t_2 \leqslant t_1$ then we set

$$t_1 - t_2 = [(m_i - k_i)_{i\in\mathbb{N}}].$$

We also may use the notations $t_1 \cap t_2$ and $t_1 \cup t_2$ instead of $\inf\{t_1, t_2\}$ and $\sup\{t_1, t_2\}$ respectively, for more convenience. A pure subgroup B of A is said to be of co-rank one if $rank(A/B) = 1$. The cotypeset of A, denoted by CT(A), is defined as

$$\mathrm{CT(A)} = \{\mathrm{t(A/B)} \mid \mathrm{B\ is\ a\ pure\ co\text{-}rank\ one\ subgroup\ of\ A}\}.$$

A torsion-free group A is called cohomogeneous if CT(A) has cardinality equal to one.

Let A is a torsion-free group of rank n and $S = \{x_1, x_2, \ldots, x_n\}$ a maximal independent set of A. For $X_i = \langle x_i \rangle_*$ and

$$Y_i = \langle x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle_*,$$

define the inner type of A to be

$$\mathrm{IT}(A) = \inf\{t(X_1), \ldots, t(X_n)\}.$$

Moreover, the outer type of A is as follows

$$\mathrm{OT}(A) = \sup\{t(A/Y_1), \ldots, t(A/Y_n)\}.$$

2. Cotypeset of rank two groups

As in [5], let A is a rank two group and $A_1, A_2, \cdots$ be an indexing of the pure rank one subgroups of A with $t_i = t(A_i)$, $\sigma_i = t(A/A_i)$ for each i. Define $T_A = (t_1, t_2, \cdots), CT_A = (\sigma_1, \sigma_2, \cdots)$ are two countable infinite sequences of types (repetition of types is allowed). We say two type sequences T and T' are equivalent, $T \approx T'$, if one is a permutation of the other, and by this, T_A and CT_A are unique up to equivalence.

Proposition 1. *Let A be a rank 2 group with $T_A = (t_1, t_2, \cdots)$ and $CT_A = (\sigma_1, \sigma_2, \cdots)$.*

(1) *There is a type t_0 such that $t_0 = inf\{t_i, t_j\}$ for each $i \neq j$ and if $T(A)$ is finite then $t_0 = t_i$ for some $i \geqslant 1$.*

(2) *There is a type σ_0 such that $\sigma_0 = sup\{\sigma_i, \sigma_j\}$ for each $i \neq j$ and if $CT(A)$ is finite then $\sigma_0 = \sigma_i$ for some $i \geqslant 1$.*

(3) *$t_i \leqslant \sigma_j$ for each $i \neq j$ and $t_0 \leqslant \sigma_0$.*

(4) *$\sigma_i - t_j = \sigma_j - t_i$ for each $i \neq j$ with $i \geqslant 0$ and $j \geqslant 0$.*

(5) *If $t_0 = t(\mathbb{Z})$ then $\sigma_i = \sigma_0 - t_i$ for each i.*

Proof. See ([5], Proposition 1.1). □

Using above Proposition and nothing the known fact from [13], which the typeset of any non-nil rank two torsion-free group has the cardinality at most three, it would be straight forward too check:

Proposition 2. *The cotypeset of a non-nil rank two torsion-free group A has one of this forms:*

(1) *If $\mathrm{T(A)} = \{\mathrm{t}\}$ and B is a pure subgroup of A with $t(B) = t$, then $\mathrm{CT(A)} = \{\mathrm{t(A/B)}\}$.*

(2) *If $\mathrm{T(A)} = \{\mathrm{t_1, t_2}\}$ with $t_1 < t_2$ and A_1 is a pure subgroup of A such that $t(A_1) = t_1$, then $\mathrm{CT(A)} = \{\sigma_1, \sigma_2\}$ such that $\sigma_1 = t(A/A_1), \sigma_2 = \sigma_1 - t_2 + t_1$.*

