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Oscillations and stability of a discrete delay logistic model

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Колебания и устойчивость в логической модели с дискретным запаздыванием

Consider the delay difference equation

$$x_{n+1}=\frac{\alpha x_n}{1+\sum_{i=1}^{m}\beta_i x_{n-k_i}},\quad n=0,1,2,\ldots,$$

where, $\alpha \in (1, \infty)$, $\beta_1, \beta_2, \ldots, \beta_m \in (0, \infty)$ and the delays $k_1, k_2, \ldots, k_m$ are nonnegative integers. We obtain conditions for the oscillation and asymptotic stability of all positive solutions about its positive equilibrium

$$(\alpha-1)/\sum_{i=1}^{m}\beta_i.$$

We also prove that all positive solutions of equation are bounded away from zero and infinity and that when $m = 2$, $k_1 = 0$ and $k_2 = 1$, the positive equilibrium is a global attractor.

Рассматривается разностное уравнение с запаздыванием

$$x_{n+1}=\frac{\alpha x_n}{1+\sum_{i=1}^{m}\beta_i x_n - k_i},\quad n=0,1,2,\ldots,$$

где $\alpha \in (1, \infty)$, $\beta_1, \beta_2, \ldots, \beta_m \in (0, \infty)$ и запаздывания $k_1, k_2, \ldots, k_m$ — неотрицательные целые числа. Получены условия колеблемости и асимптотической устойчивости всех положительных решений относительно положения равновесия

$$(\alpha-1)/\sum_{i=1}^{m}\beta_i.$$

Доказано, что все положительные решения уравнения ограничены вне нуля и бесконечности, и если $m = 2$, $k_1 = 0$ и $k_2 = 1$, то положение равновесия является глобальным аттрактором.

Розглядається різницеве рівняння з запізненням

$$x_{n+1}=\frac{\alpha x_n}{1+\sum_{i=1}^{m}\beta_i x_n - k_i},\quad n=0,1,2,\ldots,$$

де $\alpha \in (1, \infty)$, $\beta_1, \beta_2, \ldots, \beta_m \in (0, \infty)$ і запізнення $k_1, k_2, \ldots, k_m$ — невід'ємні цілі числа. Одержані умови коливності та асимптотичної стійкості всіх додатніх розв'язків відносно положення рівноваги

$$(\alpha - 1)/\sum_{i=1}^{m} \beta_i.$$

Доведено, що всі додатні розв'язки рівняння обмежені поза нулем і нескінченністю, і коли $m = 2$, $k_1 = 0$ і $k_1 = 1$, то положення рівноваги є глобальним атрактором.

1. Introduction and Preliminaries. Consider the delay difference equation

$$x_{n+1} = \frac{\alpha x_n}{1 + \sum_{i=1}^{m} \beta_i x_{n-k_i}}, \quad n = 0, 1, 2, \ldots \tag{1}$$

where

$$\alpha \in (1, \infty), \quad \beta_1, \beta_2, \ldots, \beta_m \in (0, \infty), \; k_1, k_2, \ldots, k_m \in \mathbf{N} = \{0, 1, 2, \ldots\}. \tag{2}$$

The special case of (1) with $m = 1$ was proposed by E. C. Pielou in her books [1, p. 22; 2, p. 79] as a discrete analog of the delay logistic equation

$$\dot{N}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{P}\right].$$

Pielou's interest was to show that the tendency to oscillate is a property of the populations themselves and is independent of any extrinsic factors. The blow-fly, *lucilia cuprina*, is an example of a biological population, which, according to the laboratory studies of Nicholson [3], behaves in the manner of (1) with $m = 1$.

The line of our investigation in this paper is parallel to that in [4] where the special case of (1) with $m = 1$ was studied.

Let $k = \max\{k_i : i = 1, 2, \ldots, m\}$. If $a_{-k}, \ldots, a_0$ are $k + 1$ given constants, then Equation (1) has a unique solution satisfying the initial conditions

$$x_n = a_n, \quad n = -k, \ldots, 0. \tag{3}$$

For initial values of the form

$$a_n \geqslant 0, \quad n = -k, \ldots, -1 \text{ and } a_0 > 0 \tag{4}$$

the initial value problem (1) and (3) has a unique solution which is positive for $n \geqslant 0$.

In this paper, we will only investigate solutions of Equatin (1) whose initial values satisfy (4). Such solutions, in this paper, are called *positive solutions*.

