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Projectivity and flatness over the graded ring of normalizing elements

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Abstract. Let k be a field, H a cocommutative bialgebra, A a commutative left H-module algebra, $Hom(H,A)$ the k-algebra of the k-linear maps from H to A under the convolution product, $Z(H,A)$ the submonoid of $Hom(H,A)$ whose elements satisfy the cocycle condition and G any subgroup of the monoid $Z(H,A)$. We give necessary and sufficient conditions for the projectivity and flatness over the graded ring of normalizing elements of A. When A is not necessarily commutative we obtain similar results over the graded ring of weakly semi-invariants of A replacing $Z(H,A)$ by the set $\chi(H,Z(A)^H)$ of all algebra maps from H to $Z(A)^H$, where $Z(A)$ is the center of A.

0. Introduction

It is well known that projectivity and flatness over the ring of invariants are important in the theory of Hopf-Galois extensions. These properties reflect the notions of principal bundles and homogeneous spaces in a noncommutative setting. In [8], when C is a bialgebra, A is a C-comodule algebra and G is any subgroup of the monoid of the grouplike elements of the A-coring $A \otimes C$, we have adapted to the graded set-up the methods and techniques of [5] to give necessary and sufficient conditions for the projectivity and flatness over the graded ring $\mathcal{S}(A)$ of semi-coinvariants of

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A. When A and C are commutative, we obtained similar results for the graded ring $\mathcal{N}(A)$ of conormalizing elements of A. In the present paper, we are concerned with the dual situation. Let H be a cocommutative bialgebra, A a commutative left H-module algebra. Then $Hom_k(H, A)$ is a commutative algebra under the convolution product. Let us denote by $Z(H, A)$ the submonoid of the algebra $Hom_k(H, A)$ whose elements satisfy the cocycle condition. Let G be any subgroup of the monoid $Z(H, A)$. We give necessary and sufficient conditions for the projectivity and flatness over the graded ring of normalizing elements of A. In an appendix, we establish similar results for the graded ring $\mathcal{S}(A)$ of weakly semi-invariants of A replacing $Z(H, A)$ by the set $\chi(H, Z(A)^H)$ of all k-algebra maps from H to the subring of invariants of the center $Z(A)$ of A. In this case we do not assume that A is commutative. If H is finite dimensional, our results are not new: we can derive them from [8] (see Proposition 3.8). This article is the continuation of the papers [3], [6] and [7]. In [3], with S. Caenepeel, we gave necessary and sufficient conditions for projectivity and flatness over the endomorphism ring of a finitely generated module. In [6] and [7], we obtained similar results for the endomorphism ring of a finitely generated comodule over a coring and for the colour endomorphism ring of a finitely generated G-graded comodule, where G is an abelian group with a bicharacter. For other related results we refer to [2], where, with S. Caenepeel, we gave necessary and sufficient conditions for the projectivity of a relative Hopf module over the subring of coinvariants.

Throughout we will be working over a field k. All algebras and coalgebras are over k. Except where otherwise stated, all unlabelled tensor products and Hom are tensor products and Hom over k, and all modules are left modules.

1. Preliminaries from graded ring theory

We will use the following well-known results of graded ring theory [13]. Let G be a group, B a G-graded ring and ${}_{gr-B}\mathcal{M}$, the category of left G-graded B-modules.

- Let N be a left G-graded B-module. For every x in G, $N(x)$ is the graded B-module obtained from N by a shift of the gradation by x. As vector spaces, N and $N(x)$ coincide, and the actions of B on N and $N(x)$ are the same, but the gradations are related by $N(x)_y = N_{xy}$ for all $y \in G$.

- An object of ${}_{gr-B}\mathcal{M}$ is projective (resp. flat) in ${}_{gr-B}\mathcal{M}$ if and only if it is projective (resp. flat) in ${}_{B}\mathcal{M}$, the category of left B-modules.
- An object of ${}_{gr-B}\mathcal{M}$ is free in ${}_{gr-B}\mathcal{M}$ if it has a B-basis consisting of homogeneous elements, equivalently, if it is isomorphic to some $\oplus_{i\in I}B(x_i)$, where I is an index set and $(x_i)_{i\in I}$ is a family of elements of G.
- Any object of ${}_{gr-B}\mathcal{M}$ is a quotient of a free object in ${}_{gr-B}\mathcal{M}$, and any projective object in ${}_{gr-B}\mathcal{M}$ is isomorphic to a direct summand of a free object of ${}_{gr-B}\mathcal{M}$.
- An object of ${}_{gr-B}\mathcal{M}$ is flat in ${}_{gr-B}\mathcal{M}$ if and only if it is the inductive limit of finitely generated free objects in ${}_{gr-B}\mathcal{M}$.

2. Main results

Let k be a field. For a bialgebra H with comultiplication Δ_H and counit ϵ_H we will use the version of Sweedler's sigma notation

$$\Delta_H(h) = h_1 \otimes h_2, \text{ for all } h \in H.$$

For unexplained concepts and notation on bialgebras and actions of bialgebras on rings, we refer the reader to [11], [12] and [14]. A bialgebra H is said to be cocommmutative if

$$h_1 \otimes h_2 = h_2 \otimes h_1 \quad \forall h \in H.$$

For every H-module M we denote by M^H the k-submodule of M whose elements are H-invariant, that is,

$$M^H = \{m \in M : h.m = \epsilon_H(h)m, \text{ for all } h \in H\}.$$

Note that M^H is a trivial H-submodule of M.

A k-algebra A is an H-module algebra if A is an H-module satisfying

$$h.(ab) = (h_1.a)(h_2.b) \quad \text{and} \quad h.1_A = \epsilon_H(h)1_A \quad \forall a, b \in A, \quad h \in H.$$

Let A be an H-module algebra. Then the smash product algebra $A\#H$ is the k-algebra which is equal to $A \otimes H$ as a k-vector space, and has its multiplication given by

$$(a \otimes h)(a' \otimes h') = a(h_1.a') \otimes h_2h', \quad \forall a, a' \in A, \quad h, h' \in H.$$

An element a of A is normal if for every $u \in A$ we have $au = va$ and $ua = v'a$ for some elements $v, v' \in A$.

An element a of A is H-normal if a is a normal element of A and for every $h \in H$ we have $h.a = u_h a$ for some element $u_h \in A$.

An $A\#H$-module M is both an A-module and an H-module such that the A- and H-actions are compatible in the sense that

$$h.(am) = (h_1.a)(h_2.m) \quad \forall h \in H, a \in A, m \in M.$$

It is easy to see that A is an $A\#H$-module whenever A is an H-module algebra. Let us denote by ${}_{A\#H}\mathcal{M}$ the category of $A\#H$-modules. The morphisms of ${}_{A\#H}\mathcal{M}$ are left A-linear and left H-linear maps. Note that A^H is a subalgebra of A called the subring of invariants of A.

From now A is an H-module algebra and $Hom(H, A)$ is the vector space of k-linear maps from H to A. Let us equip $Hom(H, A)$ with the convolution product; i.e.,

$$(\phi \star \phi')(h) = \phi(h_1)\phi'(h_2) \quad \forall \phi, \phi' \in Hom(H, A).$$

It is well known that $Hom(H, A)$ with this product is an algebra with identity ϵ_H. An element ϕ of $Hom(H, A)$ satisfies the cocycle condition if

$$\phi(hh') = [h_1.\phi(h')]\phi(h_2) \quad \text{for all} \quad h, h' \in H \quad (\star),$$

When A is commutative and H is cocommutative, it is easy to see that an element ϕ of $Hom(H, A)$ satisfies the cocycle condition if the k-linear map

$$A\#H \to A\#H, a \otimes h \mapsto a\phi(h_1) \otimes h_2 \quad \text{is an algebra endomorphism.}$$

If $\phi \in Hom(H, A)$ satisfies the cocycle condition then $\phi(h) = \phi(h)\phi(1_H)$ for all $h \in H$. Therefore $\phi(1_H) \neq 0$ if $\phi \neq 0$.

