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On closures in semitopological inverse semigroups with continuous inversion

Oleg Gutik

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Abstract. We study the closures of subgroups, semilattices and different kinds of semigroup extensions in semitopological inverse semigroups with continuous inversion. In particularly we show that a topological group G is H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if G is compact, a Hausdorff linearly ordered topological semilattice E is H-closed in the class of semitopological semilattices if and only if E is H-closed in the class of topological semilattices, and a topological Brandt λ^0-extension of S is (absolutely) H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is S. Also, we construct an example of an H-closed non-absolutely H-closed semitopological semilattice in the class of semitopological semilattices.

1. Introduction and preliminaries

We shall follow the terminology of [2, 8, 12, 27, 30].

A subset A of an infinite set X is called *cofinite in* X if $X \setminus A$ is finite.

Given a semigroup S, we shall denote the set of idempotents of S by $E(S)$. A *semilattice* is a commutative semigroup of idempotents. For

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a semilattice E the semilattice operation on E determines the partial order $\leqslant$ on E:

$$e \leqslant f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural*. An element e of a partially ordered set X is called *minimal* if $f \leqslant e$ implies $f = e$ for $f \in X$. An idempotent e of a semigroup S without zero (with zero 0_S) is called *primitive* if e is a minimal element in $E(S)$ (in $(E(S)) \setminus \{0_S\}$). A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set.

A semigroup S with the adjoined unit [zero] will be denoted by S^1 [S^0] (cf. [8]). Next, we shall denote the unit (identity) and the zero of a semigroup S by 1_S and 0_S, respectively. Given a subset A of a semigroup S, we shall denote by $A^* = A \setminus \{0_S\}$ and $|A| =$ the cardinality of A. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Such an element y is called *inverse* of x and it is denoted by x^{-1}.

If $h\colon S \to T$ is a homomorphism (or a map) from a semigroup S into a semigroup T and if $s \in S$, then we denote the image of s under h by $(s)h$. A semigroup homomorphism $h\colon S \to T$ is called *annihilating* if $(s)h = (t)h$ for all $s, t \in S$.

Let S be a semigroup with zero and λ a cardinal $\geqslant 1$. We define the semigroup operation on the set $B_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\}$ as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. If $S = S^1$ then the semigroup $B_\lambda(S)$ is called the *Brandt λ-extension of the semigroup S* [13]. Obviously, if S has zero then $\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) \mid 0_S$ is the zero of $S\}$ is an ideal of $B_\lambda(S)$. We put $B^0_\lambda(S) = B_\lambda(S)/\mathcal{J}$ and the semigroup $B^0_\lambda(S)$ is called the *Brandt λ^0-extension of the semigroup S with zero* [19].

Next, if $A \subseteq S$ then we shall denote $A_{\alpha\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ if A does not contain zero, and $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in \lambda$.

We shall denote the semigroup of $\lambda\times\lambda$-matrix units by B_λ and the subsemigroup of $\lambda\times\lambda$-matrix units of the Brandt λ^0-extension of a monoid S with zero by $B^0_\lambda(1)$. We always consider the Brandt λ^0-extension only of a monoid with zero. Obviously, for any monoid S with zero we have $B_1^0(S) = S$. Note that every Brandt λ-extension of a group G is isomorphic to the Brandt λ^0-extension of the group G^0 with adjoined zero. The Brandt λ^0-extension of the group with adjoined zero is called a *Brandt semigroup* [8, 27]. A semigroup S is a Brandt semigroup if and only if S is a completely 0-simple inverse semigroup [7, 25] (cf. also [27, Theorem II.3.5]). We also observe that the semigroup B_λ of $\lambda \times \lambda$-matrix units is isomorphic to the Brandt λ^0-extension of the two-element monoid with zero $S = \{1_S, 0_S\}$ and the trivial semigroup S (i. e. S is a singleton set) is isomorphic to the Brandt λ^0-extension of S for every cardinal $\lambda \geqslant 1$.