A sequence of real numbers $\{x_n\}$ is said to *oscillate about* 0, or simply, *oscillate* if the terms x_n are neither all eventually positive nor all eventually negative. Otherwise the sequence is called *nonoscillatory*. A sequence $\{x_n\}$ is said to *oscillate about* x^* if the sequence $\{x_n - x^*\}$ oscillates about 0.

The following known results will be useful in the proofs of our theorems. The lemma 1 which is exracted from [5], provides necessary and sufficient conditions for the oscillation of a certain nonlinear difference equation in terms of the oscillation of all solutions of an associated linear difference equation. The lemma 2 which was established in [6], provides necessary and sufficient conditions for the oscillation of all solutions of a linear difference equation in terms of its associated characteristic equation.

L e m m a 1 [5]. *Assume that* $k_1, k_2, \ldots, k_m \in \mathbf{N}$ *and* $f \in \mathbb{C}[\mathbb{R}^m, \mathbb{R}]$ *is such that,*

$$f(u_1, \ldots, u_m) \geqslant 0 \quad \text{for } u_1, \ldots, u_m \geqslant 0,$$

$$f(u_1, \ldots, u_m) \leqslant 0 \quad \textit{for } u_1, \ldots, u_m \leqslant 0,$$

and

$$f(u, \ldots, u) = 0 \textit{ if and only if } u = 0.$$

Now, suppose that there exists a $\delta > 0$ *such that for* $i = 1, 2, \ldots, m$, f *has continuous first partial derivatives,* $D_i f$, *for all* $u_1, u_2, \ldots, u_m \in [-\delta, \delta]$ *with*

$$D_i f(0, 0, \ldots, 0) = p_i \in (0, \infty)$$

and

$$\sum_{i=1}^{m} (p_i + k_i) \neq 1;$$

furthermore, assume that either

$$f(u_1, u_2, \ldots, u_m) \leqslant \sum_{i=1}^{m} p_i u_i, \quad \text{for } u_1, u_2, \ldots, u_m \in [0, \delta], \tag{5}$$

or

$$f(u_1, u_2, \ldots, u_m) \geqslant \sum_{i=1}^{m} p_i u_i, \text{ for } u_1, u_2, \ldots, u_m \in [-\delta, 0]. \tag{6}$$

Then, every solution of the nonlinear difference equation

$$x_{n+1} - x_n + f(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_m}) = 0 \tag{7}$$

oscillates if and only if every solution of the associated linear difference equation

$$y_{n+1} - y_n + \sum_{i=1}^{m} p_i y_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots$$

oscillates.

Lemma 2 [6]. *Let* $p_1, p_2, \ldots, p_k$ *be real numbers. Then the following two statements are equivalent.*

a) Every solution of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, 2, \ldots \tag{8}$$

oscillates.

b) The characteristic equation of (8)

$$\lambda^k + p_1 \lambda^{k-1} + \ldots + p_k = 0$$

has no positive roots.

Finally, the next lemma, from [7], provides sufficient conditions for the oscillation of all solutions of a linear delay difference equation. The conditions here are explicitly given in terms of the coefficients and the delays of the equation.

Lemma 3 [7]. *Assume that for* $i = 1, 2, \ldots, m$, $k_i \in \mathbf{N}$ *and* $p_i \in (0, \infty)$. *Also, suppose that one of the following conditions holds:*
oscillates.

$$a) \sum_{i=1}^{m} p_i \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}} > 1;$$

$$b) \left[\prod_{i=1}^{m} p_i\right]^{1/m} \frac{(\nu + 1)^{\nu+1}}{\nu^{\nu}} > 1, \text{ where } \nu = \frac{1}{m} \sum_{i=1}^{m} k_i.$$

Then every solution of the difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots$$

oscillates.

2. Oscillatory Behavior. The main result in this section is the following theorem, which states that Equation (1) has the same oscillatory behavior as an associated linear difference equation. Then by applying

this theorem and Lemma 3, we obtain sufficient conditions for the oscillation of all positive solutions of (1) in terms of its coefficients and delays.

Theorem 1. *Assume that* (2) *holds. Then every positive solution of* (1) *oscillates about its positive equilibrium*

$$(\alpha - 1)/\sum_{i=1}^{m} \beta_i$$

if and only if every solution of the linear difference equation

$$y_{n+1} - y_n + \left[(\alpha - 1)/\alpha \sum_{i=1}^{m} \beta_i\right] \sum_{i=1}^{m} \beta_i y_{n-k_i} = 0 \quad (9)$$

oscillates about 0.

