UDC 512.54

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# Some soluble groups of finite rank and some related matrix groups

# Некоторые разрешимые группы конечного ранга и некоторые связанные с ними группы матриц

Let $G$ be a torsion-free soluble group of finite rank and $F$ any field. The group algebra $FG$ is an Ore domain; let $D$ denote its division ring of quotients. It seems likely that $D$ is always locally residually finite-dimensional over $F$. This is certainly so in the non-modular case. Here in some special situations we settle the modular case. We include some applications to groups of matrices.

Пусть $G$ — разрешимая группа без кручения конечного ранга и $F$ — поле. Групповая алгебра $FG$ является областью Оре; пусть $D$ — ее тело частных. Представляется вероятным, что $D$ всегда локально резидуально конечномерно над $F$. Это несомненно так в немодулярном случае. В данной работе в некоторых специальных ситуациях мы рассматриваем модулярный случай. Приводим некоторые приложения к группе матриц.

Нехай $G$ — розв'язна група без кручення скінченного рангу і $F$ — поле. Групова алгебра $FG$ є областю Оре і нехай $D$ — її тіло часток. Ймовірно, що $D$ завжди є локально резидуально скінченновимірне над $F$. Це безперечно так в немодулярному випадку. В деяких спеціальних ситуаціях ми розглядаємо модулярний випадок. Наводимо деякі застосування до груп матриць.

Let $G$ be a torsion-free soluble group of finite rank and $F$ any field. Then the group algebra $FG$ is an Ore domain [1] and therefore has a division ring $D$ of quotients. In [2] we considered this division ring $D$ and more particularly groups of matrices over $D$. Primarily [2] is devoted to what may be called the non-modular case; that is either where char $F = 0$ or where char $F = p > 0$ and no finitely generated subgroup of $G$ involves a Prüfer $p^\infty$-group. Here we make a start on the apparently much more complicated modular case. Thus in what follows $F$ is always a field of positive characteristic $p$. At this stage we can do little more than cope with a few examples.

The fundamental theorem in [2] is that in the non-modular case $D$ is locally residually finite-dimensional over $F$. The major open question in this area at present must be whether this remains true in the modular case. We are able to settle only a few very special cases. Our techniques suffice for the following.

Theorem 1. *Let $F$ be a field of characteristic $p > 0$, $r$ a positive integer and set $G = \langle a, b \mid a^b = a^r \rangle$. Then the division ring $D$ of quotients of the group algebra $FG$ is locally residually finite-dimensional over $F$.*

Part of Theorem 1 is covered by [2, 3], new information being given only when $p|r$. The results below and of [2] depend upon the construction of localizable ideals. In [2] this is accomplished by restricting ourselves to ideals with the Artin-Rees property. Artin-Rees ideals require the ambient ring to be Noetherian. The group algebra $FG$ of the soluble group $G$ is Noetherian only if $G$ is polycyclic (e. g. [4] 5), which is usually not the case here or in [2]. Thus we required in [2] an initial reduction to Noetherian rings. This was done via a preliminary localization at the augmentation ideal of a torsion-free nilpotent normal subgroup. This approach is not available for the proof of Thoerem 1 above.

Example. Let $F$ be a field of characteristic $p > 0$, set $G = \langle a, b \mid a^b = a^p \rangle$, $A = \langle a^G \rangle$, $\mathfrak{a}$ equal to the augmentation ideal of $A$ in $FA$ and $C = FA \setminus \mathfrak{a}$. Then $C$ is a divisor subset of $FA$ and $FG$ and we form the rings $R = FG.C^{-1}$ and $S = FA.C^{-1}$ of quotients. Then $R$ and $S$ are neither left nor right Noetherian.

Note that the group of the example above is covered by Theorem 1. Our proof of Theorem 1 is unsatisfactory, in that it gives no pointers towards a proof of a general result. Again we construct a localizable ideal, but the Ore condition we check by direct calculation, using the very special circumstances pertaining there. For a general theorem it seems we would need localization techniques that work in very non-Noetherian situations. In a similar way we can also prove the following.

Theorem 2. *Let $F$ be a locally finite field of characteristic $p$, let $q$ be a power of $p$ and set $G = \langle b \rangle A$, where $A$ is a torsion-free abelian group that is an extension of a free abelian group of finite rank by a $p$-group and $b$ normalizes $A$ and acts on it by $a^b = a^q$ for all $a \in A$. Then the division ring $D$ of quotients of $FG$ is locally residually finite-dimensional over $F$.*

Clearly the groups of Theorem 2 include those of Theorem 1. We discuss briefly below, see the Proposition, what can be said if, in the situation of Theorem 2, the group $A$ is just torsion-free abelian of finite rank. As in [2] there are some ready consequences of local residual finite-dimensionality.