Denote by $Z(H, A)$ the subset of $Hom(H, A)$ whose elements satisfy the cocycle condition and send 1_H to 1_A.

For any $a \in A$, we denote by a_M the k-endomorphism of M which defines the action of a on M; i.e. $a_M(m) = am$ for all $m \in M$.

Let M be an $A\#H$-module and denote by h_M the endomorphism of M that corresponds to the action of $h \in H$ on M. For each $\phi \in Hom(H, A)$, set (see [9], where H is a cocommutative Hopf algebra)

$$\rho_\phi(h) = \phi(h_2)_M \circ (h_1)_M \quad \text{for all} \quad h \in H.$$

Then ρ_ϕ is a k-linear map from H to $End(M)$. For any $a \in A$ we have

$$\rho_\phi(h)(am) = \phi(h_3)(h_1.a)(h_2m).$$

A simple computation gives

$$\rho_\phi(hh')(m) = \phi(h_2h'_2)(h_1h'_1m)$$

and

$$\rho_\phi(h) \circ \rho_\phi(h')(m) = \phi(h_3)[h_1.\phi(h'_2)](h_2h'_1m)$$

for all $h, h' \in H$ and $m \in M$.

If we assume that A is commutative, H is cocommutative and ϕ belongs to $Z(H, A)$, then the two formulas just mentioned above show that ρ_ϕ is an algebra homomorphism. So in the case where A is commutative, H is cocommutative and ϕ belongs to $Z(H, A)$, we can define for every $A\#H$-module M a new $A\#H$-module M^ϕ, the underlying A-module of which is the same as that of M, while the action of H is new and is given by the rule

$$h._\phi m = \rho_\phi(h)m = \phi(h_2)(h_1m) \quad \forall h \in H, m \in M.$$

We call M^ϕ the twisted $A\#H$-module obtained from M and ϕ.

Let A be commutative and H be cocommutative. Then $Z(H, A)$ is a submonoid of $Hom(H, A)$ under the convolution product. The monoid $Z(H, A)$ is commutative since the algebra $Hom(H, A)$ is commutative. For every $A\#H$-module M, we have

$$M^{\epsilon_H} = M, \quad (M^\phi)^\psi = M^{\phi\star\psi}, \quad A^\phi \otimes_A M = M^\phi \quad \forall \phi, \psi \in Z(H, A).$$

In the remainder of the section, we assume that A is commutative, H is a cocommutative bialgebra and G is any subgroup of the monoid $Z(H, A)$.

The case of main interest is when H is a Hopf algebra. In this case, $Z(H, A)$ is a group and we can take G to be any subgroup of the group $Z(H, A)$. For every $\phi \in G$, we will denote by $\bar{\phi}$ its inverse with respect to the convolution product.

Let M be an $A\#H$-module and ϕ an element of G. Set

$$M_\phi = \{m \in M; hm = \phi(h)m \quad \text{for all } h \in H\}.$$

Then

$$A_\phi = \{a \in A; h.a = \phi(h)a \quad \text{for all } h \in H\}.$$

Clearly, $M_{\epsilon_H} = M^H$ and M_ϕ is a k-vector subspace of M. We have $1_A \in A_\phi$ if and only if $\phi = \epsilon_H$. An element of M_ϕ will be called an H-normal element of M with respect to G. Thus an H-normal element of A with respect to G is a particular H-normal element of A.

Lemma 2.1. *For every $A\#H$-module M and every $\phi \in G$, we have*

$$M_\phi \simeq {}_{A\#H}Hom(A^\phi, M) \quad \textit{as vector spaces.}$$

Proof. Let us define $F : Hom(A^\phi, M) \to M$ by $F(f) = f(1_A)$. If f is $A\#H$-linear, we have

$$\begin{aligned} h(F(f)) = h(f(1_A)) &= f(h._\phi 1_A) = f[\phi(h_2)(h_1.1_A)] \\ &= f[\phi(h_2)\epsilon_H(h_1)1_A] \\ &= f[\phi(h)1_A] \\ &= \phi(h)f(1_A) = \phi(h)(F(f)). \end{aligned}$$

So $F(f) \in M_\phi$, and F is a k-linear map from ${}_{A\#H}Hom(A^\phi, M)$ to M_ϕ. Let $m \in M_\phi$ and set $G(m)(a) = am$. Then $G(m) \in {}_AHom(A^\phi, M)$. We have

$$\begin{aligned} G(m)(h._\phi a) = (h._\phi a)m &= \phi(h_2)(h_1.a)m = (h_1.a)\phi(h_2)m \\ &= (h_1.a)(h_2 m) = h(am) = h[G(m)(a)]. \end{aligned}$$

So $G(m) \in {}_{A\#H}Hom(A^\phi, M)$. It is obvious that F and G are inverse of each other. □

If ϕ and ψ are elements of G and if M is an $A\#H$-module, we have $A_\phi M_\psi \subseteq M_{\phi\star\psi}$. In particular, $A_\phi A_\psi \subseteq A_{\phi\star\psi}$ and every M_ϕ is an A^H-module. It is obvious that if M and M' are $A\#H$-modules, and $f : M \to M'$ is an $A\#H$-linear map, then $f(M_\phi) \subseteq M'_\phi$ for all ϕ in G.

For more information about the vector spaces M_ϕ and M^ϕ, we refer to [9], where H is a Hopf algebra and $G = Z(H, A)$.

For every $A\#H$-module M, let us denote by $\mathcal{N}(M)$ the direct sum of the family $(M_\phi)_{\phi\in G}$ in the category of vector spaces. Then $\mathcal{N}(A)$ is the direct sum of the family $(A_\phi)_{\phi\in G}$ in the category of vector spaces. We have

$$\mathcal{N}(M) = \oplus_{\phi\in G} M_\phi \quad \text{and} \quad \mathcal{N}(A) = \oplus_{\phi\in G} A_\phi.$$

This means that $M_\phi \cap M_\psi = 0$ if $\phi \neq \psi$. We call $\mathcal{N}(M)$ the set of the H-normal elements of M with respect to G.

It is easy to see that $\mathcal{N}(A)$ is a commutative G-graded algebra which we will call the graded algebra of H-normal (or normalizing) elements of A with respect to G and $\mathcal{N}(M)$ is a G-graded $\mathcal{N}(A)$-module called the graded $\mathcal{N}(A)$-module of H-normal (or normalizing) elements of M with respect to G. We will denote by ${}_{gr-\mathcal{N}(A)}\mathcal{M}$ the category of G-graded $\mathcal{N}(A)$-modules. The morphisms of this category are the graded morphisms, that is, the $\mathcal{N}(A)$-linear maps of degree ϵ_H.