Before we present the proof of Theorem 1, we need the following lemma which provides a useful inequality for the proof of Theorem 1.

Lemma 4. *Assume that* $\alpha > 1$ *and let* $c_1, c_2, \ldots, c_m$ *be positive numbers such that* $c_1 + c_2 + \ldots + c_m = 1$. *Then,*

$$\log\left[\frac{1 + (\alpha - 1)(c_1 e^{u_1} + \ldots + c_m e^{u_m})}{\alpha}\right] \geqslant \frac{\alpha - 1}{\alpha}(c_1 u_1 + \ldots + c_m u_m) \quad (10)$$

for all $u_1, u_2, \ldots, u_m \in \mathbb{R}$.

Proof (by induction on m). First, we consider the case $m = 1$. Let f be defined for any $u \in \mathbb{R}$ by the equation

$$f(u) = \log\left[\frac{1 + (\alpha - 1)e^u}{\alpha}\right] - \frac{\alpha - 1}{\alpha} u.$$

Then,

$$f'(u) = \frac{(\alpha - 1)e^u}{1 + (\alpha - 1)e^u} - \frac{\alpha - 1}{\alpha}$$

and so,

$$f'(u) \geqslant 0 \text{ if and only if } u \geqslant 0. \quad (11)$$

Since $f(0) = 0$, from (11) we see that $f(u) \geqslant 0$ for all $u \in \mathbb{R}$. Thus, (10) holds for $m = 1$.

Next, we assume that the statement of the lemma is true for $m \in \mathbb{N}$. We will show that it is also true for $m + 1$.

For $i = 1, 2, \ldots, m - 1$, set $\tilde{c}_i = c_i$ and $\tilde{u}_i = u_i$. Also, let

$$\tilde{c}_m = c_m + c_{m+1} \text{ and } \tilde{u}_m = \log\left[\frac{c_m e^{u_m} + c_{m+1} e^{u_{m+1}}}{\tilde{c}_m}\right].$$

Since we are assuming that $c_1 + c_2 + \ldots + c_{m+1} = 1$, we have $\tilde{c}_1 + \tilde{c}_2 + \ldots + \tilde{c}_m = 1$. Thus, by the induction hypothesis,

$$\log\left[\frac{1 + (\alpha - 1)(\tilde{c}_1 e^{\tilde{u}_1} + \ldots + \tilde{c}_m e^{\tilde{u}_m})}{\alpha}\right] \geqslant \frac{\alpha - 1}{\alpha}(\tilde{c}_1 \tilde{u}_1 + \ldots + \tilde{c}_m \tilde{u}_m)$$

or equivalently,

$$\log\left[\frac{1 + (\alpha - 1)(c_1 e^{u_1} + \ldots + c_m e^{u_m} + c_{m+1} e^{u_{m+1}})}{\alpha}\right] \geqslant$$

$$\geqslant \frac{\alpha - 1}{\alpha}\left[c_1 u_1 + \ldots + c_{m-1} u_{m-1} + \tilde{c}_m \log\left(\frac{c_m e^{u_m} + c_{m+1} e^{u_{m+1}}}{\tilde{c}_m}\right)\right]. \quad (12)$$

Now, since log is a concave function and

$$\frac{c_m}{\tilde{c}_m} + \frac{c_{m+1}}{\tilde{c}_m} = 1,$$

we have that

$$\log\left(\frac{c_m}{\tilde{c}_m} e^{u_m} + \frac{c_{m+1}}{\tilde{c}_m} e^{u_{m+1}}\right) \geqslant \frac{c_m}{\tilde{c}_m} e^{u_m} + \frac{c_m + 1}{\tilde{c}_m} e^{u_{m+1}}. \tag{13}$$

Combining (12) and (13), we obtain

$$\log\left[\frac{1 + (\alpha - 1)(c_1 e^{u_1} + \ldots + c_{m+1} e^{u_{m+1}})}{\alpha}\right] \geqslant \frac{\alpha - 1}{\alpha}\,[c_1 u_1 + \ldots + c_{m+1} u_{m+1})$$

and the proof is complete.