(3) *If $\mathrm{T(A)} = \{\mathrm{t_0, t_1, t_2}\}$ with $t_0 < t_1, t_2$ and A_1, A_2, A_3 are rank one pure subgroups of A in which $t(A_1) = t_1, t(A_2) = t_2, t(A_3) = t_3$, then $\mathrm{CT(A)} = \{\sigma_1, \sigma_2, \sigma_3\}$ such that $\sigma_3 = t(A/A_3), \sigma_1 = \sigma_3 + t_0 - t_1, \sigma_2 = \sigma_3 + t_0 - t_2$.*

Moreover, we could easily show that:

Corollary 1. *If A is a non-nil rank two group which is completely decomposable, then we have:*

(1) *If $|\mathrm{T(A)}| = 1$ or 2, then $\mathrm{T(A)} = \mathrm{CT(A)}$.*

(2) *If $\mathrm{T(A)} = \{\mathrm{t_1, t_2, t_1 \cap t_2}\}$, then $\mathrm{CT(A)} = \{\mathrm{t_1, t_2, t_1 \cup t_2}\}$.*

Lemma 1. *Let $T = (t_1, t_2, \cdots)$ and $C = (\sigma_1, \sigma_2, \cdots)$ be type sequences with $t_0 = inf\{t_i, t_j\}$ and $\sigma_0 = sup\{\sigma_i, \sigma_j\}$ whenever $i \neq j$. There is a rank two group A with $T_A = T$ and $CT_A = C$ if and only if there is a rank two group B with $T_B = (t_1 - t_0, t_2 - t_0, \cdots)$, $CT_B = (\sigma_1 - t_0, \sigma_2 - t_0, \cdots)$, $IT(B) = t(\mathbb{Z})$, $OT(B) = \sigma_0 - t_0$.*

Proof. See ([5], Lemma 1.3). □

Proposition 3. *Let $S = \{t_i \mid i \geqslant 1\}$ be a set of types with $t_0 = inf\{t_i, t_j\}$ whenever $i \neq j$. If there exists characteristic $h_i \in t_i$ for $i \geqslant 0$ with $h_0 = inf\{h_i, h_j\}$ for each $i \neq j$ then there exists a rank two group A with $T(A) = S$ and $OT(A) = [sup\{h_i \mid i \geqslant 1\}]$.*

Proof. See ([5], Corollary 2.14). □

Theorem 1. *Let $S = \{t_1, t_2, \cdots\}$ be a set of types with $t_0 = inf\{t_i, t_j\}$ for each $i \neq j$ and $t_0 \in S$ if S is finite. Then*

(1) *There exists $s_i \in t_i$ for $i \geqslant 0$ such that $s_0 = min\{s_i, s_j\}$ for $i \neq j$.*

(2) *There exists a rank two group A with $T(A) = S, IT(A) = t_0$, $OT(A) = [sup\{s_i \mid i \geqslant 1\}]$ and $CT(A) = \{OT(A) - (t_i - t_0) \mid i \geqslant 1\}$.*

Proof. (1) Let $n \geqslant 3$ be an arbitrary integer and let $s_i \in t_i$ for $0 \leqslant i \leqslant n-1$ with $s_0 = \min\{s_i, s_j\}$ for $1 \leqslant i \neq j \leqslant n-1$. Now choose $s_n \in t_n$ such that $s_0 = \min\{s_i, s_n\}$ for $1 \leqslant i \leqslant n-1$.