If N is an object of ${}_{gr-\mathcal{N}(A)}\mathcal{M}$, $N = \oplus_{\phi\in G} N_\phi$, then $A \otimes_{\mathcal{N}(A)} N$ is an object of ${}_{A\#H}\mathcal{M}$: the A-module structure is the obvious one and the H-action is defined by

$$h(a \otimes n_\phi) = \phi(h_2)(h_1.a) \otimes n_\phi, \quad a \in A, h \in H, n_\phi \in N_\phi.$$

Thus we get an induction functor,

$$A \otimes_{\mathcal{N}(A)} (-) : {}_{gr-\mathcal{N}(A)}\mathcal{M} \to {}_{A\#H}\mathcal{M}; \quad N \mapsto A \otimes_{\mathcal{N}(A)} N.$$

To each element $\phi \in G$, we associate a functor

$$(-)^\phi : {}_{A\#H}\mathcal{M} \to {}_{A\#H}\mathcal{M}; \quad M \mapsto M^\phi :$$

this functor $(-)^\phi$ is an isomorphism with inverse $(-)^{\overline{\phi}}$. Since A is commutative, we can also associate to each $\phi \in G$ a functor

$$(-)_\phi : {}_{A\#H}\mathcal{M} \to {}_{A^H}\mathcal{M}; \quad M \mapsto M_\phi.$$

We define the normalizing functor to be

$$\mathcal{N}(-) : {}_{A\#H}\mathcal{M} \to {}_{gr-\mathcal{N}(A)}\mathcal{M}, \quad M \mapsto \mathcal{N}(M) = \oplus_{\phi\in G} M_\phi,$$

which is a covariant left exact functor.

Lemma 2.2. $(A \otimes_{\mathcal{N}(A)} (-), \quad \mathcal{N}(-))$ *is an adjoint pair of functors: in other words, for any* $M \in {}_{A\#H}\mathcal{M}$ *and* $N \in {}_{gr-\mathcal{N}(A)}\mathcal{M}$*, we have an isomorphism of vector spaces*

$${}_{A\#H}Hom(A \otimes_{\mathcal{N}(A)} N, M) \cong {}_{gr-\mathcal{N}(A)}Hom(N, \mathcal{N}(M)).$$

Proof. Let $N = \oplus_{\phi\in G} N_\phi$ be an object of ${}_{gr-\mathcal{N}(A)}\mathcal{M}$, M an object of ${}_{A\#H}\mathcal{M}$ and $f \in {}_{A\#H}Hom(A \otimes_{\mathcal{N}(A)} N, M)$. Let $n_\phi \in N_\phi$, that is, n_ϕ is a homogeneous element of N of degree ϕ. Then $1_A \otimes_{\mathcal{N}(A)} n_\phi$ is an element of $\left(A \otimes_{\mathcal{N}(A)} N\right)_\phi$ and $f(1_A \otimes_{\mathcal{N}(A)} n_\phi) \in M_\phi$. Let us define k-linear maps

$$u : {}_{A\#H}Hom(A \otimes_{\mathcal{N}(A)} N, M) \to Hom(N, \mathcal{N}(M))$$

by $u(f)(n_\phi) = f(1_A \otimes_{\mathcal{N}(A)} n_\phi)$ and

$$v : {}_{gr-\mathcal{N}(A)}Hom(N, \mathcal{N}(M)) \to Hom(A \otimes_{\mathcal{N}(A)} N, M)$$

by $v(g)(a \otimes_{\mathcal{N}(A)} n_\phi) = ag(n_\phi)$. Note that $g(n_\phi) \in M_\phi$ since g is an $\mathcal{N}(A)$-linear map of degree ϵ_H from N to $\mathcal{N}(M)$. It is easy to show that $u(f) \in {}_{gr-\mathcal{N}(A)}Hom(N, \mathcal{N}(M))$, that is, $u(f)$ is $\mathcal{N}(A)$-linear of degree ϵ_H. It is clear that $v(g)$ is A-linear. Let us show that it is H-linear. Take $h \in H$. We have

$$\begin{aligned} v(g)(h(a \otimes n_\phi)) &= v(g)[\phi(h_2)(h_1.a) \otimes n_\phi] \\ &= \phi(h_2)(h_1.a)[g(n_\phi)] \\ &= (h_1.a)\phi(h_2)[g(n_\phi)] \\ &= (h_1.a)(h_2.[g(n_\phi)]) \\ &= h.(ag(n_\phi)) \\ &= h[v(g)(a \otimes n_\phi)]. \end{aligned}$$

It follows that $v(g) \in {}_{A\#H}Hom(A \otimes_{\mathcal{N}(A)} N, M)$. Now we have

$$u[v(g)](n_\phi) = v(g)(1_A \otimes_{\mathcal{N}(A)} n_\phi) = g(n_\phi)$$

and

$$v[u(f)](a \otimes_{\mathcal{N}(A)} n_\phi) = a[u(f)(n_\phi)] = a[f(1_A \otimes_{\mathcal{N}(A)} n_\phi)] = f(a \otimes_{\mathcal{N}(A)} n_\phi).$$

Hence u and v are inverse of each other. □

Let us denote by F' the functor $A \otimes_{\mathcal{N}(A)} (-)$. The unit and counit of the adjunction pair $\big(F', \quad \mathcal{N}(-)\big)$ are the following: for $N \in {}_{gr-\mathcal{N}(A)}\mathcal{M}$ and $M \in {}_{A\#H}\mathcal{M}$:

$$u_N : N \to \mathcal{N}(A \otimes_{\mathcal{N}(A)} N), \quad u_N(n_\phi) = 1_A \otimes_{\mathcal{N}(A)} n_\phi; \phi \in G$$

$$c_M : A \otimes_{\mathcal{N}(A)} \mathcal{N}(M) \to M, \quad c_M(a \otimes_{\mathcal{N}(A)} m) = am.$$

The adjointness property means that we have

$$\mathcal{N}(c_M) \circ u_{\mathcal{N}(M)} = id_{\mathcal{N}(M)}, \quad c_{F'(N)} \circ F'(u_N) = id_{F'(N)} \qquad (\star\star).$$

Lemma 2.3. *The functor $\mathcal{N}(-)$ commutes with direct sums. It commutes with direct limits if $A\#H$ is left noetherian.*

Proof. We know that A is finitely generated as an $A\#H$-module (its generator is 1_A). So for every $\phi \in G$, A^ϕ is finitely generated as an $A\#H$-module. It follows that the functor ${}_{A\#H}Hom(A^\phi, -)$ commute with arbitrary direct sums for every $\phi \in G$. Let $(M_i)_{i\in I}$ be a family of objects in ${}_{A\#H}\mathcal{M}$. Using Lemma 2.1, we have

$$\begin{aligned} \mathcal{N}(\oplus_{i\in I} M_i) &= \oplus_{\phi\in G}(\oplus_{i\in I} M_i)_\phi \\ &= \oplus_{\phi\in G}[{}_{A\#H}Hom(A^\phi, \oplus_{i\in I} M_i)] \\ &= \oplus_{\phi\in G} \oplus_{i\in I} [{}_{A\#H}Hom(A^\phi, M_i)] \\ &= \oplus_{\phi\in G} \oplus_{i\in I} (M_i)_\phi \\ &= \oplus_{i\in I} \oplus_{\phi\in G} (M_i)_\phi \\ &= \oplus_{i\in I}\mathcal{N}(M_i), \end{aligned}$$

and we get the first assertion. Assume that $A\#H$ is left noetherian. Then every A^ϕ is finitely presented as an $A\#H$-module since every A^ϕ is finitely generated as an $A\#H$-module and $A\#H$ is left noetherian. It follows that the functor ${}_{A\#H}Hom(A^\phi, -)$ commutes with arbitrary direct limits for every $\phi \in G$. Let $(M_i)_{i\in I}$ be a directed family of objects in ${}_{A\#H}\mathcal{M}$. Using Lemma 2.1, we have

$$\begin{aligned} \mathcal{N}(\varinjlim M_i) &= \oplus_{\phi\in G}(\varinjlim M_i)_\phi \\ &= \oplus_{\phi\in G}[{}_{A\#H}Hom(A^\phi, \varinjlim M_i)] \\ &= \oplus_{\phi\in G} \varinjlim[{}_{A\#H}Hom(A^\phi, M_i)] \\ &= \oplus_{\phi\in G} \varinjlim(M_i)_\phi \\ &= \varinjlim \oplus_{\phi\in G}(M_i)_\phi \\ &= \varinjlim \mathcal{N}(M_i). \end{aligned}$$ □

Let A be projective in ${}_{A\#H}\mathcal{M}$. Then each A^ϕ is projective in ${}_{A\#H}\mathcal{M}$ because the functor $(-)^\phi$ is an isomorphism. So by Lemma 2.1, the functor $(-)_\phi$ is exact for every $\phi \in G$. It follows that the functor $\mathcal{N}(-)$ is exact when A is projective in ${}_{A\#H}\mathcal{M}$.