Corollary. *Let $n$ be a positive integer and suppose $D$ is one of the division rings of Theorems 1 and 2.*

a) *A unipotent subgroup of $GL(n, D)$ is unitriangularizable; that is, $U$ is a stability subgroup in the sense of [5].*

b) *Let $H$ be a subgroup of the group of units of some finitely generated subring of the matrix ring $D^{n\times n}$; for example let $H$ be any finitely generated subgroup of $GL(n, D)$. Then $H$ is residually finite.*

c) *Let $H$ be as in Part b). Then there are positive integers $e$ and $f$ such that every element of $H$ of finite order has order dividing $e$ and every finite $p'$-subgroup of $H$ has order dividing $f$.*

d) *Let $P$ be a periodic subgroup of $GL(n, D)$. Then $P$ is locally finite and $P/O_p(P)$ is isomorphic to a subgroup of $GL(n, F)$. If $P$ contains no elements of order $p$ then some conjugate of $P$ in $GL(n, D)$ lies in $GL(n, F)$.*

With one exception the above corollary is exactly that suggested by the characteristic-$p$ cases of the corollaries of [2]. The exception is if $H$ is as in c). We do not claim that $H$ is a finite extension of a poly residually-finite-$p$ group (or better). The reason is obvious; if $G$ is the group of the example above then $G$ is finitely generated subgroup of $GL(1, D)$ and yet $G$ is not a finite extension of a poly residually-finite-$p$ group.

All rings and algebras below have identities and ring and algebra homomorphisms preserve these identities. Let $R$ be a ring. A divisor subset of $R$ is a multiplicative submonoid of $R$ not containing 0 and satisfying the left and right Ore conditions with respect to $R$. If $\mathbf{a}$ is an ideal of $R$ then $C_R(\mathbf{a})$ denotes the set of elements of $R$ that are regular modulo $\mathbf{a}$.

Lemma 1. *Let $F$ be a field, $r$ a positive integer and $G = \langle a, b \mid a^b = a^r \rangle$. Set $A = \langle a^G \rangle$ and suppose $\mathbf{a}$ is a $G$-invariant maximal ideal of $FA$ with $A/\mathbf{a}$ finite. Then $C_{FG}(\mathbf{a}G) = FG \setminus \mathbf{a}G$ is a divisor subset of $FG$.*

Proof. $FG = \oplus_i b^i FA$, $\mathbf{a}G = \oplus_i b^i \mathbf{a}$ and $FG/\mathbf{a}G = (FA/\mathbf{a})\langle b \rangle$ is a skew group ring of the infinite cyclic group $\langle b \rangle$ over the field $FA/\mathbf{a}$. As such it is a domain, so $C_{FG}(\mathbf{a}G) = FG \setminus \mathbf{a}G = Q$ say. Clearly $Q$ is a multiplicative submonoid of $FG$ not containing 0. Thus only the Ore conditions are in doubt and we check the right Ore condition. The left one is checked in a similar manner.

$F\langle a \rangle / (\mathbf{a} \cap F\langle a \rangle)$ is a domain generated over $F$ by a finite image of $\langle a \rangle$. Thus it is a field and $\mathbf{a} \cap F\langle a \rangle$ is a maximal ideal of the principal ideal domain $F\langle a \rangle$. Hence $\mathbf{a} \cap F\langle a \rangle = \alpha . F\langle a \rangle$ for some atom (=prime element) ($\alpha$ от $F\langle a \rangle$.

Let $x \in FG \setminus \{0\}$ and $q \in Q$. Since $FG$ is an Ore domain ([1] or [5] 1.4.4) there are non-zero elements $y$ and $z$ of $FG$ with $xz = qy$. Now $x = \sum b^i \xi_i$, $y = \sum b^i \eta_i$, $z = \sum b^i \zeta_i$ and $q = \sum b^i \chi_i$ for some $\xi_i$, $\eta_i$, $\zeta_i$ and $\chi_i$ all in $FA$. Set $\xi_{ij} = b^{-j} \xi_i b^j \in FA$ and $\eta_{ij} = b^{-j} \eta_i b^j \in FA$ for each $i$ and $j$. Then $xz = \sum b^{i+j} \xi_{ij} \zeta_j$, $qy = \sum b^{i+j} \chi_{ij} \eta_j$ and for each $k \in \mathbb{Z}$ we have

$$\sum_{i+j=k} \xi_{ij} \zeta_j = \sum_{i+j=k} \chi_{ij} \eta_j \ldots . \qquad (*)$$

By replacing $a$ by $b^l a b^{-l}$ for some suitably large $l$ we may assume that all the $\xi_i$, $\eta_i$, $\zeta_i$, $\chi_i$, $\xi_{ij}$ for $j$ with $\zeta_j \neq 0$ and $\chi_{ij}$ for $j$ with $\eta_j \neq 0$ lie in $F\langle a \rangle$.