Let $\{S_\iota \colon \iota \in \mathscr{I}\}$ be a disjoint family of semigroups with zero such that 0_ι is zero in S_ι for any $\iota \in \mathscr{I}$. We put $S = \{0\} \cup \bigcup \{S_\iota^* \colon \iota \in \mathscr{I}\}$, where $0 \notin \bigcup \{S_\iota^* \colon \iota \in \mathscr{I}\}$, and define a semigroup operation " $\cdot$ " on S in the following way

$$s \cdot t = \begin{cases} st, & \text{if } st \in S_\iota^* \text{ for some } \iota \in \mathscr{I}; \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup S with the operation " $\cdot$ " is called an *orthogonal sum* of the semigroups $\{S_\iota \colon \iota \in \mathscr{I}\}$ and in this case we shall write $S = \sum_{\iota \in \mathscr{I}} S_\iota$.

A non-trivial inverse semigroup is called a *primitive inverse semigroup* if all its non-zero idempotents are primitive [27]. A semigroup S is a primitive inverse semigroup if and only if S is an orthogonal sum of Brandt semigroups [27, Theorem II.4.3].

In this paper all topological spaces are Hausdorff. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\operatorname{cl}_Y(A)$ we denote the topological closure of A in Y.

A (*semi*)*topological semigroup* is a Hausdorff topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an *inverse topological semigroup*. A *topological inverse semigroup* is an inverse topological semigroup with continuous inversion. We observe that the inversion on a (semi)topological inverse semigroup is a homeomorphism (see [10, Proposition II.1]). A *semitopological group* is a Hausdorff topological space with a separately continuous group operation. A semitopological group with continuous inversion is a *quasitopological group*. A *paratopological group* is called a group with a continuous group operation. A paratopological group with continuous inversion is a *topological group*.

Let $\mathfrak{STSG}_0$ be a class of semitopological semigroups. A semigroup $S \in \mathfrak{STSG}_0$ is called *H-closed in* $\mathfrak{STSG}_0$, if S is a closed subsemigroup of any topological semigroup $T \in \mathfrak{STSG}_0$ which contains S both as a subsemigroup and as a topological space. The H-closed topological semigroups were introduced by Stepp in [31], and there they were called *maximal semigroups.* A semitopological semigroup $S \in \mathfrak{STSG}_0$ is called *absolutely H-closed in the class* $\mathfrak{STSG}_0$, if any continuous homomorphic image of S into $T \in \mathfrak{STSG}_0$ is H-closed in $\mathfrak{STSG}_0$. An algebraic semigroup S is called:

- *algebraically complete in* $\mathfrak{STSG}_0$, if S with any Hausdorff topology τ such that $(S, \tau) \in \mathfrak{STSG}_0$ is H-closed in $\mathfrak{STSG}_0$;
- *algebraically h-complete in* $\mathfrak{STSG}_0$, if S with discrete topology $\mathfrak{d}$ is absolutely H-closed in $\mathfrak{STSG}_0$ and $(S, \mathfrak{d}) \in \mathfrak{STSG}_0$.

Absolutely H-closed topological semigroups and algebraically h-complete semigroups were introduced by Stepp in [32], and there they were called *absolutely maximal* and *algebraic maximal*, respectively.

Recall [1], a topological group G is called *absolutely closed* if G is a closed subgroup of any topological group which contains G as a subgroup. In our terminology such topological groups are called H-closed in the class of topological groups. In [28] Raikov proved that a topological group G is absolutely closed if and only if it is Raikov complete, i.e. G is complete with respect to the two-sided uniformity. A topological group G is called *h-complete* if for every continuous homomorphism $h\colon G \to H$ the subgroup $f(G)$ of H is closed [9]. In our terminology such topological groups are called absolutely H-closed in the class of topological groups. The h-completeness is preserved under taking products and closed central subgroups [9]. H-closed paratopological and topological groups in the class of paratopological groups studied in [29].

