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On tame semigroups generated by idempotents with partial null multiplication

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Abstract. Let I be a finite set without 0 and J a subset in $I \times I$ without diagonal elements (i,i). We define $S(I,J)$ to be the semigroup with generators e_i, where $i \in I \cup 0$, and the following relations: $e_0 = 0$; $e_i^2 = e_i$ for any $i \in I$; $e_ie_j = 0$ for any $(i,j) \in J$. In this paper we study finite-dimensional representations of such semigroups over a field k. In particular, we describe all finite semigroups $S(I,J)$ of tame representation type.

Introduction

We study finite-dimensional representations over a field k of semigroups generated by idempotents.

Let I be a finite set without 0 and J a subset in $I \times I$ without diagonal elements (i,i). We define $S(I,J)$ to be the semigroup with generators e_i, where $i \in I \cup 0$, and the following relations:

1) $e_0 = 0$;

2) $e_i^2 = e_i$ for any $i \in I$;

3) $e_ie_j = 0$ for any pair $(i,j) \in J$.

The set of all semigroups of the form $S(I,J)$ is denoted by $\mathcal{I}$. We call $S(I,J) \in \mathcal{I}$ *a semigroup generated by idempotents with partial null multiplication*.

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In this paper we give a criterion for $S(I,J)$ to be of finite representation type and a criterion for a finite $S(I,J)$ to be tame (note that any semigroup $S(I,J)$ of finite type is finite).

1. Formulation of the main results

Throughout the paper, k denotes a field. All vector space are finite-dimensional vector space over k. Under consideration maps, morphisms, etc., we keep the right-side notation.

Let S be a semigroup and let $M_n(k)$ denotes the algebra of all $n \times n$ matrices with entries in k. *A matrix representation of* S (*of degree* n) *over* k is a homomorphism T from S to the multiplicative semigroup of $M_n(k)$. If there is an identity (resp. zero) element $a \in S$, we assume that the matrix $T(a)$ is identity (resp. zero). Since $M_n(k)$ can be considered as the algebra of all linear transformations of any fixed n-dimensional vector space, we can consider representations of the semigroup S in terms of vector spaces and linear transformations. Thus, *a representation of* S *over* k is a homomorphism φ from S to the multiplicative semigroup of the algebra $End_k U$ with U being a finite-dimensional vector space. Two representation $\varphi : S \to End_k U$ and $\varphi' : S \to End_k U'$ are called *equivalent* if there is a linear map $\sigma : U \to U'$ such that $\varphi\sigma = \varphi'$.

A representation $\varphi : S \to End_k U$ of S is also denoted by (U, φ). By the dimension of (U, φ) one means the dimension of U. The representations of S form a category which will be denoted by $\mathrm{rep}_k\, S$ (it has as morphisms from (U, φ) to (U', φ) the maps σ such that $\varphi\sigma = \varphi'$). Since representations $X, Y \in S$ are equivalent iff they are isomorphic as objects of $\mathrm{rep}_k\, S$, we will use both the terms.

In an analogous way we can define representations of the semigroup S over a (not necessarily finite-dimensional) k-algebra Λ; in this case we must take free Λ-modules of finite rank instead of finite-dimensional vector spaces.

We say that a semigroup is *of finite representation type over* k if it has only finitely many equivalent classes of indecomposable representations (over k), and *of infinite type* if otherwise. Further, we say that a semigroup is *of tame* (respectively, *wild*) *type*, or simply *tame* (respectively, *wild*), if so is the problem of classifying its representations (precise definitions are given below).

Let $S = S(I,J) \in \mathcal{I}$ and $\overline{J} = \{(i,j) \in (I \times I) \setminus J \,|\, i \neq j\}$. We may assume, without loss of generality, that $I = \{1, 2, \ldots, m\}$. With the semigroup $S = S(I,J)$ we associate the quadratic form $f_S(z) : \mathbb{Z}^m \to \mathbb{Z}$

in the following way:

$$f_S(z) = \sum_{i \in I} z_i^2 - \sum_{(i,j) \in \overline{J}} z_i z_j.$$

We call $f_S(z)$ *the quadratic form of the semigroup* S.

In this paper we prove the following theorems.

Theorem 1. *A semigroup $S(I, J)$ is of finite representation type over k if and only if its quadratic form is positive (then $S(I, J)$ is finite).*

Theorem 2. *Let $S(I, J)$ be a finite semigroup. Then $S(I, J)$ is tame over k if its quadratic form is nonnegative, and wild if otherwise.*

2. Connections between representations of $S(I, J)$ and representations of quivers

We first recall the notion of representations of a quiver [1].