In $F\langle a \rangle$ suppose $\alpha$ divides each $\zeta_i$ but does not divide some $\eta_j$. Now $(\mathbf{a}G)^b = \mathbf{a}G$. Thus for any $k$ we have $b^{-k} q b^k = \sum b^i \chi_{ik} \notin \mathbf{a}G$. Hence for each $k$ with $\eta_k \neq 0$ there exists $i$ with $\alpha$ not dividing $\chi_{ik}$ in $F\langle a \rangle$. Let $m$ be the minimal $j$ with $\alpha$ not dividing $\eta_j$. Then $\eta_m \neq 0$. Let $l$ be the minimal $i$ with $\alpha$ not dividing $\chi_{im}$. Set $k = l + m$ and suppose $i + j = k$. If $j < m$ then $\alpha \mid \eta_j$. Suppose $j > m$ with $\eta_j \neq 0$. Then $i < l$ and $\alpha \mid \chi_{im}$. Now $\alpha^b \in \mathbf{a} \cap F\langle a \rangle = \alpha . F\langle a \rangle$. Thus $\alpha \mid \alpha^b$ and so $\alpha \mid \chi_{in}$ for all $n \geqslant m$. In particular $\alpha \mid \chi_{ij}$. Consequently $(*)$ yields that $\alpha \mid \chi_{lm} \eta_m$. But $\alpha$ is an atom that does not divide $\chi_{lm}$ or $\eta_m$. This contradiction shows that $\alpha$ divides every $\eta_j$ whenever it divides all the $\zeta_i$.

Using the previous paragraph, whenever $\alpha \mid \zeta_i$ for all i we may cancel $\alpha$ from $y$ and $z$. Thus we may choose $y$ and $z$ as above such that $\alpha$ does not divide every $\zeta_i$ in $F\langle a \rangle$. Suppose $z \notin Q$. Then $z \in \mathbf{a}G$ and each $\zeta_i \in \mathbf{a} \cap F\langle a \rangle = \alpha . F\langle a \rangle$, by definition of $\alpha$. This contradiction of the choice of $z$ shows that $z \in Q$. We have now shown that $Q$ is a right Ore subset of $FG$, as claimed.

Lemma 2. *Let $A$ be a torsion-free abelian minimax group and $\xi$ an automorphism of $A$ such that $\mathbf{Q} \otimes_{\mathbb{Z}} A$ is irreducible as $\mathbf{Q}\langle \xi^i \rangle$ module for every positive integer $i$ (i. e. assume $A$ is a plinth for $\langle \xi \rangle$). Let $F$ be a field of positive characteristic with a maximal finite subfield $E$. Then the intersection on the $\xi$-invariant maximal ideals $\mathbf{m}$ of $FA$ with $A/\mathbf{m}$ finite is $\{0\}$.*

Proof. If $F$ is finite then this lemma is 3.1 of [2]. Suppose $\mathbf{n}$ is a $\xi$-invariant maximal ideal of $EA$ with $A/\mathbf{n}$ finite. Then $EA/\mathbf{n}$ is a finite and hence separable field. Of course $F$ is flat over $E$, so $FA/F\mathbf{n} \cong F \otimes_E (EA/\mathbf{n})$ is a field ([6, p. 197], Theorem 21 (2)) and consequently $\mathfrak{m} = F\mathfrak{n}$ is a $\xi$-invariant maximal ideal of $FA$. Clearly $|A/\mathfrak{m}| \leqslant |A/\mathfrak{n}| < \infty$. By [2] 3.1 we have $\bigcap \mathfrak{n} = 0$. If $B$ is a basis of $F$ over $E$ then $FA = \oplus_B bEA$, $\mathfrak{m} = \oplus_B b\mathfrak{n}$ and $\bigcap_{\mathfrak{n}} \mathfrak{m} = \oplus_B b(\bigcap \mathfrak{n}) = \{0\}$. The proof is complete.

The conclusion of Lemma 2 is false if, for example, $F$ is the algebraic closure of a finite field. Note that any finitely generated field of positive characterictic satisfies the requirements on $F$ in Lemma 2.