(2) By (1) and Proposition 3, let $\chi'_0 = \sup\{s_i \mid i \geqslant 1\}$, $\sigma'_0 = [\chi'_0]$ and $\gamma_i = \sigma'_0 - t_i$ for $i \geqslant 0$. Note that $\gamma_i = [\chi'_0 - s_i]$ for each $i \geqslant 0$. Now $\Gamma = \{\gamma_1, \gamma_2, \cdots\}$ with $\gamma_0 = \sup\{\gamma_i, \gamma_j\}$ if $i \neq j$, because $t_0 = \inf\{t_i, t_j\}$ hence $\sigma'_0 - t_0 = \sup\{\sigma'_0 - t_i, \sigma'_0 - t_j\}$. Moreover, $\gamma_0 \in \Gamma$ if Γ is finite. In fact if Γ is finite then S must be finite. This means $t_0 \in S$ which yields $t_0 = t_j$ for some $t_j \in S$. Now we have $\gamma_0 = \sigma'_0 - t_0 = \sigma'_0 - t_j = \gamma_j$, for some $\gamma_j \in \Gamma$. Define $\sigma_i = \gamma_0 - \gamma_i$ for $i \geqslant 0$. The next step is to show that there exists a rank two group B with $T(B) = \{\sigma_i \mid i \geqslant 1\}$ and $CT(B) = \{\gamma_i \mid i \geqslant 1\}$. For each $i \geqslant 1$, let $\chi_i = (\chi'_0 - s_0) - (\chi'_0 - s_i) \in \sigma_i = \gamma_0 - \gamma_i$. Note that

1) If $\chi_i(p)$, the $p-$component of χ_i, is equal to ∞, for some $i \geqslant 1$, then $s_i(p) = \infty, \quad s_0(p) < \infty$ and $\chi'_0(p) = \infty$.

2) $\chi_i(p) = s_i(p) - s_0(p)$.

3) $\min\{\chi_i, \chi_j\} = (0, 0, \cdots)$ whenever $i \neq j$. This is a consequence of 2) and the fact that $s_0 = \min\{s_i, s_j\}$.

By 3) and Proposition 3, there exists a rank two group B with $T(B) = \{\sigma_i \mid i \geqslant 1\}, \mathrm{OT}(B) = [\sup\{\chi_i \mid i \geqslant 1\}]$. Moreover, from 3) we deduce that $\mathrm{IT}(B) = t(\mathbb{Z})$. Now 2) implies

$$\begin{aligned} \sup\{\chi_i \mid i \geqslant 1\} &= \sup\{s_i - s_0 \mid i \geqslant 1\} \\ &= \sup\{s_i \mid i \geqslant 1\} - s_0 \\ &= \chi'_0 - s_0, \end{aligned}$$

therefore $\mathrm{OT}(B) = \gamma_0$. Now by Proposition 1 (5), we deduce

$$CT(B) = \{\gamma_0 - \sigma_i \mid i \geqslant 1\} = \{\gamma_i \mid i \geqslant 1\}.$$

The last equality holds because of 3). In fact:

$$\gamma_0 - \sigma_i = [(\chi'_0 - s_0) - (s_i - s_0)] = [\chi'_0 - s_i] = \gamma_i.$$

Consequently, in view of Lemma 1, there exists a rank two group A with

$$\begin{aligned} T(A) &= \{\sigma_i + t_0 \mid i \geqslant 1\} = \{t_i \mid i \geqslant 1\}, \quad \mathrm{IT}(A) = t_0, \\ CT(A) &= \{\gamma_i + t_0 \mid i \geqslant 1\} = \{\sigma'_0 - t_i + t_0 \mid i \geqslant 1\}, \\ \mathrm{OT}(A) &= \mathrm{OT}(B) + t_0 = \gamma_0 + t_0 = \sigma'_0. \end{aligned}$$

□

3. Cotypeset of finite rank groups

We begin this section with an example of a cohomogeneous group of any arbitrary finite rank that is homogeneous too. First we need the following definition and two propositions:

Definition 1. A torsion-free group A is called coseparable if, given any pure subgroup B of A such that A/B reduced of finite rank, B contains a summand C of A which has a completely decomposable finite rank complement. Moreover, a torsion-free group A is finitely cohesive exactly if for every pure finite corank subgroup B of A, A/B is divisible.

Proposition 4. *A finite rank group is coseparable exactly if it is completely decomposable.*

Proof. See ([9], Proposition 1.2). □

Remark 1. By above definition, a finitely cohesive group A is cohomogeneous with

$$\mathrm{CT(A)} = \{(\infty, \infty, \cdots)\}.$$

Proposition 5. *Finitely cohesive groups are coseparable.*

Proof. See ([9], Proposition 1.5). □

Example 1. Let A be a finitely cohesive group of finite rank. Then by Proposition 5, A is coseparable and so completely decomposable group by Proposition 4. This yields $\mathrm{T(A)} = \{(\infty, \infty, \cdots)\}$ and so A is a homogeneous group.