In [32] Stepp studied H-closed topological semilattice in the class of topological semigroups. There he proved that an algebraic semilattice E is algebraically h-complete in the class of topological semilattices if and only if every chain in E is finite. In [23] Gutik and Repovš established the closure of a linearly ordered topological semilattice in a topological semilattice. They proved the criterium of H-closedness of a linearly ordered topological semilattice in the class of topological semilattices and showed that every H-closed topological semilattice is absolutely H-closed in the class of topological semilattices. Also, such semilattices studied in [6, 14]. In [3] the structure of closures of the discrete semilattices $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$ is described. Here the authors constructed an example of an H-closed topological semilattice in the class of topological semilattices which is not absolutely H-closed in the class of topological semilattices. The constructed example gives a negative answer on Question 17 from [32].

Definition 1.1 ([19])**.** Let $\mathfrak{STSG}_0$ be a class of semitopological semigroups. Let $\lambda \geqslant 1$ be a cardinal and $(S,\tau) \in \mathfrak{STSG}_0$. Let τ_B be a topology on $B^0_\lambda(S)$ such that

a) $(B^0_\lambda(S), \tau_B) \in \mathfrak{STSG}_0$;

b) the topological subspace $(S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}})$ is naturally homeomorphic to (S,τ) for some $\alpha \in \lambda$.

Then $(B^0_\lambda(S), \tau_B)$ is called a *topological Brandt λ^0-extension of (S,τ) in* $\mathfrak{STSG}_0$.

In the paper [24] Gutik and Repovš established homomorphisms of the Brandt λ^0-extensions of monoids with zeros. They also described a category whose objects are ingredients in the constructions of the Brandt λ^0-extensions of monoids with zeros. Here they introduced finite, compact topological Brandt λ^0-extensions of topological semigroups and countably compact topological Brandt λ^0-extensions of topological inverse semigroups in the class of topological inverse semigroups, and established the structure of such extensions and non-trivial continuous homomorphisms between such topological Brandt λ^0-extensions of topological monoids with zero. There they also described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt λ^0-extensions of topological monoids with zeros. These investigations were continued in [20–22], where established countably compact topological Brandt λ^0-extensions of topological monoids with zeros and pseudocompact topological Brandt λ^0-extensions of semitopological monoids with zeros their corresponding categories. In the papers [4,15,16,19,26] where studies H-closed and absolutely H-closed topological Brandt λ^0-extensions of topological semigroups in the class of topological semigroups.

In Section 2 we study the closure of a quasitopological group in a semitopological inverse semigroup with continuous inversion. In particularly we show that a topological group G is H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if G is compact.

Section 3 is devoted to the closure of a semitopological semilattice in a semitopological inverse semigroup with continuous inversion. We show that a Hausdorff linearly ordered topological semilattice E is H-closed in the class of semitopological semilattices if and only if E is H-closed in the class of topological semilattices. Also, we construct an example of an H-closed semitopological semilattice in the class of semitopological semilattices which is not absolutely H-closed in the class of semitopological semilattices.

In Section 4 we show that a topological Brandt λ^0-extension of S is (absolutely) H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is S. Also, we study the preserving of (absolute) H-closedness in the class of semitopological inverse semigroups with continuous inversion by orthogonal sums.

2. On the closure of a quasitopological group in a semitopological inverse semigroup with continuous inversion

Proposition 2.1. *Every left topological inverse semigroup with continuous inversion is semitopological semigroup.*

Proof. We write an arbitrary right translation $\rho_a\colon S\to S\colon x\mapsto xa$ of a left topological inverse semigroup S with continuous inversion $\mathbf{inv}\colon S\to S$ on three steps in the following way:

$$\rho_a(x)=xa=\left(a^{-1}x^{-1}\right)^{-1}=(\mathbf{inv}\circ\lambda_{a^{-1}}\circ\mathbf{inv})(x).$$

This implies the continuity of right translations i S. □

It is well known that the closure of an inverse subsemigroup of a topological inverse semigroup is again a topological inverse semigroup (see: [10, Proposition II.1]). The following proposition extends this result to semitopological inverse semigroups with continuous inversion.