Lemma 3. *Let $F$ be a field and $G$ a group such that the group algebra $FG$ is an Ore domain with division ring $D = F(G)$ of quotients.*

a) *$D$ is locally residually finite-dimensional over $F$ if (and only if) for each finitely generated subgroup $H$ of $G$ and each element $t \in FH \setminus \{0\}$ there exists an $F$-algebra homomorphism of the subalgebra $FH[t^{-1}]$ of $D$ into a finite-dimensional (non-zero) $F$-algebra.*

b) *If $E$ is any subfield of $F$ then $EG$ has a division ring $E(G)$ of quotients.*

c) *$D$ is locally residually finite-dimensional over $F$ if $E(G)$ is locally residually finite-dimensional over $E$ for every $E$ in some local system $L$ of subfields of $F$.*

Proof. a) Let $X$ be a finite subset of $D$ and $x$ a non-zero element of $F[X] \leqslant D$. Enlarge $X$ so that $x \in X$. By the Ore condition there is a common right denominator $d \in FG$ of the elements of $X$, so $FG \supseteq Xd$. There is a finitely generated subgroup $H$ of $G$ with $d \in FH$ and $FH \supseteq Xd$. Set $t = xd^2 \in FH \setminus \{0\}$. By hypothesis there is an $F$-algebra homomorphism $\varphi$ of $FH[t^{-1}]$ into a finite dimensional $F$-algebra. Now

$$F[X] \leqslant FH[X] \leqslant FH[d^{-1}] \leqslant FH[t^{-1}],$$

since $d^{-1} = t^{-1}(xd) \in FH[t^{-1}]$. Thus $F[X]\varphi$ is defined and is a finite-dimensional $F$-algebra. Also $x^{-1} = d^2t^{-1}$ is a unit of $FH[t^{-1}]$ and hence $x\varphi \neq 0$. The proof is complete.

b) We check the right Ore condition. If $a, c \in EG \setminus \{0\}$ there exist $b$, $d \in FG \setminus \{0\}$ with $ad = cb$. If $B$ is any basis of $F$ over $E$ then $d = \sum_{x \in B} xd_x$ for suitable $d_x \in EG$ and similarly $b = \sum_{x \in B} xb_x$. For some $y$ in $B$ we have $d_y$ non-zero. Using that $B$ is central in $FG$ we obtain $0 \neq ad_y = cb_y$. Thus $EG$ is a right Ore domain. The left Ore condition is checked in the obvious way.

c) Let $H$ be a finitely generated subgroup of $G$ anq $t$ a non-zero element of $FH$. There exists a subfield $E$ of $F$ in $L$ with $t \in EH$. By b) the subring $EG$ of $D$ has a division ring $L$ of quotients and $L$ is naturally a subring of $D$. Also $FG = F \oplus_E EG$ and thus $D \geqslant FL \cong F \oplus_E L$. Hence we have $EH[t^{-1}] \leqslant L$ and $D \geqslant FH[t^{-1}] \cong F \oplus_E EH[t^{-1}]$. By hypothesis there is an $E$-algebra homomorphism $\theta$ of $EH[t^{-1}]$ onto some finite-dimensional $E$-algebra $B$. Then $1 \oplus \theta$ is an $F$-algebra homomorphism of $F \oplus_E EH[t^{-1}] \cong FH(t^{-1})$ onto the finite-dimensional $F$-algebra $F \oplus_E B$. The claim now follows from Part a).

The Proof of Theorem 1. Note first that $FG$ is an Ore domain ([1] or [5] 1.4.4). By Lemma 3 we may assume that $F$ is a finitely generated field. Let $t$ be a non-zero element of $FG$. Then $t = \sum_i b^i \tau_i$ for some elements $\tau_i$ of $FA$, not all zero. Hence Lemma 2 yields the exstence of a $G$-invariant maximal ideal $\mathfrak{m}$ of $FA$ not containing all the $\tau_i$ sucih that $A/\mathfrak{m}$ is finite. We can localize $FG$ at $\mathfrak{m}G$ by Lemma 1. Clearly $FG[t^{-1}] \leqslant FG.C_{FG}(\mathfrak{m}G)^{-1}$.

There is a natural map $\pi$ of the latter onto $(FG/\mathfrak{m}G).C_{FG/\mathfrak{m}G}(0)^{-1} = D_0$ say. Now $FG/\mathfrak{m}G = (FA/\mathfrak{m})\langle b\rangle$ is a skew group algebra. Since $A/\mathfrak{m}$ is finite there is a positive power $b^n$ of $b$ centralizing $FA/\mathfrak{m}$. Then $(FA/\mathfrak{m})\langle b^n\rangle$ is a group algebra with field $D_1$ say of quotients. Clearly $b$ normalizes $D_1$, so $D_1[b]$ has finite dimension $n$ over $D_1$ and hence is Artinian. Thus $D_0 = D_1[b]$ embeds into $D_1^{n\times n}$. Since $D_1$ is locally residually finite-dimensional over $F$, so is $D_0$. Hence there is a homomorphism of $FG[t^{-1}]\pi$ into a finite-dimensional $F$-algebra. In view of Lemma 3, the proof of Theorem 1 is complete.