Proof of Theorem 1. The change of variables

$$x_n = \left[(\alpha - 1)/\sum_{i=1}^{m} \beta_i\right] e^{z_n}$$

transform (1) to the difference equation

$$z_{n+1} - z_n + f(z_{n-k_1}, \ldots, z_{n-k_m}) = 0, \tag{14}$$

where

$$f(u_1, \ldots, u_m) = \log\left[\frac{1 + \left[(\alpha - 1)/\sum_{i=1}^{m} \beta_i\right] \sum_{i=1}^{m} \beta_i e^{u_i}}{\alpha}\right] =$$

$$= \log\left[\frac{1 + (\alpha - 1) \sum_{i=1}^{m} \tilde{\beta}_i e^{u_i}}{\alpha}\right]$$

with

$$\tilde{\beta}_i = \beta_i / \sum_{i=1}^{m} \beta_i \quad \text{for } i = 1, 2, \ldots, m.$$

Observe that by Lemma 4,

$$f(u_1, \ldots, u_m) \geqslant \frac{\alpha - 1}{\alpha} \sum_{i=1}^{m} \tilde{\beta}_i u_i$$

and so, in particular, condition (6) of Lemma 1 is satisfied. One can see that indeed all the hypotheses of Lemma 1 are satisfied and that the linear equation associated with (14) is Eduation (9). The proof of the theorem is therefore a consequence of Lemma 1.

For the purpose of illustrating the use of Theorem 1, we present two of its corollaries. The first is a generalization of Theorem 1 in [4].

Corollary 1. *Assume* $\alpha \in (1, \infty)$, $\beta_1, \beta_2 \in (0, \infty)$ *and* $k \in \{1, 2, \ldots\}$. *Then, every positive solution of the difference equation*

$$x_{n+1} = \frac{\alpha x_n}{1 + \beta_1 x_n + \beta_2 x_{n-k}}, \quad n = 0, 1, 2, \ldots \tag{15}$$

oscillates about its positive equilibrium $(\alpha - 1)/(\beta_1 + \beta_2)$, *if and only if*

$$\frac{\alpha^k (\alpha - 1) \beta_2 (\beta_1 + \beta_2)^k}{(\alpha\beta_2 + \beta_1)^{k+1}} > \frac{k^k}{(k+1)^{k+1}}. \tag{16}$$

Corollary 1 follows immediately from Theorem 1 and the following lemma.

L e m m a 5. *Assume that* $p,\ q \in (0,\ \infty)$ *and* $k \in \{1,\ 2,\ \ldots\}$. *Then every solution of the difference equation*

$$x_{n+1} - qx_n + px_{n-k} = 0 \tag{17}$$

oscillates about 0 *if and only if*

$$\frac{p}{q^{k+1}} > \frac{k^k}{(k+1)^{k+1}}\ . \tag{18}$$

P r o o f. The characteristic equation of (17) is

$$F(\lambda) = \lambda^{k+1} - q\lambda^k + p = 0.$$

Since $F(0) > 0$, $F(+\infty) = \infty$ and $F(\lambda)$ has only one critical point, namely $\bar{\lambda} = qk/(k+1)$, it follows that $\min\{F(\lambda) : \lambda \in [0,\ \infty)\} = F(\bar{\lambda})$. Thus $F(\lambda)$ has no positive roots if and only if $F(\bar{\lambda}) > 0$. This is equivalent to (18). Now the proof is completed by applying Lemma 2.

The second corollary of Theorem 1 provides sufficient conditions for the oscillation of all positive solutions of (1). It is a direct application of Lemma 3 and Theorem 1.

C o r o l l a r y 2. *Assume* (2) *holds. Then every positive solution of Equation* (1) *oscillates about its positive equilibrium* $(\alpha - 1)/\sum_{i=1}^{m} \beta_i$ *provided that one of the following conditions holds*:

$$a)\ \left[(\alpha - 1)/\alpha \sum_{i=1}^{m} \beta_i\right] \sum_{i=1}^{m} \beta_i \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}} > 1; \tag{19}$$

$$b)\ \left((\alpha - 1)/\alpha \sum_{i=1}^{m} \beta_i\right)\left(\prod_{i=1}^{m} \beta_i\right)^{1/m} \frac{(\nu + 1)^{\nu+1}}{\nu^\nu} > 1,\ \textit{where}\ \nu = \frac{1}{m}\sum_{i=1}^{m} k_i. \tag{20}$$

Note that both (19) and (20) are «sharp» sufficient conditions for oscillation in the sense that when $m = 1$ each of them is reduced to

$$\frac{\alpha - 1}{\alpha} > \frac{k^k}{(k+1)^{k+1}}$$

which, by Corollary 1, is a necessary and sufficient condition for the oscillation of all positive solutions of (1).