Lemma 2.4. *Let M be an $A\#H$-module. Then*

(1) $(M^\phi)_\psi = M_{\overline{\phi}\star\psi} \quad \forall \phi, \psi \in G$;

(2) $\mathcal{N}(M)(\phi) = \mathcal{N}(M^{\overline{\phi}})$ *for every* $\phi \in G$;

(3) *The k-linear map* $f : A \otimes_{\mathcal{N}(A)} \mathcal{N}(A^\phi) \to A^\phi$; $a \otimes_{\mathcal{N}(A)} u \mapsto au$ *is an isomorphism in* ${}_{A\#H}\mathcal{M}$.

Proof. (1) Let $m \in M_{\overline{\phi}\star\psi}$. Then $hm=(\overline{\phi}\star\psi)(h)m$, i.e., $hm=\overline{\phi}(h_1)\psi(h_2)m$. Since M is equal to M^{ϕ} as an A-module and H is cocommutative, we get

$$h._{\phi}m = \phi(h_2)(h_1m) = \phi(h_3)\overline{\phi}(h_1)\psi(h_2)m = \epsilon_H(h_1)\psi(h_2)m = \psi(h)m.$$

This means that $m \in (M^{\phi})_{\psi}$. Now let $m \in (M^{\phi})_{\psi}$. Then $m \in M^{\phi}$ and $h._{\phi}m = \psi(h)m$. It follows that

$$(\overline{\phi}\star\psi)(h)m = \overline{\phi}(h_1)\psi(h_2)m = \overline{\phi}(h_1)(h_2._{\phi}m) = \overline{\phi}(h_1)\phi(h_3)(h_2m) = hm,$$

because H is cocommutative. This means that $m \in M_{\overline{\phi}\star\psi}$. Thus we showed that $m \in M_{\overline{\phi}\star\psi}$ if and only if $m \in (M^{\phi})_{\psi}$.

(2) We have $\mathcal{N}(M)(\phi) = \oplus_{\psi\in G}M_{\phi\star\psi}$ and using (1), we have

$$\mathcal{N}(M^{\overline{\phi}}) = \oplus_{\psi\in G}((M^{\overline{\phi}})_{\psi}) = \oplus_{\psi\in G}M_{\overline{\overline{\phi}}\star\psi} = \oplus_{\psi\in G}M_{\phi\star\psi}.$$

(3) Assume that u is homogeneous of degree ψ in $\mathcal{N}(A^{\phi})$. This means that $u \in (A^{\phi})_{\psi} = A_{\overline{\phi}\star\psi}$. Since H is cocommutative and A is commutative, we have

$$\begin{aligned} h.(au) = (h_1.a)(h_2._{\phi}u) &= (h_1.a)\phi(h_3)(h_2.u) \\ &= (h_1.a)\phi(h_3)[(\overline{\phi}\star\psi)(h_2)]u \\ &= (h_1.a)\phi(h_4)\overline{\phi}(h_2)\psi(h_3)u \\ &= (h_1.a)\psi(h_2)u = \psi(h_2)(h_1.a)u. \end{aligned}$$

On the other hand, we have

$$f(h.(a\otimes_{\mathcal{N}(A)}u)) = f\big(\psi(h_2)(h_1.a)\otimes_{\mathcal{N}(A)}u\big) = \psi(h_2)(h_1.a)u.$$

Therefore, f is H-linear. Clearly, f is A-linear.

Note that $a\otimes_{\mathcal{N}(A)}u = au\otimes_{\mathcal{N}(A)}1_A$ for every $a \in A$. Then f is an isomorphism of $A\#H$-modules: the inverse of f is defined by $a \mapsto a\otimes_{\mathcal{N}(A)}1_A$. □

Lemma 2.5. *For every index set I,*

1. $c_{\oplus_{i\in I}A^{\overline{\phi_i}}}$ *is an isomorphism;*
2. $u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}$ *is an isomorphism;*
3. *if A is projective in ${}_{A\#H}\mathcal{M}$, then u is a natural isomorphism; in other words, the induction functor $F' = A\otimes_{\mathcal{N}(A)}(-)$ is fully faithful.*

Proof. (1) It is straightforward to check that the canonical isomorphism

$$A \otimes_{\mathcal{N}(A)} (\oplus_{i\in I}\mathcal{N}(A)(\phi_i)) \simeq \oplus_{i\in I} A^{\overline{\phi_i}} \quad \text{is just} \quad c_{\oplus_{i\in I}A^{\overline{\phi_i}}} \circ (id_A \otimes \kappa),$$

where κ is the isomorphism $\oplus_{i\in I}\mathcal{N}(A)(\phi_i) \cong \mathcal{N}(\oplus_{i\in I}A^{\overline{\phi_i}})$, (see Lemmas 2.3 and 2.4). So $c_{\oplus_{i\in I}A^{\overline{\phi_i}}}$ is an isomorphism.

(2) Putting $M = \oplus_{i\in I}A^{\overline{\phi_i}}$ in ($\star\star$), we find

$$\mathcal{N}(c_{\oplus_{i\in I}A^{\overline{\phi_i}}}) \circ u_{\mathcal{N}(\oplus_{i\in I}A^{\overline{\phi_i}})} = id_{\mathcal{N}(\oplus_{i\in I}A^{\overline{\phi_i}})}.$$

From Lemmas 2.3 and 2.4, we get

$$\mathcal{N}(c_{\oplus_{i\in I}A^{\overline{\phi_i}}}) \circ u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)} = id_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}.$$

From (1), $\mathcal{N}(c_{\oplus_{i\in I}A^{\overline{\phi_i}}})$ is an isomorphism, hence $u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}$ is an isomorphism.

(3) Since A is projective in ${}_{A\#H}\mathcal{M}$, we know that the functor $\mathcal{N}(A)$ is exact. Take a free resolution $\oplus_{j\in J}\mathcal{N}(A)(\phi_j) \to \oplus_{i\in I}\mathcal{N}(A)(\phi_i) \to N \to 0$ of a left graded $\mathcal{N}(A)$-module N. Since u is natural and the tensor product commutes with arbitrary direct sums, using Lemma 2.4, we have a commutative diagram

$$\begin{array}{ccccccc}
\oplus_{j\in J}\mathcal{N}(A)(\phi_j) & \longrightarrow & \oplus_{i\in I}\mathcal{N}(A)(\phi_i) & \longrightarrow & N & \longrightarrow & 0 \\
\Big\downarrow {\scriptstyle u_{\oplus_{j\in J}\mathcal{N}(A)(\phi_j)}} & & \Big\downarrow {\scriptstyle u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}} & & \Big\downarrow {\scriptstyle u_N} & & \\
\mathcal{N}(\oplus_{j\in J}A^{\overline{\phi_j}}) & \longrightarrow & \mathcal{N}(\oplus_{i\in I}A^{\overline{\phi_i}}) & \longrightarrow & \mathcal{N}(A \otimes_{\mathcal{N}(A)} N) & \longrightarrow & 0
\end{array}$$