Let $Q = (Q_0, Q_1)$ be a (finite) quiver, where Q_0 is the set of its vertices and Q_1 is the set of its arrows $\alpha : x \to y$.

A representation of the quiver $Q = (Q_0, Q_1)$ over a field k is a pair $R = (V, \gamma)$ formed by a collection $V = \{V_x \,|\, x \in Q_0\}$ of vector spaces V_x and a collection $\gamma = \{\gamma_\alpha \,|\, \alpha : x \to y \text{ runs through } Q_1\}$ of linear maps $\gamma_\alpha : V_x \to V_y$. A morphism from $R = (V, \gamma)$ to $R' = (V', \gamma')$ is given by a collection $\lambda = \{\lambda_x \,|\, x \in Q_0\}$ of linear maps $\lambda_x : V_x \to V'_x$, such that $\gamma_\alpha \lambda_y = \lambda_x \gamma'_\alpha$ for any arrow $\alpha : x \to y$.

The category of representations of $Q = (Q_0, Q_1)$ will be denoted by $\operatorname{rep}_k Q$.

In an analogous way we can define representations of the quiver Q over a (not necessarily finite-dimensional) k-algebra Λ; in this case we must take free Λ-modules of finite rank instead of finite-dimensional vector spaces.

A quiver Q is said to be *of finite representation type over k* if $\operatorname{rep}_k Q$ has only finitely many isomorphism classes of indecomposable representations (over k), and *of infinite representation type* if otherwise. Further, Q is said to be *of tame* (respectively, *wild*) *representation type*, or simply *tame* (respectively, *wild*), if so is the problem of classifying its representations (precise definitions are given below).

Now we proceed to investigate connections between representations of $S(I, J)$ and representations of quivers.

We identify a linear map α of $U = U_1 \oplus \ldots U_p$ into $V = V_1 \oplus \ldots V_q$ with the matrix (α_{ij}), $i = 1, \ldots, p$, $j = 1, \ldots, q$, where $\alpha_{ij} : U_i \to V_j$ are the linear maps induced by α (then the sum and the composition of maps are given by the matrix rules).

For a finite set X and $Y \subseteq X \times X$, we denote by $Q(X,Y)$ the quiver with vertex set X and arrows $a \to b$, $(a,b) \in Y$.

Let $S = S(I,J)$, where, as before, $I = \{1,2,\dots,m\}$. Define the functor F from $\operatorname{rep}_k Q(I,\overline{J})$ to $\operatorname{rep}_k S(I,J)$ as follows. $F = F(I,J)$ assigns to each object $(V,\gamma) \in \operatorname{rep}_k Q(I,\overline{J})$ the object $(V',\gamma') \in \operatorname{rep}_k S(I,J)$, where $V' = \oplus_{i\in I} V_i$, $(\gamma'(e_i))_{jj} = \mathbf{1}_{V_j}$ if $i = j$, $(\gamma'(e_i))_{ij} = \gamma_{ij}$ if $(i,j) \in \overline{J}$, and $(\gamma'(e_i))_{js} = 0$ in all other cases. F assigns to each morphism λ of $\operatorname{rep}_k Q(I,\overline{J})$ the morphism $\oplus_{i\in I}\lambda_i$ of $\operatorname{rep}_k S(I,J)$.

Proposition 1. *The functor* $F = F(I,J) : \operatorname{rep}_k Q(I,\overline{J}) \to \operatorname{rep}_k S(I,J)$ *is full and faithful.*

Proof. It is obvious that the functor F is faithful. It remains to prove that it is full. Let δ be a morphism from $(V,\gamma)F = (V',\gamma')$ to $(W,\sigma)F = (W',\sigma')$. In other words, δ is a linear map of V' into W' such that $\gamma'(e_s)\delta = \delta\sigma'(e_s)$ for $s = 1,\dots,m$. We will consider these equalities as matrix ones (taking into account that $V' = \oplus_{i\in I} V_i$ and $W' = \oplus_{i\in I} W_i$) and denote by $[s,i,j]$ the scalar equality $(\gamma'(e_s)\delta)_{ij} = (\delta\sigma'(e_s))_{ij}$, induced by the (matrix) equality $\gamma'(e_s)\delta = \delta\sigma'(e_s)$.