3. B o u n d e d n e s s a n d A s y m p t o t i c S t a b i l i t y. In this section we prove that every positive solution of Equation (1) is bounded away from zero and infinity. We also present sufficient conditions for the asymptotic stability of the positive equilibrium of (1).

T h e o r e m 2. *Assume that* (2) *holds. Then every positive solution of Equation* (1) *is bounded away from zero and infinity.*

P r o o f. Let $\{x_n\}$ be a positive solution of Equation (1). Then $x_{n+1} \leqslant \alpha x_n$, for $n \geqslant 0$, and so

$$x_{n-\rho} \geqslant \frac{1}{\alpha^\rho} x_n, \quad \text{for} \quad n \geqslant \rho.$$

Hence, for $n \geqslant k = \max\{k_1, k_2, \ldots, k_m\}$,

$$x_{n+1} = \frac{\alpha x_n}{1 + \sum_{i=1}^{m} \beta_i x_{n-k}} < \frac{\alpha x_n}{\left(\sum_{i=1}^{m} \frac{\beta_i}{\alpha^{k_i}}\right) x_n} = \frac{\alpha}{\sum_{i=1}^{m} \frac{\beta_i}{\alpha^{k_i}}},$$

and so, $\{x_n\}$ is bounded from above.

Set

$$B = \frac{\alpha}{\sum_{i=1}^{m} \frac{\beta_i}{\alpha^{k_i}}} \text{ and } b = \frac{\alpha}{1 + B \sum_{i=1}^{m} \beta_i}.$$

Then

$$x_{n+1} \geqslant \frac{\alpha x_n}{1 + \left(\sum_{i=1}^{m} \beta_i\right) B} = b x_n, \quad \text{for } n \geqslant k,$$

and so

$$x_{n-\rho} \leqslant \frac{1}{b^{\rho}} x_n, \quad \text{for} \quad n \geqslant k + \rho.$$

Hence

$$x_{n+1} \geqslant \frac{\alpha x_n}{1 + \left(\sum_{i=1}^{m} \frac{\beta_i}{b^{k_i}}\right) x_n}, \quad \text{for } n \geqslant 2k.$$

Let

$$\nu = \min \left\{ \frac{\alpha - 1}{\left(\sum_{i=1}^{m} \frac{\beta_i}{b^{k_i}}\right)}, \quad x_{2k} \right\}.$$

Now we claim that $x_n \geqslant \nu$ for $n \geqslant 2k$. Otherwise, there exists $n \geqslant 2k$ such that $x_{n+1} < \nu \leqslant x_n$. By using the monotonicity of the function $\frac{\alpha x}{1 + \delta x}$ with $\delta > 0$ we see that

$$x_{n+1} \geqslant \frac{\alpha x_n}{1 + \left(\sum_{i=1}^{m} \frac{\beta_i}{b^{k_i}}\right) x_n} > \frac{\alpha x_{n+1}}{1 + \left(\sum_{i=1}^{m} \frac{\beta_i}{b^{k_i}}\right) x_{n+1}}$$

which yields the contradiction $x_{n+1} \geqslant \nu$. The proof of Theorem 2 is complete.

Concerning the (local) asymptotic stability of Equation (1) note that its linearized equation about its positive equilibrium

$$(\alpha - 1)/\sum_{i=1}^{m} \beta_i$$

is (9) which can be written in the form

$$y_{n+1} - y_n + \sum_{i=1}^{m} p_i y_{n-k_i} = 0 \tag{21}$$

with

$$p_i = \left[(\alpha - 1)/\alpha \sum_{i=1}^{m} \beta_i\right] \beta_i \text{ for } i = 1, 2, \dots, m.$$

The characteristic equation of (21) is

$$c(\lambda) = \lambda^{k+1} - \lambda^k + \sum_{i=1}^{m} p_i \lambda^{k-k_i} = 0 \tag{22}$$

where $k = \max\{k_1, k_2, \dots, k_m\}$. The following lemma provides a sufficient condition for the zeros of (22) to lie inside the unit circle.

L e m m a 6. *Assume that* $p_1, p_2, \ldots, p_m \in (0, \infty)$ *and* $l, k_1, k_2, \ldots, k_m$ *are such that* $l \geqslant k_i$ *for* $i = 1, 2, \ldots, m$ *and*

$$\sum_{i=1}^{m} p_i \left[\frac{l}{l+1}\right]^{l-k_i} < \frac{l^l}{(l+1)^{l+1}}. \tag{23}$$

Then all the roots of the equation

$$\lambda^{l+1} - \lambda^l + \sum_{i=1}^{m} p_i \lambda^{l-k_i} = 0$$

lie inside the unit circle.