Now we present the main results of this section.

Theorem 2. *Let A is a torsion-free group of finite rank n, A set $\{x_1, x_2, \cdots, x_n\}$ a maximal independent set of A and $A_1, A_2, \cdots$ is an indexing of the rank one pure subgroups of A. Define*

$$U_A = \{m_1x_1 + \cdots + m_nx_n \mid m_1, m_2, \cdots, m_n \in \mathbb{Z}, (m_1, m_2, \cdots, m_n) = 1\}$$

which is a subset of $\bigoplus_{i=1}^n \mathbb{Z}x_i \subseteq A$. Then

(1) *For each $i \geqslant 1$ there exists a unique $a_i \in U_A \bigcap A_i$. Moreover,*

$$A_i \bigcap (\bigoplus_{i=1}^{n} \mathbb{Z}x_i) = \mathbb{Z}(a_i), \quad t(a_i) = t(A_i).$$

(2) *$OT(A) = [sup\{\chi_A(a) \mid a \in U_A\}]$.*

Proof. (1) Let a'_i be a non-zero element of A_i with $t(a'_i) = t_i = t(A_i)$. Then $A_i = \langle a'_i \rangle_*$ and $A_i \bigcap U_A \neq 0$. Now for all $i \geqslant n$, let $k, k_{i1}, k_{i2}, \cdots, k_{in}$ be some integers such that

$$0 \neq ka'_i = \sum_{j=1}^{n} k_{ij}x_j.$$

Suppose $(k_{i1}, k_{i2}, \cdots, k_{in}) = l$; if $l = 1$ then ka'_i has the stated properties in (1). If $l \neq 1 \in \mathbb{Z}$ then we could write $k_{ij} = lk'_{ij}, (j = 1, 2, \cdots, n)$ and $ka'_i = l(\sum_{j=1}^n k'_{ij}x_j)$. Now by letting $a_i = \sum_{j=1}^n k'_{ij}x_j$ we obtain $a_i \in U_A \bigcap A_i$. To show that a_i is unique, let $a''_i \in U_A \bigcap A_i$, then from $a_i, a''_i \in A_i$, there exist some integers m, n such that $(n, m) = 1$ and $ma''_i = na_i$. Now using the fact that $a''_i, a_i \in U_A$ we conclude the result and the other parts of (1) are easy to proof.

(2) We write

$$(A/\bigoplus_{i=1}^{n} \mathbb{Z}x_i) = \bigoplus_{p} [\mathbb{Z}(p^{i_{1p}}) \oplus \cdots \oplus \mathbb{Z}(p^{i_{np}})]$$

such that $0 \leqslant i_{1p} \leqslant \cdots \leqslant i_{np} \leqslant \infty$ for each p. Then $\mathrm{IT}(A) = [(i_{1p})]$ and $\mathrm{OT}(A) = [(i_{np})]$, (See [14]). If $a + (\bigoplus_{i=1}^n \mathbb{Z}x_i)$ is an element of the $p-$component of $A/\bigoplus_{i=1}^n \mathbb{Z}x_i$, then the order of $a + (\bigoplus_{i=1}^n \mathbb{Z}x_i)$ is the least j such that $p^j a = mu$ for some $u \in U_A$ and $m \in \mathbb{Z}$ with $(m, p) = 1$. Since i_{np} is the maximum of such j, in view of $j \leqslant h_p^A(u)$, we have

$$i_{np} \leqslant \sup\{h_p^A(a) \mid a \in U_A\}.$$

But
$$\frac{A}{\bigoplus_{i=1}^{n}\mathbb{Z}x_i} \supseteq \frac{A_i+(\bigoplus_{i=1}^{n}\mathbb{Z}x_i)}{\bigoplus_{i=1}^{n}\mathbb{Z}x_i} \cong \frac{A_i}{\mathbb{Z}a_i} = \bigoplus_p \mathbb{Z}(p^{l_p}),$$
such that $i_{np} \geqslant l_p = h_p^A(a_i)$. This means $\sup\{h_p^A(a) \mid a \in U_A\} \leqslant i_{np}$ and therefore
$$\mathrm{OT}(A) = [(i_{np})] = [\sup\{\chi_A(a) \mid a \in U_A\}]. \qquad \square$$