Remark 1. In the above proof, $D_0$ is a division ring and $F\langle b^n\rangle$ lies in its centre. Thus $D_0$ is a finite-dimensional division algebra over rhe rational function field over $F$ in one variable. Let $X$ be a non-empty finite subset of some finitely generated subring of $D$ and set $r = \Pi_{x\neq y}(x-y)$, where $x$ and $y$ run over $X$ in some order. The above proof of Theorem 1 shows that there is a homomorphism $\varphi$ of $R$ into some finite-dimensional division algebra $L$ of characteristic $p$ such that $r\varphi\neq 0$. Then $\varphi|_X$ is one-to-one and so for a given $R$ the set of all such $\varphi$ form a super-residual system. We make use of this remark in the proof of the corollary.

Lemma 4. *Let $R = SG$ be a skew group ring of the group $G$ over ring $S$. If $R$ is right Noetherian then so is $S$.*

Proof. If $N$ is a right ideal of $S$ then $M = \oplus_{g\in G} Ng$ is a right ideal of $R$ and $M \cap S = N$. Thus an infinite ascending chain of right ideals of $S$ generates an infinite ascending chain of right ideals of $R$. The lemma follows.

The Proof that the Example is Non-Noetherian. $R = \oplus_{i\in\mathbb{Z}} Sb^i$ is a skew group ring. Now $S$ is not Noetherian, for $A/\langle a\rangle = P$ is a Prüfer $p^\infty$-group, so $FP$ is a local ring with maximal ideal the augmentation ideal. Thus $S/(a-1)S$ is isomorphic to $FP$, which is not Noetherian, since $P$ is not [4]. Therefore $S$ is not Noetherian and consequently $R$ is neither left nor right Noetherian by Lemma 4.

Lemma 5. *Let $F$ be a field and $A$ a free abelian group of finite rank:*

a) *the group algebra $FA$ is a unique factorization domain;*

b) *if $A = \langle a\rangle \times B$ with $a\neq 1$, then $a-1$ is an atom of $FA$ with $FA/(a-1)FA \cong FB$;*

c) *if $A = \langle a\rangle \times B$ and $b\in B$ with, $a, b\neq 1$ then $ab-1$ and $a-1$ are non-associate atoms of $FA$.*

Proof. a). $FA$ is a localization of the polynomial ring $F[X_1, \ldots, X_n]$ for $n = \text{rank } A$. The latter is a unique factorization domain (e. g. [7], Page 32, Theorem 10), so $FA$ is too.

b). $FA/(a-1)FA \cong FB$. The latter is a domain, so $a-1$ is an atom.

c). The units of $FA$ have the form $\alpha x$ for $\alpha\in F^*$ and $x\in A$. Consequently the only associates of $a-1$ of the form $x-1$ with $x\in A$ are $a-1$ and $a^{-1}-1$.

The Proof of theorem 2. By Lemma 3 we may assume that $F$ is a finite field. If $n$ is a positive integer then $F\langle b^n\rangle A$ has a division ring $E\leqslant D$ of quotients, $D = E[b]$ and $D$ embeds into $E^{n\times n}$. If $E$ is locally residually finite-dimensional over $F$ then so is $D$. Thus we may assume that $|F|$ divides $q$. It follows that $b$ acts on $FA$ and its quotient field $K$ as a Frobenius map; that is $x^b = x^q$ for all $x$ in $K$.

A has a free abelian subgroup $A_0$ of finite rank with $A/A_0$ a $p$-group. Set $A_i = b^iA_0b^{-i}$. Then $A_i\leqslant A_{i+1}$ for all $i$ and $A = \cup A_i$. Let $r\in FG\setminus\{0\}$. By Lemma 3 it sufices to construct an $F$-algebra homomorphism of $FG[r^{-1}]\leqslant$ $\leqslant D$ into some finite dimensional $F$-algebra. If rank $A_0 = 0$ then $A = \langle 1\rangle$ and the result is easy. We assume otherwise.

We may choose $A_0$ so that $r = \sum_i b^i\rho_i$ where all the $\rho_i$ lie in $FA_0$.