P r o o f. Set

$$c(\lambda) = \lambda^{l+1} - \lambda^l + \sum_{i=1}^{m} p_i \lambda^{l-k_i}$$

and note that condition (23) implies $c\,(l/(l+1)) < 0$. Since $c\,(1) > 0$, $c\,(\lambda)$ has at least one root inside the interval $(l/(l+1), 1)$. Next, observe that on $|\lambda| = l\,(l+1)$ condition (23) implies that

$$|\lambda^l - \lambda^{l+1}| \geqslant \frac{l^l}{(l+1)^{l+1}} > \sum_{i=1}^{m} p_i \left(\frac{l}{l+1}\right)^{l-k_i} \geqslant \left|\sum_{i=1}^{m} p_i \lambda^{l-k_i}\right|.$$

Hence, by Rouche's theorem, $\lambda^{l+1} - \lambda^l$ and $c\,(\lambda)$ have the same number of zeros inside $|\lambda| = l/(l+1)$. Therefore, $c\,(\lambda)$ has exactly l zeros inside $|\lambda| = l/(l+1)$ and the proof is complete.

Therefore, we have the following theorem concerning the asymptotic stability of Equation (1) about its positive equilibrium.

T h e o r e m 3. *Assume that* (2) *holds. Then the positive equilibrium of Equation* (1) *is asymptotically stable provided that*

$$\left[(\alpha - 1)/\alpha \sum_{i=1}^{m} \beta_i\right] \frac{(k+1)^{k+1}}{k^k} \sum_{i=1}^{m} \beta_i \left(\frac{k}{k+1}\right)^{k-k_i} < 1 \tag{24}$$

where $k = \max\{k_1, k_2, \ldots, k_m\}$.

Before we close this section, note that from (9) the linearized equation of

$$x_{n+1} = \frac{\alpha x_n}{1 + \beta_1 x_n + \beta_2 x_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{25}$$

is

$$y_{n+1} + p y_n + q y_{n-1} = 0, \quad n = 0, 1, 2, \ldots, \tag{26}$$

where

$$p = -\frac{\alpha\beta_2 + \beta_1}{\alpha(\beta_1 + \beta_2)} \quad \text{and} \quad q = \frac{(\alpha - 1)\beta_2}{\alpha(\beta_1 + \beta_2)}. \tag{27}$$

It is well known (see for example [7]) that Equation (26) is asymptotically stable if and only if $q - p + 1 > 0$, $p + q + 1 > 0$ and $1 - q > 0$.) As these conditions are satisfied when p and q are given by (27), Equation (25) is asymptotically stable. This observation anticipates the result of the next section.

4. G l o b a l A t t r a c t i v i t y. Our aim in this section is to establish a global attractivity result about the positive equilibrium of Equation (25). With regards to this matter, it was proved in Theorems 3 and 4 of [4] that when $\beta_1 = 0$ or $\beta_2 = 0$, the positive equilibrium of (25) is a global attractor of all positive solutions. The next result extends these two theorems from [4] to the case where $\beta_1\beta_2 \neq 0$.

T h e o r e m 4. *Assume that* $\alpha \in (1, \infty)$ *and* $\beta_1, \beta_2 \in (0, \infty)$. *Then,* *every positive solution of Equation* (25) *converges to the positive equilibrium* $(\alpha - 1)/(\beta_1 + \beta_2)$.

P r o o f. We tailor this proof after the proof of theorem 4 in [4].

The change of variables

$$x_n = \frac{\alpha - 1}{\beta_1 + \beta_2} + \frac{y_n}{\beta_2}, \quad \text{for } n \geqslant -1 \tag{28}$$

transforms (25) to

$$y_{n+1} = \frac{(\alpha + \beta) y_n - (\alpha - 1) y_{n-1}}{(1 + \beta)(\alpha + \beta y_n + y_{n-1})}, \quad n = 0, 1, 2, \dots, \tag{29}$$

where $\beta = \beta_1/\beta_2$. Furthermore, the solutions of (25) are positive, and so, by (28),

$$y_n > -(\alpha - 1)/(1 + \beta) \tag{30}$$

and

$$\alpha + \beta y_n + y_{n-1} > 1. \tag{31}$$

Also note, that for $n = 0, 1, 2, \dots$

$$y_{n+1} = \frac{(\alpha + \beta)[y_n - [\alpha/(\alpha + \beta) y_{n-1}] + y_{n-1}}{(1 + \beta)(\alpha + \beta y_n + y_{n-1})}, \tag{32}$$