Theorem 3. *Let A is a torsion-free group of finite rank n and A_1, A_2, $\cdots, A_n$ are rank one subgroups such that $\{x_i|x_i \in A_i\}_{i=1}^n$ is an independent set of A. If $\sigma_i = t\left(\frac{A}{\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*}\right)$ and $t(A_i) = t_i$, then $\sigma_i - t_i = \sigma_j - t_j$ for all $i \neq j \in \{1, 2, \cdots, n\}$.*

Proof. There is an exact sequence
$$0 \longrightarrow A_i \longrightarrow \frac{A}{\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*} \longrightarrow \frac{A}{\bigoplus_{j=1}^{n}A_j} \longrightarrow 0$$
for all $i = 1, 2, \cdots, n$. Choose $a_i \in A_i$ and $y_i \in \frac{A}{\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*}$ with $a_i \longmapsto y_i$. Then
$$0 \longrightarrow \frac{A_i}{\mathbb{Z}a_i} \longrightarrow \frac{A/\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*}{\mathbb{Z}y_i} \longrightarrow \frac{A}{\bigoplus_{j=1}^{n}A_j} \longrightarrow 0 \qquad (*)$$
is exact. Now since $A/\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*$ is a rank one torsion-free group, $\frac{A/\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*}{\mathbb{Z}y_i}$ is torsion, so we have
$$\frac{A}{\bigoplus_{j=1}^{n}A_j} \cong \bigoplus_p \mathbb{Z}(p^{k_p}), \quad \frac{A_i}{\mathbb{Z}a_i} \cong \bigoplus_p \mathbb{Z}(p^{l_p})$$
and
$$\frac{A/\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*}{\mathbb{Z}y_i} \cong \bigoplus_p \mathbb{Z}(p^{n_p}).$$
On the other hand the exactness of $(*)$ implies that
$$\bigoplus_p \mathbb{Z}(p^{k_p}) \cong \frac{\bigoplus_p \mathbb{Z}(p^{n_p})}{\bigoplus_p \mathbb{Z}(p^{l_p})},$$
hence $k_p = n_p - l_p$. Moreover,
$$n_p = h_p^{A/\langle\bigoplus_{i\neq j=1}^{n}A_j\rangle_*}(y_i) - h_p^{\mathbb{Z}y_i}(y_i), \quad l_p = h_p^{A_i}(a_i) - h_p^{\mathbb{Z}a_i}(a_i)$$

and $h_p^{\mathbb{Z}y_i}(y_i) = 0 = h_p^{\mathbb{Z}a_i}(a_i)$. Therefore

$$k_p = h_p^{A/\langle \bigoplus_{i \neq j=1}^{n} A_j \rangle_*}(y_i) - h_p^{A_i}(a_i),$$

which means $[(k_p)] = \sigma_i - t_i$. Similarly $[(k_p)] = \sigma_j - t_j$ and this completes the proof. □

Proposition 6. *In any torsion-free abelian group of finite rank A with finite typeset, the intersection type and inner type coincide and this type is realized. This means there exists a rank one subgroup B of A such that* $\mathrm{IT(A)} = \mathrm{t(B)}$.

Proof. See ([10], Corollary 1.3). □

Proposition 7. *Let A is a group of rank two and X, Y be different pure rational subgroups of A. Then*

$$t(A/X) - t(Y) = t(A/Y) - t(X).$$

Moreover, the outer type is realized if the inner type is realized; more precisely if $t(B) = \mathrm{IT(A)}$ for some subgroup B of A then $t(A/B) = \mathrm{OT(A)}$.