From an equation $[j,i,j]$ with $j \neq i$ it follows that $\delta_{ij} = 0$, and consequently δ is a diagonal matrix: $\delta = \delta_{11} \oplus \delta_{22} \oplus \cdots \oplus \delta_{mm}$. Further, if $\alpha : i \to j$ is an arrow of the quiver $Q(I,\overline{J})$, then from the equation $[i,i,j]$ we have that $\gamma_\alpha\delta_{jj} = \delta_{ii}\sigma_\alpha$. Consequently, a collection $\overline{\delta} = \{\delta_{ss} \mid s = 1,\dots,m\}$ is a morphism from (V,γ) to (W,σ). Since $\delta = \delta_{11} \oplus \delta_{22} \oplus \cdots \oplus \delta_{mm}$, we have that $\delta = \lambda F$, where $\lambda = \overline{\delta}$, as claimed. □

Proposition 2. *If the quiver* $Q(I,\overline{J})$ *has no oriented cycles, then each object of* $\operatorname{rep}_k S(I,J)$ *is isomorphic to an object of the form* $XF(I,J) \oplus (W,0)$*, where* X *is an object of* $\operatorname{rep}_k Q(I,\overline{J})$ *(*W *is a vector space of dimension* $d \geq 0$ *and* $0 : W \to W$ *is the zero map).*

Proof. For simplicity, the quiver $Q(I,\overline{J})$ is denoted by $Q = (Q_0,Q_1)$. The proof will be by induction on m, the case $m = 0,1$ being trivial.

Now let $m > 1$ and let $R = (U,\varphi)$ be a representation of $S(I,J)$. Fix $s \in Q_0$ such that there is no arrow $i \to s$; obviously, one can assume that $s = m$. We consider the subsemigroup S' of S generated by e_i, $i \in I' \cup 0$, where $I' = \{1,\dots,m-1\}$. Then $S' = S(I',J')$ with $J' = \{(i,j) \in I \times I \mid i,j \in I'\}$, and $Q' = Q(I',\overline{J'})$ is the full subquiver of Q with vertex set $Q'_0 = I'$.

Denote by $R' = (U,\varphi')$ the restriction of R to S' ($\varphi'(x) = \varphi(x)$ for any $x \in S'$). It follows by induction that $R' \cong \overline{R'} = X'F(I',J') \oplus (W',0)$, where X' is a representation of the quiver $Q(I', \overline{J'})$. Let $\overline{R'} = (\overline{U}, \overline{\varphi'})$ and $X' = (V', \gamma')$ with $V' = \{V'_i \,|\, i \in Q'_0\}$ and $\gamma' = \{\gamma'_\alpha \,|\, \alpha : i \to j \text{ runs through } Q'_1\}$. Since $R' \cong \overline{R'}$, there exists a linear map $\sigma : U \to \overline{U} = V'_1 \oplus V'_2 \oplus \ldots \oplus V'_{m-1} \oplus W'$ such that $\varphi'\sigma = \overline{\varphi'}$. Then the representation $R = (U, \varphi)$ is equivalent to the representation $\overline{R} = (\overline{U}, \overline{\varphi})$, where $\overline{\varphi}(e_i) = \overline{\varphi'}(e_i)$ for any $i = 1, \ldots, m-1$ and $\overline{\varphi}(e_m) = \varphi(e_m)\sigma$ (because, for $i \neq m$, $\overline{\varphi'}(e_i) = \varphi'(e_i)\sigma = \varphi(e_i)\sigma$, and so $\overline{\varphi}(x) = \varphi(x)\sigma$ for any $x \in S$).

We consider the representation $\overline{R} = (\overline{U}, \overline{\varphi})$ in more detail. We set $V_m = W'$ and consider $\overline{\varphi}$ as a matrix (taking into account that $\overline{U} = V_1 \oplus V_2 \oplus \ldots \oplus V_{m-1} \oplus V_m$). For $(p, q) \in J$, we denote by $[p, q, i, j]$ the scalar equality $[\overline{\varphi}(e_p)\overline{\varphi}(e_q)]_{ij} = 0$, induced by the (matrix) equality $\overline{\varphi}(e_p)\overline{\varphi}(e_q) = 0$ (the last equation holds since $e_p e_q = 0$ in $S(I, J)$). It follows from $[m, q, i, q]$ (for any fixed $q \neq m$) that $(\overline{\varphi}(e_m))_{iq} = 0$, and consequently $(\overline{\varphi}(e_m))_{ij} = 0$ for any $(i, j) \in I \times I'$.

We first consider two special cases: a) $\overline{\varphi}_{mm} = 0$; b) $\overline{\varphi}_{mm} = \mathbf{1} = \mathbf{1}_{\mathbf{V_m}}$.