The top row is exact. The bottom row is exact, since the sequence

$$\oplus_{j\in J}A^{\overline{\phi_j}} \longrightarrow \oplus_{i\in I}A^{\overline{\phi_i}} \longrightarrow A \otimes_{\mathcal{N}(A)} N \longrightarrow 0$$

is exact in ${}_{A\#H}\mathcal{M}$ (because $A \otimes_{\mathcal{N}(A)} (-)$ is right exact) and $\mathcal{N}(-)$ is an exact functor. By (2), $u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}$ and $u_{\oplus_{j\in J}\mathcal{N}(A)(\phi_j)}$ are isomorphisms. It follows from the five lemma that u_N is an isomorphism. □

Theorem 2.6. *For $P \in {}_{gr-\mathcal{N}(A)}\mathcal{M}$, we consider the following statements.*

(1) *$A \otimes_{\mathcal{N}(A)} P$ is projective in ${}_{A\#H}\mathcal{M}$ and u_P is injective;*

(2) *P is projective as a graded $\mathcal{N}(A)$-module;*

(3) *$A \otimes_{\mathcal{N}(A)} P$ is a direct summand in ${}_{A\#H}\mathcal{M}$ of some $\oplus_{i\in I}A^{\overline{\phi_i}}$, and u_P is bijective;*

(4) *there exists* $Q \in {}_{A\#H}\mathcal{M}$ *such that* Q *is a direct summand of some* $\oplus_{i\in I} A^{\overline{\phi_i}}$, *and* $P \cong \mathcal{N}(Q)$ *in* ${}_{gr-\mathcal{N}(A)}\mathcal{M}$;

(5) $A \otimes_{\mathcal{N}(A)} P$ *is a direct summand in* ${}_{A\#H}\mathcal{M}$ *of some* $\oplus_{i\in I} A^{\overline{\phi_i}}$.

Then (1) $\Rightarrow$ *(2)* $\Leftrightarrow$ *(3)* $\Leftrightarrow$ *(4)* $\Rightarrow$ *(5).*

If A *is projective in* ${}_{A\#H}\mathcal{M}$, *then (5)* $\Rightarrow$ *(3)* $\Rightarrow$ *(1).*

Proof. (2) $\Rightarrow$ (3). If P is projective as a right graded $\mathcal{N}(A)$-module, then we can find an index set I and $P' \in {}_{gr-\mathcal{N}(A)}\mathcal{M}$ such that $\oplus_{i\in I}\mathcal{N}(A)(\phi_i) \cong P \oplus P'$. Obviously $\oplus_{i\in I} A^{\overline{\phi_i}} \cong \oplus_{i\in I}(A \otimes_{\mathcal{N}(A)} \mathcal{N}(A)(\phi_i)) \cong (A \otimes_{\mathcal{N}(A)} P) \oplus (A \otimes_{\mathcal{N}(A)} P')$. Since u is a natural transformation, we have a commutative diagram:

$$\begin{array}{ccc} \oplus_{i\in I}\mathcal{N}(A)(x_i) & \xrightarrow{\cong} & P \oplus P' \\ {\scriptstyle u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}}\downarrow & & \downarrow{\scriptstyle u_P \oplus u_{P'}} \\ \mathcal{N}(\oplus_{i\in I} A^{\overline{\phi_i}}) & \xrightarrow[\cong]{} & \mathcal{N}(A \otimes_{\mathcal{N}(A)} P) \oplus \mathcal{N}(A \otimes_{\mathcal{N}(A)} P') \end{array}$$

From the fact that $u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}$ is an isomorphism (Lemma 2.5), it follows that u_P (and $u_{P'}$) are isomorphisms.

(3) $\Rightarrow$ (4). Take $Q = A \otimes_{\mathcal{N}(A)} P$.

(4) $\Rightarrow$ (2). Let $f : \oplus_{i\in I} A^{\overline{\phi_i}} \to Q$ be a split epimorphism in ${}_{A\#H}\mathcal{M}$. Then the map $\mathcal{N}(f) : \mathcal{N}(\oplus_{i\in I} A^{\overline{\phi_i}}) \cong \oplus_{i\in I}\mathcal{N}(A)(\phi_i) \to \mathcal{N}(Q) \cong P$ is split surjective in ${}_{gr-\mathcal{N}(A)}\mathcal{M}$, hence P is projective as a right graded $\mathcal{N}(A)$-module.

(4) $\Rightarrow$ (5). We already proved that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). Since (5) is contained in (3), we get (4) $\Rightarrow$ (5).

(1) $\Rightarrow$ (2). Take an epimorphism $f : \oplus_{i\in I}\mathcal{N}(A)(\phi_i) \to P$ in ${}_{gr-\mathcal{N}(A)}\mathcal{M}$. Then

$$F(f) =: A \otimes_{\mathcal{N}(A)} (\oplus_{i\in I}\mathcal{N}(A)(\phi_i)) \cong \oplus_{i\in I} A^{\overline{\phi_i}} \to A \otimes_{\mathcal{N}(A)} P$$

is surjective because the functor $A \otimes_{\mathcal{N}(A)} (-)$ is right exact, and splits in ${}_{A\#H}\mathcal{M}$ since $A \otimes_{\mathcal{N}(A)} P$ is projective in ${}_{A\#H}\mathcal{M}$. Consider the commutative diagram

$$\begin{array}{ccccc} \oplus_{i\in I}\mathcal{N}(A)(\phi_i) & \xrightarrow{f} & P & \longrightarrow & 0 \\ {\scriptstyle u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}}\downarrow & & {\scriptstyle u_P}\downarrow & & \\ \mathcal{N}(\oplus_{i\in I} A^{\overline{\phi_i}}) & \xrightarrow[\mathcal{N}F(f)]{} & \mathcal{N}(A \otimes_{\mathcal{N}(A)} P) & \longrightarrow & 0 \end{array}$$

The bottom row is split exact, since any functor, in particular $\mathcal{N}(-)$ preserves split exact sequences. By Lemma 2.5(2), $u_{\oplus_{i\in I}\mathcal{N}(A)(\phi_i)}$ is an isomorphism. A diagram chasing argument tells us that u_P is surjective. By assumption, u_P is injective, so u_P is bijective. We deduce that the top row is isomorphic to the bottom row, and therefore splits. Thus $P \in {}_{gr-\mathcal{N}(A)}\mathcal{M}$ is projective.

(5) $\Rightarrow$ (3). Under the assumption that A is projective in ${}_{A\#H}\mathcal{M}$, (5) $\Rightarrow$ (3) follows from Lemma 2.5(3).

(3) $\Rightarrow$ (1). By (3), $A\otimes_{\mathcal{N}(A)} P$ is a direct summand of some $\oplus_{i\in I}A^{\overline{\phi_i}}$. If A is projective in ${}_{A\#H}\mathcal{M}$, then $\oplus_{i\in I}A^{\overline{\phi_i}}$ is projective in ${}_{A\#H}\mathcal{M}$. So $A\otimes_{\mathcal{N}(A)} P$ being a direct summand of a projective object of ${}_{A\#H}\mathcal{M}$ is projective in ${}_{A\#H}\mathcal{M}$. □

Theorem 2.7. *Assume that $A\#H$ is left noetherian. For $P \in {}_{gr-\mathcal{N}(A)}\mathcal{M}$, the following assertions are equivalent.*

(1) *P is flat as a graded $\mathcal{N}(A)$-module;*

(2) *$A\otimes_{\mathcal{N}(A)} P = \varinjlim Q_i$, where $Q_i \cong \oplus_{j\leqslant n_i}A^{\overline{\phi_{ij}}}$ in ${}_{A\#H}\mathcal{M}$ for some positive integer n_i, and u_P is bijective;*

(3) *$A\otimes_{\mathcal{N}(A)} P = \varinjlim Q_i$, where $Q_i \in {}_{A\#H}\mathcal{M}$ is a direct summand of some $\oplus_{j\in I_i}A^{\overline{\phi_{ij}}}$ in ${}_{A\#H}\mathcal{M}$, and u_P is bijective;*

(4) *there exists $Q = \varinjlim Q_i \in {}_{A\#H}\mathcal{M}$, such that $Q_i \cong \oplus_{j\leqslant n_i}A^{\overline{\phi_{ij}}}$ for some positive integer n_i and $\mathcal{N}(Q)\cong P$ in ${}_{gr-\mathcal{N}(A)}\mathcal{M}$;*

(5) *there exists $Q = \varinjlim Q_i \in {}_{A\#H}\mathcal{M}$, such that Q_i is a direct summand of some $\oplus_{j\in I_i}A^{\overline{\phi_{ij}}}$ in ${}_{A\#H}\mathcal{M}$, and $\mathcal{N}(Q)\cong P$ in ${}_{gr-\mathcal{N}(A)}\mathcal{M}$.*

If A is projective in ${}_{A\#H}\mathcal{M}$, these conditions are also equivalent to conditions (2) and (3), without the assumption that u_P is bijective.