Proof. See ([10], Lemma 2.4). □

Theorem 4. *Let A is a group such that any rank two torsion-free quotient of A is non-nil. Then $\mathrm{CT(A)}$ is closed under the union of its elements.*

Proof. Let $s, t \in \mathrm{CT(A)}$ be two arbitrary elements. Then there exist pure subgroups B, C of A such that A/B and A/C are of rank one and $t(A/B) = s, t(A/C) = t$. Now $D = \frac{A}{B \cap C}$ is a torsion-free group of rank two and $\frac{B}{B \cap C}, \frac{C}{B \cap C}$ are two co-rank one pure subgroups of D such that

$$t\left(\frac{D}{B/B \cap C}\right) = s, \qquad t\left(\frac{D}{C/B \cap C}\right) = t.$$

Hence $s, t \in \mathrm{CT(D)}$. Moreover, our assumption implies that $\mathrm{CT(D)} \subseteq \mathrm{CT(A)}$. Now it is sufficient to prove $s \cup t \in \mathrm{CT(D)}$. By Proposition 6 the inner type of D is realized in $\mathrm{T(D)}$, because $\mathrm{T(D)}$ is finite, so there is a pure subgroup Y of D with $r(Y) = 1, \mathrm{IT(D)} = \mathrm{t(Y)}$. Now by Proposition 7 we have $t(D/Y) = \mathrm{OT(D)} = \mathrm{s \cup t} \in \mathrm{CT(D)}$ and this completes the proof. □

Theorem 5. *Let $A = A_1 \oplus A_2$ be a group of rank three with A_2 a non-nil group of rank two. Then*

$$\mathrm{CT(A)} \supseteq \{\mathrm{t(A_1)}\} \cup \mathrm{CT(A_2)}$$

and $\mathrm{T(A)}$ *contains at most three maximal elements.*

Proof. The first part is obtained from the fact that for any pure co-rank one subgroup B of A_2, $A_1 \oplus B$ is a pure co-rank one subgroup of A such that

$$\frac{A}{A_1 \oplus B} \cong \frac{A_2}{B}.$$

Moreover,

$$\mathrm{T(A)} = \{\mathrm{t(A_1)}\} \cup \mathrm{T(A_2)} \cup \{\mathrm{t(A_1)} \cap \mathrm{t} \mid \mathrm{t} \in \mathrm{T(A_2)}\}.$$

But $\mathrm{T(A_2)}$ has at most two maximal elements since A_2 is a non-nil rank two group. □

Remark 2. At the proof of above theorem, if Y is any pure fully invariant subgroup of A with $r(A/Y) = 1$ and $Y \neq A_2$, then $Y \cap A_2 \neq 0$ and $Y \cap A_1 = A_1$. In fact if $Y \cap A_1 \neq A_1$,

$$\frac{A}{Y} = \frac{A_1 \oplus A_2}{(Y \cap A_1) \oplus (Y \cap A_2)}$$

is not a torsion-free group, (because $A_1/(Y \cap A_1)$ is torsion) which yields a contradiction.

Now $0 \neq Y \cap A_2$ is a pure subgroup of rank one of A_2. Let $pa_2 = y$ for some $a_2 \in A_2, y \in Y \cap A_2$ and a prime number p, then there exist an element $a \in Y$ such that $a = a_1 + a_2'$ for some $a_1 \in A_1$ and $a_2' \in A_2$ in which $pa_2 = y = pa = pa_1 + pa_2'$, because Y is a pure subgroup of A, but this yields $a_2 = a_2'$ and $a_1 = 0$. Therefore $a_2' \in Y \cap A_2$ and this completes this part of proof. So we have $Y = A_1 \oplus (Y \cap A_2)$ and $Y \cap A_2$ is a co-rank one pure subgroup of A_2.

But if Y is not a fully invariant subgroup, similar result couldn't be true.