In case a) $(\overline{\varphi})^2 = \overline{\varphi}$ implies $\overline{\varphi} = 0$ and so $\overline{R} = XF(I, J) \oplus (W, 0)$ with $X = (V, \gamma)$, where $V = \{V'_1 \ldots, V'_{m-1}, 0\}$, $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = 0$ for $\alpha \notin Q'_1$ and $W = W'$.

In case b) an equality $[p, m, p, m]$ for $(p, m) \notin \overline{J}$ implies $(\overline{\varphi})_{pm} = 0$ and so $\overline{R} = XF(I, J) \oplus (W, 0)$ with $X = (V, \gamma)$, where $V = \{V'_1 \ldots, V'_{m-1}, 0\}$, $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = 0$ for $\alpha \notin Q'_1$ and $W = W'$.

Now we consider the general case. Since $(\overline{\varphi}_{mm})^2 = \overline{\varphi}_{mm}$, there is an invertible map $\nu = (\nu_1, \nu_2) : V_m \to W_1 \oplus W_2$ such that

$$\overline{\varphi}_{mm}(\nu_1, \nu_2) = (\nu_1, \nu_2) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{1} = \mathbf{1}_{W_1}$. Then the representation $\overline{R'} = (\overline{U}, \overline{\varphi'})$ is isomorphic to the the representation $\widehat{R'} = (\widehat{U}, \widehat{\varphi'})$, where $\widehat{U} = \widehat{U}_1 \oplus \widehat{U_2} \oplus \ldots \oplus \widehat{U}_{m+1}$ with $\widehat{U}_i = V_i$ for $i = 1, \ldots m-1$, $\widehat{U}_m = W_1$, $\widehat{U}_{m+1} = W_2$, and $\widehat{\varphi'}(e_i) = \overline{\varphi'}(e_i)$ for $i = 1, \ldots m-1$, $(\widehat{\varphi'}(e_m))_{ij} = (\overline{\varphi'}(e_m))_{ij}$ for $(i, j) \in I' \times I'$, $(\widehat{\varphi'}(e_m))_{ij} = 0$ for $i = m, m+1, j \in I'$, $(\widehat{\varphi'}(e_m))_{m,mj} = \mathbf{1} = \mathbf{1}_{W_1}$, $(\widehat{\varphi'}(e_m))_{m,m+1} = 0$, $(\widehat{\varphi'}(e_m))_{m+1,m} = 0$, $(\widehat{\varphi'}(e_m))_{m+1,m+1} = 0$ (for instance, one can take the isomorphism $\beta : \widehat{R'} \to \overline{R'}$ with $\widehat{\varphi'}(e_i) = \mu^{-1}\overline{R'}\mu$, where $\mu = \mathbf{1}_{U_1} \oplus \ldots \oplus \mathbf{1}_{U_{m-1}} \oplus \nu$.

From $(\widehat{\varphi'}(e_i))^2 = \widehat{\varphi'}(e_i)$ it follows that $\widehat{\varphi'}(e_m))_{i,m+1} = 0$ for any $i \in I'$ (see the partial case a)); (then $\widehat{\varphi'}(e_m))_{i,m+1} = 0$ for any $i = 1, \ldots, m+1$). From the scalar equalities $[p, m, p, m]$ for $(p, m) \notin \overline{J}$ implies $(\widehat{\varphi})_{pm} = 0$ (see the partial case b)). Thus, $\overline{R} = (\widehat{U}, \widehat{\varphi}) \cong R = (U, \varphi)$ has the form $XF(I, J) \oplus (W, 0)$, where $X = (V, \gamma)$ with $V = \{\widehat{U}_i \,|\, i \in Q_0\}$, $\gamma = \{\gamma_\alpha \,|\, \alpha : i \to j \text{ runs through } Q_1\}$ with $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = \widehat{\varphi}(e_m)_{ij}$ for $\alpha \notin Q'_1$ (then $j = m$), and $W = \widehat{W_{m+1}}$, as claimed. □

Denote by $\operatorname{rep}_k^\circ S(I,J)$ the full subcategory of $\operatorname{rep}_k S(I,J)$ consisting of all objects that have no objects $(W,0)$, with $W \neq 0$, as direct summands.

We have as an immediate consequence of Propositions 1 and 2 the following statement.