Proposition 2.2. *The closure of an inverse subsemigroup T in a semitopological inverse semigroup S with continuous inversion is an inverse semigroup.*

Proof. By Proposition 1.8(ii) from [30, Chapter I, Proposition 1.8(ii)] the closure $\mathrm{cl}_S(T)$ of T in a semitopological semigroup S is a semitopological semigroup. Then the continuity of the inversion $\mathbf{inv}\colon S\to S$ and Theorem 1.4.1 from [11] imply that $\mathbf{inv}(\mathrm{cl}_S(T))\subseteq\mathrm{cl}_S(\mathbf{inv}(T))=\mathrm{cl}_S(T)$ and hence we get that $\mathbf{inv}(\mathrm{cl}_S(T))=\mathrm{cl}_S(T)$. This implies that $\mathrm{cl}_S(T)$ is an inverse subsemigroup of S. □

We observe that the statement of Proposition 2.2 is not true in the case of inverse topological semigroup. It is complete to consider the set $\mathbb{R}^+ = [0, +\infty)$ of non-negative real numbers with usual topology and usual multiplication of real numbers. This implies that in Proposition 2.2 the condition that S has continuous inversion is essential.

In a compact topological semigroup the closure of a subgroup is a topological subgroup (see: [5, Vol. 1, Theorems 1.11 and 1.13]). Also, since for a topological inverse semigroup S the map $f\colon S \to S\colon x \to xx^{-1}$ is continuous, the maximal subgroup of S is closed, and hence the closure of a subgroup of a topological inverse semigroup is a subgroup. The previous observation implies that this is not true in the general case of topological semigroups. Also, the following example shows that the closure of a subgroup in a semitopological inverse semigroup with continuous inversion is not a subgroup.

Example 2.3. Let $\mathbb{Z}$ be the discrete additive group of integers. We put $\mathscr{A}(\mathbb{Z})$ is the one point Alexandroff compactification of the space $\mathbb{Z}$ with the remainder ∞. We extend the semigroup operation from $\mathbb{Z}$ onto $\mathscr{A}(\mathbb{Z})$ in the following way:

$$n + \infty = \infty + n = \infty + \infty = \infty, \qquad \text{for every} \quad n \in \mathbb{Z}.$$

It is well known that $\mathscr{A}(\mathbb{Z})$ with such defined operation is a semitopological inverse semigroup with continuous inversion and $\mathbb{Z}$ is not a closed subgroup of $\mathscr{A}(\mathbb{Z})$ [30].

A quasitopological group G is called *precompact* if for every open neighbourhood U of the neutral element of G there exists a finite subset F of G such that $UF = G$ [2].

The following proposition gives examples quasitopological groups which are non-closed subgroups of some semitopological inverse semigroups with continuous inversion.

Proposition 2.4. *For every non-precompact regular quasitopological group (G, τ) there exists a regular semitopological inverse semigroup with continuous inversion which contains (G, τ) as a non-closed subgroup.*

Proof. Since the quasitopological group (G, τ) is non-precompact there exists an open neighbourhood U of the neutral element e of the group G such that $FU \neq G$ and $UF \neq G$ for every finite subset F in G. Let $\mathscr{B}_e$ be a base of the topology τ at the neutral element e of (G, τ). Since the inversion is continuous in (G, τ), without loss of generality we may assume that all elements of the family $\mathscr{B}_e$ are symmetric, i.e., $V = V^{-1}$ for every $V \in \mathscr{B}_e$. We put

$$\mathscr{B}_U = \{V \in \mathscr{B}_e \colon \operatorname{cl}_G(V) \subseteq U\}.$$

Since the quasitopological group (G, τ) is not precompact we have that $FV \neq G$ and $VF \neq G$ for every $V \in \mathscr{B}_U$ and for every finite subset F in G.

By G^0 we denote the group G with a joined zero 0. Now, we put

$$\begin{aligned}\mathscr{P}_0 = \{W_{g,V} = \{0\} \cup G \setminus \operatorname{cl}_G(gV) \colon V \in \mathscr{B}_U, g \in G\} \\ \cup \{W_{V,g} = \{0\} \cup G \setminus \operatorname{cl}_G(Vg) \colon V \in \mathscr{B}_U, g \in G\}\end{aligned}$$

and $\tau \cup \mathscr{P}_0$ is a subbase of a topology τ_0 on G^0.