Now $FA_0$ is a unigue factorization domain by Lemma 5 with infinitely many non-associate atoms. Hence we can choose an atom $\alpha$ of $FA_0$ not dividing any non-zero $\rho_i$. Let $J$ be the localization of $FA_0$ at $\alpha FA_0$ in $K$. Then $J$ is a discrete valuation domain with maximal ideal $\alpha J$. Set $J_i = b^iJb^{-i}$. Since $b$ induces a Frobenius map on $K$, so $J_i\leqslant J_{i+1}$ for all $i$ and $I = \cup J_i$ is a subalgebra of $D$. Let $\alpha_i = b^i\alpha b^{-i}$. Then $\alpha_iJ_i$ is the maximal ideal of $J_i$ and it lies in $J_i\cap\alpha_{i+1}J_{i+1}$. Hence $\alpha_iJ_i = J_i\cap\alpha_{i+1}J_{i+1}$ for all $i$. Consequently $\mathbf{m} = \bigcup_i \alpha_iJ$ is an ideal of $I$ with $I/\mathbf{m} = \varinjlim J_i/\alpha_iJ_i$ is a field, so $\mathbf{m}$ is a maximal ideal of $I$ satisfying $\mathbf{m}\cap J = \alpha J$.

Clearly $I$ and $\mathbf{m}$ are $G$-invariant, so $\mathbf{p} = \mathbf{m}G$ is an ideal of the subring

$I\,[G]$ of $D$. Also $I\,[G]/\mathbf{p} \cong (I/\mathbf{m})\,\langle b\rangle$ is a skew group ring of $\langle b\rangle$ over the field $I/\mathbf{m}$. As such it has a division ring $L$ of quotients. Set $Q = C_{I[G]}(\mathbf{p}) = I\,[G]\setminus\mathbf{p}$. Suppose $Q$ is an Ore subset of $I\,[G]$. Then it is a divisor subset and we can form $I\,[G]\,Q^{-1} \leqslant D$. There is an obvious map $\theta$ of the latter onto $L$. Also $r \notin \mathbf{p}$ by construction, for if otherwise all the $\rho_i$ lie in $\mathbf{m} \cap FA_0 \leqslant \alpha J \cap FA_0 = \alpha FA_0$, so $r \in Q$ and $\theta$ maps $FG\,[r^{-1}] \leqslant D$ into $L$. Clearly $(FA_0)\,\theta = FA_0/\alpha FA_0$. If rank $A_0$ is at least 2 we can choose $\alpha$ by Lemma 5 so that $B_0 = A_0\,\theta$ is free abelian of rank one less that of $A_0$ and $(FA_0)\,\theta = FB_0$. We can apply induction to $FB_0\,\langle b\rangle = (FG)\,\theta \leqslant L$. If rank $A_0 = 1$ then $(FA)\,\theta$ is a finite field (using that $A/A_0$ is a $p$-group). Then the division ring of quotients of $(FG)\,\theta$ has finite dimension over the quotient field of the group algebra $F\,\langle b\rangle \leqslant (FG)\,\theta$. The conclusion then follows from the commutative case.

It remains to check the Ore conditions for $Q$. Again we check only the right one. The proof is a variant of the proof of Lemma 1. Let $x \in I\,[G]$ and $q \in Q$. Since $FG$ is an Ore domain ([1] or [5] 1.4.4) so is $I\,[G]$, see for example [5] 4.4.3. Hence there are elements $y$ and $z$ of $I\,[G]$ with $xz = qy \neq 0$. Suppose $x = \Sigma\, b^i\xi_i$, $y = \Sigma\, b^i\eta_i$, $z = \Sigma\, b^i\zeta_i$ and $q = \Sigma b^i\chi_i$, where the coefficients $\xi_i$, $\eta_i$, $\zeta_i$ and $\chi_i$ all lie in $I$. Equating coefficients in $xz = qy$ yields

$$\sum_{i+j=k} \xi_{ij}\zeta_j = \sum_{i+j=k} \chi_{ij}\eta_j \ldots \tag{*}$$

for each $k$, where as before $\xi_{ij} = b^{-j}\xi_i b^j$ and $\chi_{ij} = b^{-j}\chi_i b^j$. By replacing $A_0$ by $A_l$ for sufficiently large $l$ we may assume that all the $\xi_i, \eta_i, \zeta_i, \chi_i, \xi_{ij}$ for $j$ with $\zeta_j \neq 0$ and $\chi_{ij}$ for $j$ with $\eta_j \neq 0$ lie in $J$.