$$y_{n+1} - y_n = -\frac{(\alpha - 1) + (1 + \beta) y_n}{(1 + \beta)(\alpha + \beta y_n + y_{n-1})}(\beta y_n + y_{n-1}) \tag{33}$$

and

$$y_{n+2} = \frac{[\beta(\beta + 3\alpha - \alpha^2) + \alpha] y_n - (\alpha - 1)[(\alpha + \beta) + (1 + \beta) y_n] y_{n-1}}{(1 + \beta)^2 (\alpha + \beta y_{n+1} + y_n)(\alpha + \beta y_n + y_{n-1})} -$$

$$- \frac{(\alpha - 1)\beta(1 + \beta) y_n^2}{(1 + \beta)^2 (\alpha + \beta y_{n+1} + y_n)(\alpha + \beta y_n + y_{n-1})}. \tag{34}$$

The proof will be complete if we show that

$$\lim_{n \to \infty} y_n = 0. \tag{35}$$

In view of (33), we see that (35) holds for every solution of Equation (29) which is eventually nonnegative or eventually nonpositive. Therefore, it remains to establish (35) for every solution $\{y_n\}$ of (29) which is «strictly» oscillatory, in the sense that for every $n_0 \in \mathbf{N}$, y_n attains both positive and negative values for $n \geqslant n_0$. Such a solution consists of a «string» of consecutively negative (nonnegative) terms followed by a string of consecutively nonnegative (negative) terms, and so on. We shall call these strings *negative semicycles* and *positive semicycles*, respectively.

By using (29) and (31), we see that every semicycle contains at least two terms. From (33), we have that the term of greatest magnitude in a given semicycle is either the first or the second term. Furthermore, from (32), it follows that in every semicycle, there exists at least one term which immediately follows the term of greatest magnitude.

Now, to complete the proof, we consider the four consecutive semicycles C_{r-1}^-, C_r^+, C_{r+1}^-, and C_{r+2}^+ which we supposse are negative, positive, negative and positive semicycles, respectively, such that

$$C_{r-1} = \{y_{k+1}, y_{k+2}, \dots, y_l\}, \quad C_r = \{y_{l+1}, y_{l+2}, \dots, y_m\},$$

$$C_{r+1} = \{y_{m+1}, y_{m+2}, \dots, y_n\}, \quad C_{r+2} = \{y_{n+1}, y_{n+2}, \dots, y_q\}.$$

Let us denote by b_{r-1}^-, b_r^+, b_{r+1}^-, and b_{r+2}^+ the absolute values of the terms of greatest magnitude in the above four semicycles.

We will establish the following two estimates, from which the proof of (35) will become obvious:

$$b_{r+1}^{-} < \frac{1}{1+\beta} \cdot \frac{(\alpha-1)^2}{\alpha(1+\beta)^2+(\alpha-1)^2}\, b_{r-1}^{-} \tag{36}$$

and

$$b_{r+2}^{+} < \frac{1}{(1+\beta)^2} \cdot \frac{(\alpha-1)^2}{\alpha^2}\, b_r^{+}. \tag{37}$$

To this end, note that from (29) and (34) we have,

$$b_r^{+} = (y_{l+1} \text{ or } y_{l+2}) < -\frac{1}{1+\bar{\beta}} \cdot \frac{(\alpha-1)\, y_{l-1}}{\alpha+\beta y_l + y_{l-1}} < -\frac{1}{1+\beta} \times$$

$$\times \frac{(\alpha-1)\, y_{l-1}}{\alpha+(1+\beta)\, y_{l-1}}.$$

Next, observe that for $a,\ b,\ c > 0$ the function

$$f(x) = \frac{ax}{b+cx} \text{ is increasing for } x > -b/c. \tag{38}$$

Therefore,

$$b_r^{+} < \frac{1}{1+\beta} \cdot \frac{(\alpha-1)\, b_{r-1}^{-}}{\alpha-(1+\beta)\, b_{r-1}^{-}}. \tag{39}$$

Also, using (29) and (34), we have

$$b_{r+1}^{-} = -y_{m+1} \text{ or } -y_{m+2} < \frac{1}{1+\beta} \cdot \frac{(\alpha-1)\, y_{m-1}}{\alpha+\beta y_m + y_{m-1}} < \frac{1}{1+\beta} \cdot \frac{(\alpha-1)\, y_{m-1}}{\alpha+y_{m-1}}$$

and using (39), we obtain,

$$b_{r+1}^{-} < \frac{1}{1+\beta} \cdot \frac{(\alpha-1)\, b_r^{+}}{\alpha+b_r^{+}}. \tag{40}$$