Since (G, τ) a quasitopological group, it is sufficient to show that the semigroup operation on (G^0, τ_0) is separately continuous in the following two cases: $h \cdot 0 = 0$ and $0 \cdot h = 0$, for $h \in G$. Then for arbitrary subbase neighbourhoods $W_{g_1,V_1}, \ldots, W_{g_n,V_n}$ and $W_{V_1,g_1}, \ldots, W_{V_n,g_n}$ we have that

$$h \cdot (W_{g_1,V_1} \cap \cdots \cap W_{g_n,V_n}) \subseteq W_{hg_1,V_1} \cap \cdots \cap W_{hg_n,V_n}$$

and

$$(W_{V_1,g_1} \cap \cdots \cap W_{V_n,g_n}) \cdot h \subseteq W_{V_1,g_1h} \cap \cdots \cap W_{V_n,g_nh}.$$

Also, since translations in the quasitopological group (G, τ) are homeomorphisms, for every open subbase neighbourhood $V \in \mathscr{B}_U$ of the neutral element of G and every $g \in G$ we have that $(W_{g,V})^{-1} \subseteq W_{V^{-1},g^{-1}}$. Therefore (G^0, τ_0) is a quasitopological inverse semigroup with continuous inversion.

Now for every open subbase neighbourhoods $V_1, V_2 \in \mathscr{B}_U$ of the neutral element of G such that $\operatorname{cl}_G(V_1) \subseteq V_2$ and every $g \in G$ the following conditions holds:

$$\operatorname{cl}_G(W_{g,V_2}) \subseteq W_{g,V_1} \qquad \text{and} \qquad \operatorname{cl}_G(W_{V_2,g}) \subseteq W_{V_1,g}.$$

Hence we get that the topological space (G^0, τ_0) is regular. $\square$

Theorem 2.5. *A topological group G is H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if G is compact.*

Proof. The implication ($\Leftarrow$) is trivial.

($\Rightarrow$) Let a topological group G be H-closed in the class of semitopological inverse semigroups with continuous inversion. Suppose to the contrary: the space G is not compact. Then G is H-closed in the class of topological groups and hence it is Raĭkov complete. If G is precompact then by Theorem 3.7.15 of [2], G is compact. Hence the topological group G is not precompact. This contradicts Proposition 2.4. The obtained contradiction implies the statement of our theorem. □

Theorem 2.5 implies the following two corollaries:

Corollary 2.6. *A topological group G is absolutely H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if G is compact.*

Corollary 2.7. *A topological group G is H-closed in the class of semitopological semigroups if and only if G is compact.*

The following example shows that there exists a non-compact quasitopological group with adjoined zero which H-closed in the class of semitopological inverse semigroups with continuous inversion.

Example 2.8. Let $\mathbb{R}$ be the additive group of real numbers with usual topology. We put G is the direct quare of $\mathbb{R}$ with the product topology. It is well known that G is a topological group. Let G^0 be the group G with the adjoined zero 0. We define the topology τ on G^0 in the following way. For every non-zero element x of G^0 the base of the topology τ at x coincides with base of the product topology at x in G. For every $(x_0, y_0) \in \mathbb{R}^2$ and every $\varepsilon > 0$ we denote by

$$O_\varepsilon(x_0, y_0) = \left\{(x, y) \in \mathbb{R}^2 \colon \sqrt{(x - x_0)^2 + (y - y_0)^2} \leqslant \varepsilon\right\}$$

the usual closed ε-ball with the center at the point (x_0, y_0). We denote

$$A(x_0, y_0) = \left\{(x_0, y) \in \mathbb{R}^2 \colon y \in \mathbb{R}\right\} \cup \left\{(x, y_0) \in \mathbb{R}^2 \colon x \in \mathbb{R}\right\}$$

and

$$U_\varepsilon(x_0, y_0) = G^0 \setminus (O_\varepsilon(x_0, y_0) \cup A(x_0, y_0)) .$$

Now we put $\mathscr{P}(0) = \left\{U_\varepsilon(x, y) \colon (x, y) \in \mathbb{R}^2, \varepsilon > 0\right\}$ and $\mathscr{P}(0) \cup \mathscr{B}_G$ is a subbase of the topology τ on G^0, where $\mathscr{B}_G$ is a base of the topology of the topological group G. Simple verifications show that (G^0, τ) is a Hausdorff semitopological inverse semigroup with continuous inversion and (G^0, τ) is not a regular space.