Proof. (1) $\Rightarrow$ (2). $P = \varinjlim N_i$, with $N_i = \oplus_{j\leqslant n_i}\mathcal{N}(A)(\phi_{ij})$. Take $Q_i = \oplus_{j\leqslant n_i}A^{\overline{\phi_{ij}}}$, then

$$\varinjlim Q_i \cong \varinjlim(A\otimes_{\mathcal{N}(A)} N_i) \cong A\otimes_{\mathcal{N}(A)} (\varinjlim N_i) \cong A\otimes_{\mathcal{N}(A)} P.$$

Consider the following commutative diagram:

$$\begin{array}{ccc} P = \varinjlim N_i & \xrightarrow{\lim(u_{N_i})} & \varinjlim \mathcal{N}(A\otimes_{\mathcal{N}(A)} N_i) \\ \Big\downarrow{\scriptstyle u_P} & & \Big\downarrow{\scriptstyle f} \\ \mathcal{N}(A\otimes_{\mathcal{N}(A)} (\varinjlim N_i)) & \xrightarrow[\cong]{} & \mathcal{N}(\varinjlim(A\otimes_{\mathcal{N}(A)} N_i)) \end{array}$$

By Lemma 2.5(2), the u_{N_i} are isomorphisms. By Lemma 2.3, the natural homomorphism f is an isomorphism. Hence u_P is an isomorphism.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5). Put $Q = A \otimes_{\mathcal{N}(A)} P$. Then $u_P : P \to \mathcal{N}(A \otimes_{\mathcal{N}(A)} P)$ is the required isomorphism.

(5) $\Rightarrow$ (1). We have a split exact sequence $0 \to N_i \to P_i = \oplus_{j \in I_i} A^{\overline{\phi_{ij}}} \to Q_i \to 0$ in ${}_{A\#H}\mathcal{M}$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F\mathcal{N}(N_i) & \longrightarrow & F\mathcal{N}(P_i) & \longrightarrow & F\mathcal{N}(Q_i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_{N_i}} & & \downarrow{\scriptstyle c_{P_i}} & & \downarrow{\scriptstyle c_{Q_i}} & & \\
0 & \longrightarrow & N_i & \longrightarrow & P_i & \longrightarrow & Q_i & \longrightarrow & \mathbf{0}
\end{array}$$

We know from Lemma 2.5(1) that c_{P_i} is an isomorphism. Both rows in the diagram are split exact, so it follows that c_{N_i} and c_{Q_i} are also isomorphisms. Next consider the commutative diagram:

$$\begin{array}{ccc}
A \otimes_{\mathcal{N}(A)} (\varinjlim \mathcal{N}(Q_i)) & \xrightarrow{id_A \otimes f} & A \otimes_{\mathcal{N}(A)} \mathcal{N}(Q) \\
\uparrow{\scriptstyle h} & & \downarrow{\scriptstyle c_Q} \\
\varinjlim (A \otimes_{\mathcal{N}(A)} \mathcal{N}(Q_i)) & \xrightarrow{\lim c_{Q_i}} & Q
\end{array}$$

where h is the natural homomorphism and f is the isomorphism $\varinjlim \mathcal{N}(Q_i) \cong \mathcal{N}(\varinjlim (Q_i))$ (see Lemma 2.3). h is an isomorphism, because the functor $A \otimes_{\mathcal{N}(A)} (-)$ preserves inductive limits. $limc_{Q_i}$ is an isomorphism, because every c_{Q_i} is an isomorphism. It follows that c_Q is an isomorphism, hence $\mathcal{N}(c_Q)$ is an isomorphism. From ($\star\star$), we get $\mathcal{N}(c_Q) \circ u_{\mathcal{N}(Q)} = id_{\mathcal{N}(Q)}$. It follows that $u_{\mathcal{N}(Q)}$ is also an isomorphism. Since $\mathcal{N}(Q) \cong P$, u_P is an isomorphism. Consider the isomorphisms

$$P \cong \mathcal{N}(A \otimes_{\mathcal{N}(A)} P) \cong \mathcal{N}(A \otimes_{\mathcal{N}(A)} \mathcal{N}(Q)) \cong \mathcal{N}(Q) \cong \varinjlim \mathcal{N}(Q_i);$$

where the first isomorphism is u_P, the third is $\mathcal{N}(c_Q)$ and the last one is f. By Lemmas 2.3 and 2.4, each $\mathcal{N}(P_i) \cong \oplus_{j \in I} \mathcal{N}(A)(\phi_{ij})$ is projective as a right graded $\mathcal{N}(A)$-module, hence each $\mathcal{N}(Q_i)$ is also projective as a graded $\mathcal{N}(A)$-module, and we conclude that $P \in {}_{gr-\mathcal{N}(A)}\mathcal{M}$ is flat. The final statement is an immediate consequence of Lemma 2.5(3). $\square$

Let us examine some particular cases.

• Let H be a finite dimensional cocommutative bialgebra and A a commutative left H-module algebra. Then $H^* = Hom(H,k)$ is a commutative bialgebra: the product of H^* is the convolution product

$$(f \star f')(h) = f(h_1)f'(h_2); f, f' \in H^*, h \in H.$$

We know that A is a right H^*-comodule algebra. Denote by $\mathcal{C}$ the coring $A \otimes H^*$. Then $\mathcal{C}$ is a commutative algebra under the product

$$(a \otimes f)(a \otimes f') = aa' \otimes (f \star f'); a, a' \in A, f, f' \in H^*.$$

Denote by $G(\mathcal{C})$ the monoid of grouplike elements of $\mathcal{C}$. It is well known that the k-linear map $\eta : \mathcal{C} \to Hom(H, A)$ defined by

$$\eta(\sum a_i \otimes f_i)(h) = \sum a_i f_i(h); a_i \in A, f_i \in H^*, h_i \in H$$

is an isomorphism of k-algebras.

For the proof of the following proposition, we refer to [10], where H is a Hopf algebra.

Proposition 2.8. *With the above notations, let H be a finite dimensional cocommutative bialgebra and A a commutative left H-module algebra. Then*

$$\eta(G(\mathcal{C})) = Z(H, A).$$

Consequently, if G is a subgroup of the monoid $G(\mathcal{C})$, then $\eta(G)$ is a subgroup of the monoid $Z(H, A)$. Conversely, if G is a subgroup of the monoid $Z(H, A)$, then $\eta^{-1}(G)$ is a subgroup of the monoid $G(\mathcal{C})$.

It follows from Proposition 2.8 that our results are not new if H is finite-dimensional: they can be derived from [8]. We refer to [1] for more information on corings and comodules over corings.