Corollary 1. *If the quiver $Q(I,\overline{J})$ has no oriented cycles, then the functor $F = F(I,J)$, viewed as a functor from $\operatorname{rep}_k Q(I,\overline{J})$ to $\operatorname{rep}_k^\circ S(I,J)$, is an equivalence of categories.*

3. Proof of Theorems 1 and 2

In [1] P. Gabriel introduced *the quadratic Tits form $q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ of a quiver $Q = (Q_0, Q_1)$:*

$$q_Q(z) = \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j,$$

where $i \to j$ runs through Q_1, and proved that Q is of finite representation type if and only if its Tits form is positive.

The definitions of the quadratic Tits form of a quivers and the quadratic form of a semigroup $S(I,J) \in \mathcal{I}$ immediately imply the following lemma.

Lemma 1. *Let $S = S(I,J) \in \mathcal{I}$ and $Q = Q(I,\overline{J})$. Then the quadratic forms $f_S(z)$ and $q_Q(z)$ coincide.*

Now we prove Theorem 1. In [3] one proves that a semigroup $S(I,J)$ is finite if and only if the quiver $Q(I,\overline{J})$ has no oriented cycles (the Tits form of which are not positive [6]). Then Theorem 1 follows from Corollary 1, Lemma 1 and the above-mentioned Gabriel's results.

Before we begin to prove Theorem 2, we recall precise definitions of tame and wild semigroups (see general definitions in [2]).

For a semigroup S and a k-algebra Λ, we denote by $R_\Lambda(S)$ the set of all representations of S over Λ. By $\mathcal{L}(\Lambda)$ we denote the category of left finite-dimensional (over k) Λ-modules.

Let S be a semigroup and $\Lambda = K_1 = k[x]$. We say that a representation $N = (U, \varphi)$ from $\operatorname{rep}_k S$ is generated by a representation $M = (V, \psi)$ from $R_\Lambda(S)$ if, for some $X \in \mathcal{L}(\Lambda)$, $N \cong M \otimes X = (V \otimes X, \psi \otimes \mathbf{1}_X)$ (the tensor products are considered over Λ).

We assume first that the field k is separable closed. The semigroup S is called *tame* if, for any fixed dimension d, there exist finitely many elements M_i of $R_\Lambda(S)$ such that, up to isomorphism, each indecomposable object of $\operatorname{rep}_k S$ (of the dimension d) is generated by M_i for some

i. Such a set $\{M_i\}$ is called *a parametrizing family of representations of S of dimension d.*

When the field k is not separable closed, the semigroup S is called *tame*, if it is tame over the separable closure $\overline{k}$ of k (in the case of infinite k one can take k itself in place of $\overline{k}$).

Now we give a definition of wild semigroups.

Let S be a semigroup and $\Lambda = K_2 = k < x, y >$ the free associative k-algebra in two noncommuting variables x and y. A representation $M = (V, \psi)$ of S over Λ is said to be *perfect* if it satisfies the following conditions:

1) the representation $M \otimes X = (V \otimes X, \psi \otimes \mathbf{1}_X)$ (of S over k) with $X \in \mathcal{L}(\Lambda)$ is indecomposable if so is X;

2) the representations $M \otimes X$ and $M \otimes X'$ are nonisomorphic if so are X and X'.

The semigroup S is called *wild over k* if it has a perfect representation over Λ.

In an analogous way one can define tame and wild quivers; the set of all representations of a quiver Q over an algebra Λ will be denote by $R_\Lambda(Q)$.

Now we prove Theorem 2.

Let $S = S(I, J)$ be a finite semigroup. Then the quiver $Q(I, \overline{J})$ has no oriented cycles (see above). From the papers [4, 5] on tame quivers and the paper [6] on integral quadratic forms it follows that a quiver Q is tame if its Tits form is nonnegative, and wild if otherwise. Then the first part of Theorem 2 follows from Lemma 1, Corollary 1 and the obvious fact that, for $\Lambda = k[x]$, the map $F_\Lambda = F_\Lambda(I, J)$ from $R_\Lambda(Q)$ to $R_\Lambda(S)$, which is defined analogously to the functor $F = F(I, J)$ on objects, "preserves" (from left to right) parametrizing families of any fixed dimension. Analogously, the second part of Theorem 2 follows from Lemma 1, Corollary 1 and the obvious fact that, for $\Lambda = k < x, y >]$, the map $F_\Lambda = F_\Lambda(I, J)$ from $R_\Lambda(Q)$ to $R_\Lambda(S)$, which is defined analogously to the functor $F = F(I, J)$ on objects, "preserves" (from left to right) perfect representations over Λ.

Theorems 1 and 2 are proved.

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