Then for any finitely many points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$ and finitely many $\varepsilon_1, \ldots, \varepsilon_n > 0$ the following conditions hold:

(*a*) $O_{\varepsilon_1}(x_1, y_1) \cup \cdots \cup O_{\varepsilon_n}(x_n, y_n)$ is a compact subset of the space (G^0, τ);

(*b*) $\operatorname{cl}_{G^0}(U_{\varepsilon_1}(x_1, y_1) \cap \cdots \cap U_{\varepsilon_n}(x_n, y_n)) \cup O_{\varepsilon_1}(x_1, y_1) \cup \cdots \cup O_{\varepsilon_n}(x_n, y_n) = G^0$.

This implies that (G^0, τ) is an H-closed topological space and hence the semigroup (G^0, τ) is H-closed in the class of semitopological inverse semigroups with continuous inversion.

3. On the closure of a semilattice in a semitopological inverse semigroup with continuous inversion

It is well known that the subset of idempotent $E(S)$ of a topological semigroup S is a closed subset of S (see: [5, Vol. 1, Theorem 1.5]). We observe that for semitopological semigroups this statement does not hold [30]. Amassing, but the subset of all idempotent $E(S)$ of a semitopological inverse semigroup S with continuous inversion is a closed subset of S.

Proposition 3.1. *The subset of idempotents $E(S)$ of a semitopological inverse semigroup S with continuous inversion is a closed subset of S.*

Proof. First we observe that for any topological space X and any continuous map $f\colon X \to X$ the set $\mathsf{Fix}(f)$ of fixed point of f is closed subset of X (see: [5, Vol. 1, Theorem 1.4] or [11, Theorem 1.5.4]). Since $e^{-1} = e$ for every idempotent $e \in S$, the continuity of inversion implies that $E(S) \subseteq \mathsf{Fix}(\mathbf{inv})$. Let be $x \in S$ such that $x \in \mathsf{Fix}(\mathbf{inv})$. Since S is an inverse semigroup we obtain that $xx = xx^{-1} \in E(S)$ and hence $\mathsf{Fix}(\mathbf{inv}) \subseteq E(S)$. This completes the proof of the proposition. □

Proposition 3.1 implies the following

Corollary 3.2. *The closure of a subsemilattice in a semitopological inverse semigroup S with continuous inversion is a subsemilattice of S.*

Since the closure of a subsemilattice in a Hausdorff topological semigroup is again a topological semilattice, an (absolutely) H-closed topological semilattice in the class of topological semilattices is (absolutely)

H-closed in the class of topological semigroups [16]. In [32] Stepp proved that an algebraic semilattice E is algebraically h-complete in the class of topological semilattices if and only if every chain in E is finite. The following example shows that for every infinite cardinal λ there exists an algebraically h-complete semilattice $E(\lambda)$ in the class of topological semilattices of cardinality λ such that $E(\lambda)$ with the discrete topology is not H-closed in the class of semitopological semigroups.

Example 3.3. Let λ be any infinite cardinal. We fix an arbitrary $a_0 \in \lambda$ and define the semigroup operation on λ by the formula:

$$xy = \begin{cases} x, & \text{if } x = y; \\ a_0, & \text{if } x \neq y. \end{cases}$$

The cardinal λ with so defined semigroup operation we denote by $E(\lambda)$. It is obvious that $E(\lambda)$ is a semilattice such that a_0 is zero of $E(\lambda)$ and any two distinct non-zero elements of $E(\lambda)$ are incomparable with respect to the natural partial order on $E(\lambda)$. Let be $a \notin E(\lambda)$. We extend the semigroup operation from $E(\lambda)$ onto $S = E(\lambda) \cup \{a\}$ in the following way:

$$aa = ax = xa = a_0, \qquad \text{for any } \ x \in E(\lambda).$$

It is obvious that S with so defined operation is not a semilattice.