We now work in $J$. Suppose $\alpha$ divides each $\zeta_i$ but not every $\eta_j$. Since $\mathbf{p}$ is $G$-invariant we have $b^{-k}qb^k = \Sigma\, b^i\chi_{ik} \in Q$ for every $k$. Thus for each $k$ with $\eta_k \neq 0$ there exists $i$ with $\alpha$ not dividing $\chi_{ik}$. Let $m$ be the minimal $j$ with $\alpha$ not dividing $\eta_j$ and $l$ the minimal $i$ with $\alpha$ not dividing $\chi_{im}$ (of course $\eta_m \neq 0$). Set $k = 1 + m$ and suppose $i + j = k$. If $j < m$ then $\alpha \mid \eta_j$. Suppose $j > m$ and $\eta_j \neq 0$. Then $j < l$ and $\alpha \mid \chi_{im}$. But $\alpha^b \in \mathbf{m} \cap J = \alpha J$, so $\alpha \mid \alpha^b$ and hense $\alpha \mid \chi_{ij}$. Consequently (*) yields that $\alpha \mid \chi_{lm}\eta_m$. But $\alpha$ is an atom not dividing either of the factors. This contradiction proves that if $\alpha$ divides each $\zeta_i$ then $\alpha$ divides each $\eta_j$. By cancelling any superfluous $\alpha$'s we can choose $y$ and $z$ as above such that $\alpha$ does not divide each $\zeta_i$. If $z \notin Q$ then $z \in \mathbf{p} = \mathbf{m}\langle b\rangle$ and each $\zeta_i$ lies in $\mathbf{m} \cap J = \alpha J$. Thus $z \in Q$ and the proof of the right Ore condition is complete.

Remark 2. Again the above proof shows that for every finitely generated subring $R$ of $D$ and each finite subset $X$ of $R$ there is a homomorphism $\varphi$ of $R$ into some finite-dimensional division algebra of characteristic $p$ such that $\varphi$ is one-to-one on $X$.

In the progression from Theorem 1 to Theorem 2 and beyond, the next step is to allow $A$ to be any torsion-free abelian group of finite rank. Here we have only been able to localize at ideals arizing from certain augmentation ideals. We merely sketch the proof of this. Note first that 4.4.2 of [5] should read as follows.

Lemma 6. *Let $A$ be a torsion-free abelian group, let $F$ be a field and let $Q$ be the quotient field of $FA$. Denote the augmentation ideal of $A$ in $FA$ by* $\mathbf{a}$ *and let $H$ be a group of automorphisms of $A$ and hence of $Q$. If char $F = p > 0$ assume also that $A$ is residually a finite $p$-group. Then there exists a discrete valuation domain $J$ with maximal ideal* $\mathbf{m}$ *such that $FA \leqslant J \leqslant Q$, $FA \cap \mathbf{m} = \mathbf{a}$* and *$H$ normalizes $J$* and $\mathbf{m}$.

(In [5] the residually finite-$p$ hypothesis is omitted. The result is used in [5] to study certain groups, there called **X**-groups, that are, in particular, finite extensions of residually torsion-free polycyclic groups, so in [5] the residual condition in implicit in the applications. In characteristic $p > 0$ the augmentation ideal of a finite $p$-group is nilpotent, so in the characteristic $p$ case of the proof of [5] 4.4.2 it is now immediate that $\bigcap_i \mathbf{a}^i = \{0\}$.)

We return to the matter in hand. Let $F$ be a finite field of characteristic $p$ and let $G = \langle b\rangle\,A$, where $A$ is a torsion-free abelian group of finite rank and for some fixed power $q$ of $|F|$ we have $a^b = a^q$ for all $a$ in $A$. Again $FG$ has a division ring $D$ of quotients. Let $A_0$ be a free abelian subgroup of $A$ of maximal rank and let $B$ be the $p'$-divisible closure of $A_0$ in $A$. Then $B$ is residually a finite $p$-group and $A/B$ is a $p$-group, necessarily divisible because of the action of $b$. Set $B_i = b^i B b^{-i}$. Then $B_i \leqslant B_{i+1}$ for each $i$ and $A = \bigcup_i B_i$.

Let $\mathbf{a}$ and $\mathbf{b}$ denote the augmentation ideals of $A$ and $B$ respectively in $FA$ and $FB$ and let $K$ and $L$ be the quotient fields of $FA$ and $FB$ in $D$. By Lemma 6 there is a discrete valuation domain $J$ with maximal ideal $\mathbf{n} = \alpha J$ such that $FB \leqslant J \leqslant L$ and $\mathbf{b} = FB \cap \mathbf{n}$. Set $I = \bigcup_i b^i J b^{-i}$ and $\mathbf{m} = \bigcup_i b^i \mathbf{n} b^{-i}$. Since $b$ induces a Frobenius map we have $J \supseteq J^b$ and $\mathbf{n} \supseteq \mathbf{n}^b$. It follows that $I$ is a subring of $K$ containing $\bigcup_i b^i FB b^{-i} = FA$, $\mathbf{m}$ is an ideal of $I$, $\mathbf{m} \cap b^i J b^{-i} = b^i \mathbf{n} b^{-i}$ for each $i$, $I/\mathbf{m} \cong \varinjlim (b^i J b^{-i}/b^i \mathbf{n} b^{-i})$ is a field and $\mathbf{m}$ is a maximal ideal of $I$.