Theorem 6. *Let $X = \bigoplus_{i=1}^{n} X_i$ is a completely decomposable group of rank n and B a torsion-free group with finite rank greater than one. If $A = X \otimes B$ and $B_i = X_i \otimes B$, then*

$$\mathrm{CT(A)} \supseteq \bigcup_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{CT(B_i)}$$

and $\mathrm{T(A)}$ *is equal to*

$$\bigcup_{i=1}^{n} \mathrm{T(B_i)} \bigcup \{\cap_{\mathrm{i\in I}} \mathrm{t_i} | \text{ I is a finite subset of} \{1, 2, \cdots, \mathrm{n}\}, \mathrm{t_i} \in \mathrm{T(B_i)}\}.$$

Proof. Let C_i is a pure rank (co-rank) one subgroup of B_i, then

$$C_i(\bigoplus_{i \neq j=1}^{n} B_j \bigoplus C_i)$$

is a pure rank (co-rank) one subgroup of A. Moreover,

$$\frac{A}{(\bigoplus_{i\neq j=1}^{n} B_j) \bigoplus C_i} \cong \frac{B_i}{C_i}$$

which yields the result. □

Theorem 7. *If $A = \bigoplus_{i=1}^{n} B_i$ with $r(B_i) \geqslant 2$, then the typeset of A is equal to*

$$\bigcup_{i=1}^{n} \mathrm{T(B_i)} \bigcup \{\cap_{\mathrm{i\in I}} \mathrm{t_i} \mid \text{ I is a finite subset of } \{1, 2, \cdots, \mathrm{n}\}, \mathrm{t_i} \in \mathrm{T(B_i)}\}$$

and

$$\mathrm{CT(A)} \supseteq \bigcup_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{CT(B_i)}.$$

Proof. Obvious. □

In this part we have some results about the cotypeset of completely decomposable groups.

Lemma 2. *Let A be a torsion-free group and $H \leqslant A$ then*

$$CT(A/H) \subseteq CT(A).$$

Proof. Obvious. □

Theorem 8. *Let $A = \bigoplus_{i\in I} A_i$ is a torsion-free group with $r(A_i) = 1$ and $t(A_i) = t_i$. Then the cotypeset of A is closed under mutually union of $t(A_i)$s. Moreover, if $t_i = t(A_i)$ and $t_j = t(A_j)$ are two incomparable types, then $(t_i \cup t_j) - t_k = t_k - (t_i \cap t_j)$ for $k = i, j$.*

Proof. We know $A/(\oplus_{(j\neq)i\in I} A_i) \cong A_j$, hence $t_j \in CT(A)$ for all $j \in I$. Now let t_i, t_j be two incomparable types and let $A' = A_i \oplus A_j$. Then A' is a pure subgroup of A and from $T(A') = \{t_i, t_j, t_i \cap t_j\} \subseteq T(A)$ and Proposition 1, we deduce that $CT(A') = \{t_i, t_j, t_i \cup t_j\}$. Let $t_i \cup t_j = t(A'/H)$ for some co-rank one subgroup H of A'. Now

$$\frac{A' \oplus (\bigoplus_{(i,j\neq)k\in I} A_k)}{H \oplus (\bigoplus_{(i,j\neq)k\in I} A_k)}$$

is a rank one torsion-free quotient of A. We let $G = H \oplus (\bigoplus_{(i,j\neq)k\in I} A_k)$, hence $A/G \cong A'/H$ which yields $t(A/G) = t(A'/H) = t_i \cup t_j \in CT(A)$. Moreover, by assuming $A' = A_i \oplus A_j$ we have $\mathrm{OT}(A') = t_i \cup t_j$ and $\mathrm{IT}(A') = t_i \cap t_j$. Now the result follows from Prposition 1(4) and the fact that $\mathrm{OT}(A') \in CT(A)$ and $\mathrm{IT}(A') \in T(A)$. $\square$

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Contact information

Fatemeh Karimi
Department of Mathematics,
Payame Noor University, PO BOX 19395-3697,
Tehran, I.R. of IRAN
E-Mail(s): karimi@pnu.ac.ir

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