We define a topology τ on S in the following way. Fix an arbitrary sequence of distinct points $\{x_n \colon n \in \mathbb{N}\}$ from $E(\lambda)$ and put $U_n(a) = \{a\} \cup \{x_i \colon i \geqslant n\}$. Put all elements of the set $E(\lambda)$ are isolated points of the space (S, τ) and the family $\mathscr{B}(a) = \{U_n(a) \colon n \in \mathbb{N}\}$ is a base of the topology τ at the point $a \in S$. Simple verifications show that (S, τ) is a metrizable 0-dimensional semitopological semigroup and $E(\lambda)$ is a dense subsemilattice of (S, τ). Also, we observe that by Theorem 9 from [32] the semilattice $E(\lambda)$ is algebraically h-complete in the class of topological semilattices.

Remark 3.4. We observe that for every infinite cardinal λ and every Hausdorff topology τ on $E(\lambda)$ such that $(E(\lambda), \tau)$ is a semitopological semilattice we have that all non-zero idempotents of $(E(\lambda), \tau)$ are isolated points and moreover $(E(\lambda), \tau)$ is a topological semilattice. Also, a simple modification of the proof in the Example 3.3 shows that a semitopological semilattice $(E(\lambda), \tau)$ is H-closed in the class of semitopological semigroups if and only if the space $(E(\lambda), \tau)$ is compact.

Suppose that E is a Hausdorff semitopological semilattice. If L is a maximal chain in E, then by Proposition IV-1.13 of [12] we have that

$L = \bigcap_{e \in L}({\uparrow}e \cup {\downarrow}e)$ is a closed subset of E and hence we proved the following proposition:

Proposition 3.5. *The closure of a linearly ordered subsemilattice of a Hausdorff semitopological semilattice E is a linearly ordered subsemilattice of E.*

It is well known that the natural partial order on a Hausdorff semitopological semilattice is semiclosed (see [12, Proposition IV-1.13]). Also, by Lemma 3 of [33] a semiclosed linear order is closed, and hence every linearly ordered set with a closed order admits the structure of a Hausdorff topological semilattice. This implies the following proposition:

Proposition 3.6. *Every linearly ordered Hausdorff semitopological semilattice is a topological semilattice.*

Propositions 3.5 and 3.6 imply

Theorem 3.7. *A Hausdorff linearly ordered topological semilattice E is H-closed in the class of semitopological semilattices if and only if E is H-closed in the class of topological semilattices.*

Theorem 3.7 and results obtained in the paper [23] imply Corollaries 3.8—3.12.

A linearly ordered semilattice E is called *complete* if every non-empty subset of S has inf and sup.

Corollary 3.8. *A linearly ordered semitopological semilattice E is H-closed in the class of semitopological semilattices if and only if the following conditions hold:*

(*i*) *E is complete;*

(*ii*) *$x = \sup A$ for $A = {\downarrow}A \setminus \{x\}$ implies $x \in \operatorname{cl}_E A$, whenever $A \neq \varnothing$; and*

(*iii*) *$x = \inf B$ for $B = {\uparrow}B \setminus \{x\}$ implies $x \in \operatorname{cl}_E B$, whenever $B \neq \varnothing$*

Corollary 3.9. *Every linearly ordered H-closed semitopological semilattice in the class of semitopological semilattices is absolutely H-closed in the class of semitopological semilattices.*

Corollary 3.10. *Every linearly ordered H-closed semitopological semilattice in the class of semitopological semilattices contains maximal and minimal idempotents.*

Corollary 3.11. *Let E be a linearly ordered H-closed semitopological semilattice in the class of semitopological semilattices and $e \in E$. Then ${\uparrow}e$ and ${\downarrow}e$ are (absolutely) H-closed topological semilattices in the class of semitopological semilattices.*

Corollary 3.12. *Every linearly ordered semitopological semilattice is a dense subsemilattice of an H-closed semitopological semilattice in the class of semitopological semilattices.*

Remark 3.13. Theorem 3.7, Example 7 and Proposition 8 from [23] imply that there exists a countable linearly ordered σ-compact 0-dimensional scattered locally compact metrizable topological semilattice which does not embeds into any compact Hausdorff semitopological semilattice.