•• Let g be a Lie algebra and $U(g)$ the enveloping algebra of g. It is well known that an algebra A is a $U(g)$-module algebra if and only if g acts on A by derivations. An element a of A is $U(g)$-normal if and only if it is g-normal, here a is a g-normal element if a is a normal element of A and for every $x \in g$ we have $x.a = u_x a$ for some u_x in A. Let A be a commutative $U(g)$-module algebra. Let us denote by $Z(g, A)$ the set of k-linear maps ϕ from g to A satisfying the cocycle condition $\phi([x, y]) = x.\phi(y) - y.\phi(x)$ for all $x, y \in g$. Clearly $Z(g, A)$ is an abelian additive group. It is easy to see that there is a bijection from $Z(g, A)$ to $Z(U(g), A)$. An element a of A is $U(g)$-normal with respect to $Z(U(g), A)$ if and only if it is g-normal with respect to $Z(g,A)$. So in the case of a Lie algebra g acting by derivations on a commutative algebra A, we can replace everywhere in our results $Z(U(g),A)$ by $Z(g,A)$.

• • • Let Γ be a group and $k\Gamma$ the group algebra of Γ. It is well known that an algebra A is a $k\Gamma$-module algebra if and only if Γ acts on A by automorphisms. An element a of A is $k\Gamma$-normal if and only if it is Γ-normal, here a is a Γ-normal element if a is a normal element of A and for every $x \in \Gamma$ we have $x.a = u_x a$ for some u_x in A. Let A be a commutative $k\Gamma$-module algebra. Let us denote by $Z(\Gamma,A)$ the set of maps ϕ from Γ to the set $U(A)$ of invertible elements of A satisfying the cocycle condition $\phi(xx') = [x.\phi(x')]\phi(x)$ for all $x, x' \in \Gamma$. Clearly $Z(\Gamma,A)$ is an abelian group $(\phi\phi')(x) = \phi(x)\phi'(x)$. It is easy to see that there is a bijection from $Z(\Gamma,A)$ to $Z(k\Gamma,A)$. An element a of A is $k\Gamma$-normal with respect to $Z(k\Gamma,A)$ if and only if it is Γ-normal with respect to $Z(\Gamma,A)$. So in the case of a group Γ acting by automorphisms on a commutative algebra A, we can replace everywhere in our results $Z(k\Gamma,A)$ by $Z(\Gamma,A)$.

• • •• Let Γ be an algebraic group and $k[\Gamma]$ the affine coordinate ring of Γ. It is well known that an affine variety X is a left Γ-module if and only if $k[X]$ is a right $k[\Gamma]$-comodule algebra. Note that $k[\Gamma]$ is a commutative Hopf algebra. Since a finite group is an algebraic group, our results are not new for a finite group acting by automorphisms on an affine variety: they can be derived from [8].

3. Appendix

We keep the conventions and notations of the preceding section. Let H be a bialgebra. Denote by $\chi(H,A^H)$ the set of all k-algebra maps from H to A^H. Clearly, $\chi(H,A^H)$ is a subset of $Hom(H,A)$. Let χ be an element of $\chi(H,A^H)$. It is easy to see that the map ρ_χ defined in the preceding section is an algebra homomorphism without the assumption that A is commutative and H is cocommutative. Likewise the set $\chi(H,A^H)$ is a monoid under the convolution product with identity ϵ_H. For χ in $\chi(H,A^H)$, we can define a new $A\#H$-module M^χ (exactly as in section 2), the underlying A-module of which is the same as that of M, while the action of H is new and is given by the rule

$$h._\chi m = \chi(h_2)(h_1 m) \quad \forall h \in H, m \in M.$$

We call M^χ the twisted $A\#H$-module obtained from M and χ.

If A is an H-module algebra, then the center $Z(A)$ of A is an H-module algebra and $Z(A)^H$ is a subalgebra of A^H. Let us denote by $\chi(H,Z(A)^H)$

the set of all k-algebra maps from H to $Z(A)^H$. It is a submonoid of $\chi(H, A^H)$.

A careful examination of the lemmas of section 2 shows that we have used the commutativity of A to get $\phi(H)$ contained in the center $Z(A)$ of A (see Lemmas 2.1 and 2.2). But this fact is always true for $\chi(H, Z(A)^H)$. Note also that it is only in Lemma 2.4 that the computations use the cocommutativity of H and that we have used Lemme 2.4 in the proof of Lemma 2.5. These remarks suggest that all the results of the preceding section are true replacing $Z(H, A)$ by $\chi(H, Z(A)^H)$ without the assumption that A is commutative.

Let us assume that G is any subgroup of the monoid $\chi(H, Z(A)^H)$.

For every $\chi \in G$ we will denote by $\bar{\chi}$ its inverse. Note that if H is a Hopf algebra, then $\chi(H, Z(A)^H)$ is a group and we can take $G = \chi(H, Z(A)^H)$ in our results. This group is commutative if H is cocommutative. Any element χ of $\chi(H, Z(A)^H)$ satisfies $\bar{\chi} = \chi S_H$ if H is a Hopf algebra with antipode S_H.

For an $A\#H$-module M and for an element χ of G, the elements of M_χ will be called the weakly H-semi-invariant elements of M.

The proofs of the following results are similar to those of the preceding section and we omit them.

Lemma 3.1. *Under the above notations, for every $A\#H$-module M and every $\chi \in G$, we have*

$$M_\chi \simeq {}_{A\#H}Hom(A^\chi, M) \quad \textit{as vector spaces.}$$

If χ and λ are elements of G and if M is an $A\#H$-module we have $A_\chi M_\lambda \subseteq M_{\chi\star\lambda}$. In particular, $A_\chi A_\lambda \subseteq A_{\chi\star\lambda}$ and every M_χ is an A^H-module.

It is obvious that if M and M' are $A\#H$-modules, and $f : M \to M'$ is $A\#H$-linear, then $f(M_\chi) \subseteq M'_\chi$ for all χ in G.

For every $A\#H$-module M, let us denote by $\mathcal{S}(M)$ the direct sum of the family $(M_\chi)_{\chi\in G}$ in the category of vector spaces. We have

$$\mathcal{S}(M) = \oplus_{\chi\in G} M_\chi \quad \text{and} \quad \mathcal{S}(A) = \oplus_{\chi\in G} A_\chi$$

We call $\mathcal{S}(M)$ (resp. $\mathcal{S}(A)$) the set of the weakly H-semi-invariant elements of M (resp. of A) with respect to G. It is easy to see that $\mathcal{S}(A)$ is a G-graded algebra and $\mathcal{S}(M)$ is a left G-graded $\mathcal{S}(A)$-module. We call

$\mathcal{S}(A)$ the graded algebra of weakly semi-invariants of A with respect to G and $\mathcal{S}(M)$ the graded $\mathcal{S}(A)$-module of weakly semi-invariants of M with respect to G. We will denote by ${}_{gr-\mathcal{S}(A)}\mathcal{M}$ the category of G-graded $\mathcal{S}(A)$-modules. The morphisms of this category are the graded morphisms, that is, the $\mathcal{S}(A)$-linear maps of degree ϵ_H. For any object $N \in {}_{gr-\mathcal{S}(A)}\mathcal{M}$, $A \otimes_{\mathcal{S}(A)} N$ is an object of ${}_{A\#H}\mathcal{M}$: the A-module structure is the obvious one and the H-action is defined by $h(a \otimes n_\chi) = \chi(h_2)(h_1.a) \otimes n_\chi$, where $a \in A$, $h \in H$ and $n_\chi \in N_\chi$. We have an induction functor,

$$A \otimes_{\mathcal{S}(A)} - : {}_{gr-\mathcal{S}(A)}\mathcal{M} \to {}_{A\#H}\mathcal{M}; \quad N \mapsto A \otimes_{\mathcal{S}(A)} N.$$

To each element $\chi \in G$, we associate a functor

$$(-)^\chi : {}_{A\#H}\mathcal{M} \to {}_{A\#H}\mathcal{M}; \quad M \mapsto M^\chi,$$

which is an isomorphim with inverse $(-)^{\overline{\chi}}$. We also associate to each $\chi \in G$ a functor

$$(-)_\chi : {}_{A\#H}\mathcal{M} \to {}_{A^H}\mathcal{M}; \quad M \mapsto M_\chi.$$

We define the weakly semi-invariant functor

$$\mathcal{S}(-) : {}_{A\#H}\mathcal{M} \to {}_{gr-\mathcal{S}(A)}\mathcal{M}, \quad M \mapsto \mathcal{S}(M) = \oplus_\chi M_\chi,$$

which is a covariant left exact functor.