Trivially $b$ normalizes $I$ and $\mathbf{m}$, so $I\langle b \rangle = \oplus\, b^i I$ is a subring of $D$ and $\mathbf{p} = \mathbf{m}\langle b \rangle = \oplus\, b^i \mathbf{m}$ is an ideal of $I\langle b \rangle$. Moreover $I\langle b \rangle/\mathbf{p}$ is a skew group ring of $\langle b \rangle$ over the field $I/\mathbf{m}$. An adaption of the proof of Theorem 2 yields the following.

P r o p o s i t i o n. $Q = C_{I\langle b \rangle}(\mathbf{p}) = I\langle b \rangle \setminus \mathbf{p}$ is a divisor subset of $I\langle b \rangle$.

L e m m a 7. *Let $n$ be a positive integer and $D$ one of the division F-algebras of Theorem* 1 *or Theorem* 2. *Suppose $F$ is a finitely generated field. Then there exist positive integers $e$ and $f$, depending only on $n$ and $F$, such that every element of $GL(n, D)$ of finite order has order dividing $e$ and every finite $p'$-subgroup of $GL(n, D)$ has order dividing $f$.*

P r o o f. Let $k$ be an element of $GL(n, D)$ of finite order. If $k$ is a $p$-element the order of $k$ divides $p^n$, see [5] 1.3.1. Suppose $k$ is now a $p'$-element. Let $K$ be any finite-dimensional divison F-algebra. Then $KG$ has a classical division ring $K(G)$ of quotients and $K \otimes_F D$ is isomorphic to a subring of $K(G)$. As such it is a domain, so by [8] 1b) some conjugate $k'$ of $k$ lies in $GL(n, F)$. The eigenvalues of $k'$ are roots of unity satisfying polypomials over $F$ of degree $n$. Let $E$ be the prime subfield of $F$ and let $X$ de a transcendence basis of $F$ over $E$. The degree $(F : E(X))$ is finite and the eigenvalues of $k'$ lie in the (necessarily unique) extension of $E$ of degree $n!\,(F : E(X))$. Then $k'$ and hence $k$ has order dividing $p^{n!(F:E(X))} - 1$. This completes the proof of the first claim. A finite $p'$-subgroup of $GL(n, D)$ is isomorphic to a linear group of degree $n$ and characteristic $p$ ([5] 2.3,1; alternatively use [8] 1b). The second claim thus follows from the first with, for example, $f = n!e^n$.

T h e P r o o f o f t h e C o r o l l a r y. a). Adapt the proof of [2] Corollary 2, Pt. a).

b). This is immediate from Theorems 1 and 2 and the corresponding result for linear groups.

c). Now $H$ lies in $GL(n, R)$ for some finitely generated subring $R$ of $D$. Then $R$ is contained in the division ring of quotients of $EG$ in $D$ for some finitely generated subfield $E$ of $F$. Consequently c) follows from Lemma 7.

d). Provided $P$ is locally finite d) follows from [8] 1b). Let $H$ be a finitely generated subgroup of $P$. We have to prove that $H$ is finite. By c) the group $H$ has finite exponent, $e$ say. Suppose $H$ lies in $GL(n, R)$ where $R$ is a finitely generated subring of $D$ and let $\varphi$ be a homomorphism of $R$ into a finite-dimensional division algebra $L$ of characteristic $p$. Then $\varphi$ determines an obvious map $\varphi_n$ of $H$ into $GL(n, L)$. By the linear case $K = H_{\varphi n}$ is finite. Hence $K/O_p(K)$ is isomorphic to a linear group of degree $n$ and characteristic $p$, see [5] 2.3.1. Then Burnside's Theorem yields that $(K : O_p(K)) \leqslant f = e^m$ for $m = n^3$.

Let $d$ be the minimal number of generators of $H$. Then $K$ can be generated by $d$ elements and so $O_p(K)$ can be generated by $c = fd - f + 1$ elements.

Thus the *i-th* lower central factor of $O_p(K)$ can be generated by $c^i$ elements. Further $O_p(K)$, being unipotent, is nilpotent of class less than $n$ and exponent dividing $p^n$. Consequently there is a function of $d$, $e$, $n$ and $p$ only bounding the order of $K$. The above proofs of Theorems 1 and 2, see remarks 1 and 2, show that the maps $\varphi_n$ of $H$ as above forma super-residual system for $H$. Therefore $H$ is also finite (with order bounded by the same function of $d$, $e$, $n$ and $p$). The proof of the Corollary is complete.


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Received 27.09.90