Now, observe that for $a,\ b,\ c > 0$, the function

$$g(x) = \frac{ax}{b-cx} \text{ is increasing for } x < b/c. \tag{41}$$

Thus, from (40) and using (39) we get,

$$b_{r+1}^{-} < \frac{1}{1+\beta} \cdot \frac{(\alpha-1) \cdot \dfrac{(\alpha-1)\, b_{r-1}^{-}}{\alpha-(1+\beta)\, b_{r-1}^{-}}}{\alpha + \dfrac{1}{1+\beta} \cdot \dfrac{(\alpha-1)\, b_{r-1}^{-}}{\alpha-(1+\beta)\, b_{r-1}^{-}}} = \frac{1}{(1+\beta)^2} \times$$

$$\times \frac{(\alpha-1)^2 b_{r-1}^{-}}{\alpha^2(1+\beta) - [\alpha(1+\beta)^2-(\alpha-1)]\, b_{r-1}^{-}}$$

and so by (30)

$$b_{r+1}^{-} < \frac{1}{(1+\beta)^2} \cdot \frac{(\alpha-1)^2 b_{r-1}^{-}}{\alpha^2(1+\beta) - [\alpha(1+\beta)^2-(\alpha-1)] \cdot \dfrac{\alpha-1}{1+\beta}} =$$

$$= \frac{1}{1+\beta} \cdot \frac{(\alpha-1)^2 b_{r-1}^{-}}{(\alpha-1)^2+\alpha(1+\beta)^2}.$$

This proves (36).

In a similar way, we prove (37). From (29) and (34), we derive

$$b_{r+2}^{+} = (y_{n+1} \text{ or } y_{n+2}) < -\frac{1}{1+\beta}\cdot\frac{(\alpha-1)\,y_{n-1}}{\alpha+\beta y_n + y_{n-1}} <$$

$$< -\frac{1}{1+\beta}\cdot\frac{(\alpha-1)\,y_{n-1}}{\alpha+(1+\beta)\,y_{n-1}}$$

and thus, by (38)

$$b_{r+2}^{+} < \frac{1}{1+\beta}\cdot\frac{(\alpha-1)\,b_{r+1}^{-}}{\alpha-(1+\beta)\,b_{r+1}^{-}}.$$

Hence, from (40)

$$b_{r+2}^{+} < \frac{1}{1+\beta}\cdot\frac{(\alpha-1)\dfrac{1}{1+\beta}\cdot\dfrac{(\alpha-1)\,b_r^{+}}{\alpha+b_r^{+}}}{\alpha-(1+\beta)\dfrac{1}{1+\beta}\cdot\dfrac{(\alpha-1)\,b_r^{+}}{\alpha+b_r^{+}}} = \frac{1}{(1+\beta)^2}\cdot\frac{(\alpha-1)^2 b_r^{+}}{\alpha^2+b_r^{+}}$$

and so,

$$b_{r+2}^{+} < \frac{1}{(1+\beta)^2}\cdot\frac{(\alpha-1)^2}{\alpha^2}\,b_r^{+}$$

which proves (37) and completes the proof of the theorem.

R e m a r k. Throughout this paper we have assumed that condition (2) holds and in particular, that $\alpha \in (1, \infty)$. Now, instead of condition (2), assume that

$$\alpha \in (0, 1], \beta_1, \beta_2, \ldots, \beta_m \in (0, \infty)$$
$$k_1, k_2, \ldots, k_m \in \mathbf{N}. \qquad (2')$$

Then, zero is the only nonnegative equilibrium of (1). Furthermore, zero is a global attractor of all positive solutions of (1). Indeed, in this case $x_{n+1} < \alpha x_n \leqslant x_n$ and so, every positive solution of (1) converges monotonically to zero.

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and thus, by (38)

$$b_{r+2}^{+} < \frac{1}{1+\beta}\cdot\frac{(\alpha-1)\,b_{r+1}^{-}}{\alpha-(1+\beta)\,b_{r+1}^{-}}.$$

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and so,

$$b_{r+2}^{+} < \frac{1}{(1+\beta)^2}\cdot\frac{(\alpha-1)^2}{\alpha^2}\,b_r^{+}$$

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