Lemma 3.2. *Under the above notations,* $(A \otimes_{\mathcal{S}(A)} (-), \quad \mathcal{S}(-))$ *is an adjoint pair of functors; in other words, for any* $M \in {}_{A\#H}\mathcal{M}$ *and* $N \in {}_{gr-\mathcal{S}(A)}\mathcal{M}$*, we have an isomorphism of vector spaces*

$${}_{A\#H}Hom(A \otimes_{\mathcal{S}(A)} N, M) \cong {}_{gr-\mathcal{S}(A)}Hom(N, \mathcal{S}(M)).$$

Let us denote by F' the functor $A \otimes_{\mathcal{S}(A)} (-)$. The unit and counit of the adjunction pair $(F', \quad \mathcal{S}(-))$ are the following: for $N \in {}_{gr-\mathcal{S}(A)}\mathcal{M}$ and $M \in {}_{A\#H}\mathcal{M}$:

$$u_N : N \to \mathcal{S}(A \otimes_{\mathcal{S}(A)} N), \quad u_N(n) = 1_A \otimes_{\mathcal{S}(A)} n$$

$$c_M : A \otimes_{\mathcal{S}(A)} \mathcal{S}(M) \to M, \quad c_M(a \otimes_{\mathcal{S}(A)} m) = am.$$

The adjointness property means that we have

$$\mathcal{S}(c_M) \circ u_{\mathcal{S}(M)} = id_{\mathcal{S}(M)}, \quad c_{F'(N)} \circ F'(u_N) = id_{F'(N)} \qquad (\star\star\star).$$

Lemma 3.3. *Under the above notations, the functor $\mathcal{S}(-)$ commutes with direct sums. It commutes with direct limits if $A\#H$ is left noetherian.*

Let A be projective in ${}_{A\#H}\mathcal{M}$. Then each A^{χ} is projective in ${}_{A\#H}\mathcal{M}$ because the functor $(-)^{\chi}$ is an isomorphism. So by Lemma 3.1, the functor $(-)_{\chi}$ is exact for every $\chi \in G$. It follows that the functor $\mathcal{S}(-)$ is exact if A is projective in ${}_{A\#H}\mathcal{M}$.

Lemma 3.4. *Under the above notations, let H be cocommutative and let M be an $A\#H$-module. Then we have*

(1) $(M^{\chi})_{\lambda} = M_{\overline{\chi}\star\lambda}$ *for every* $\chi \in G$.

(2) $\mathcal{S}(M)(\chi) = \mathcal{S}(M^{\overline{\chi}})$ *for every* $\chi \in G$*;*

(3) *The k-linear map* $f : A \otimes_{\mathcal{S}(A)} \mathcal{S}(A^{\chi}) \to A^{\chi}$*;* $a \otimes_{\mathcal{S}(A)} u \mapsto au$ *is an isomorphism in* ${}_{A\#H}\mathcal{M}$.

Lemma 3.5. *Under the above notations, let H be cocommutative. For every index set I,*

(1) $c_{\oplus_{i\in I} A^{\overline{\chi_i}}}$ *is an isomorphism;*

(2) $u_{\oplus_{i\in I}\mathcal{S}(A)(\chi_i)}$ *is an isomorphism;*

(3) *if A is projective in ${}_{A\#H}\mathcal{M}$, then u is a natural isomorphism; in other words, the induction functor $F' = A \otimes_{\mathcal{S}(A)} (-)$ is fully faithful.*

Theorem 3.6. *Let H be cocommutative. For $P \in {}_{gr-\mathcal{S}(A)}\mathcal{M}$, we consider the following statements.*

(1) *$A \otimes_{\mathcal{S}(A)} P$ is projective in ${}_{A\#H}\mathcal{M}$ and u_P is injective;*

(2) *P is projective as a graded $\mathcal{S}(A)$-module;*

(3) *$A \otimes_{\mathcal{S}(A)} P$ is a direct summand in ${}_{A\#H}\mathcal{M}$ of some $\oplus_{i\in I} A^{\overline{\chi_i}}$, and u_P is bijective;*

(4) *there exists $Q \in {}_{A\#H}\mathcal{M}$ such that Q is a direct summand of some $\oplus_{i\in I} A^{\overline{\chi_i}}$, and $P \cong \mathcal{S}(Q)$ in ${}_{gr-\mathcal{S}(A)}\mathcal{M}$;*

(5) *$A \otimes_{\mathcal{S}(A)} P$ is a direct summand in ${}_{A\#H}\mathcal{M}$ of some $\oplus_{i\in I} A^{\overline{\chi_i}}$.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If A is projective in ${}_{A\#H}\mathcal{M}$, then (5) $\Rightarrow$ (3) $\Rightarrow$ (1).

Theorem 3.7. *Let H be cocommutative. Assume that $A\#H$ is left noetherian. For $P \in {}_{gr-\mathcal{S}(A)}\mathcal{M}$, the following assertions are equivalent.*

(1) *P is flat as a graded $\mathcal{S}(A)$-module;*

(2) *$A \otimes_{\mathcal{S}(A)} P = \varinjlim Q_i$, where $Q_i \cong \oplus_{j\leqslant n_i} A^{\overline{\chi_{ij}}}$ in ${}_{A\#H}\mathcal{M}$ for some positive integer n_i, and u_P is bijective;*

(3) *$A \otimes_{\mathcal{S}(A)} P = \varinjlim Q_i$, where $Q_i \in {}_{A\#H}\mathcal{M}$ is a direct summand of some $\oplus_{j\in I_i} A^{\overline{\chi_{ij}}}$ in ${}_{A\#H}\mathcal{M}$, and u_P is bijective;*

(4) *there exists $Q = \varinjlim Q_i \in {}_{A\#H}\mathcal{M}$, such that $Q_i \cong \oplus_{j\leqslant n_i} A^{\overline{\chi_{ij}}}$ for some positive integer n_i and $\mathcal{S}(Q) \cong P$ in ${}_{gr-\mathcal{S}(A)}\mathcal{M}$;*

(5) *there exists $Q = \varinjlim Q_i \in {}_{A\#H}\mathcal{M}$, such that Q_i is a direct summand of some $\oplus_{j\in I_i} A^{\overline{\chi_{ij}}}$ in ${}_{A\#H}\mathcal{M}$, and $\mathcal{S}(Q) \cong P$ in ${}_{gr-\mathcal{S}(A)}\mathcal{M}$.*

If A is projective in ${}_{A\#H}\mathcal{M}$, these conditions are also equivalent to conditions (2) and (3), without the assumption that u_P is bijective.

Note that in all our results, G can be any subgroup of the set of characters $\chi(H)$ of H, that is the set of all k-algebra maps from H to k.

For further information about the vector space M_χ and the above functors we refer to [4], where H is a finite-dimensional Hopf algebra and χ is a character of H.

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