At the finish of this section we construct an H-closed semitopological semilattice in the class of semitopological semilattices which is not absolutely H-closed in the class of semitopological semilattices.

A filter $\mathscr{F}$ on a set X is called *free* if $\bigcap \mathscr{F} = \varnothing$.

Example 3.14 ([3])**.** Let $\mathbb{N}$ denote the set of positive integers. For each free filter $\mathscr{F}$ on $\mathbb{N}$ consider the topological space $\mathbb{N}_{\mathscr{F}} = \mathbb{N} \cup \{\mathscr{F}\}$ in which all points $x \in \mathbb{N}$ are isolated while the sets $F \cup \{\mathscr{F}\}$, $F \in \mathscr{F}$, form a neighbourhood base at the unique non-isolated point $\mathscr{F}$.

The semilattice operation min of $\mathbb{N}$ extends to a continuous semilattice operation min on $\mathbb{N}_{\mathscr{F}}$ such that $\min\{n, \mathscr{F}\} = \min\{\mathscr{F}, n\} = n$ and $\min\{\mathscr{F}, \mathscr{F}\} = \mathscr{F}$ for all $n \in \mathbb{N}$. By $\mathbb{N}_{\mathscr{F},\min}$ we shall denote the topological space $\mathbb{N}_{\mathscr{F}}$ with the semilattice operation min. Simple verifications show that $\mathbb{N}_{\mathscr{F},\min}$ is a topological semilattice. Then by Theorem 2(i) of [3] the topological semilattice $\mathbb{N}_{\mathscr{F},\min}$ is H-closed in the class of topological semilattices and hence by Theorem 3.7 it is H-closed in the class of semitopological semilattices.

Later by $E_2 = \{0, 1\}$ we denote the discrete topological semilattice with the semilattice operation min.

Theorem 3.15. *Let $\mathscr{F}$ be a free filter on $\mathbb{N}$ and $F \in \mathscr{F}$ be a set with infinite complement $\mathbb{N} \setminus F$. Then the closed subsemilattice $E = (\mathbb{N}_{\mathscr{F},\min} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{1\})$ of the direct product $\mathbb{N}_{\mathscr{F},\min} \times E_2$ is H-closed not absolutely H-closed in the class of semitopological semilattices.*

Proof. The definition of the topological semilattice $\mathbb{N}_{\mathscr{F},\min} \times E_2$ implies that E is a closed subsemilattice of $\mathbb{N}_{\mathscr{F},\min} \times E_2$.

Suppose the contrary: the topological semilattice E is not H-closed in the class of semitopological semilattices. Since the closure of a subsemilattice in a semitopological semilattice is a semilattice (see [30, Chapter I, Proposition 1.8(ii)]) we conclude that there exists a semitopological semilattice S which contains E as a dense subsemilattice and $S \setminus E \neq \varnothing$. We fix an arbitrary $a \in S \setminus E$. Then for every open neighbourhood $U(a)$ of the point a in S we have that the set $U(a) \cap E$ is infinite. By Theorem 2(i) of [3] and Theorem 3.7, the subspace $\mathbb{N}_{\mathscr{F},\min} \times \{0\}$ of E with the induced semilattice operation from E is an H-closed in the class of semitopological semilattices. Therefore there exists an open neighbourhood $U(a)$ of the point a in S such that $U(a) \cap E \subseteq (\mathbb{N} \setminus F) \times \{1\}$ and hence the set $U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$ is infinite.

Since the subset $\mathbb{N}_{\mathscr{F},\min} \times \{0\}$ is an ideal of E, the H-closedness of $\mathbb{N}_{\mathscr{F},\min} \times \{0\}$ in the class of semitopological semilattices implies that $\mathbb{N}_{\mathscr{F},\min} \times \{0\}$ is a closed ideal in S and hence we have that $x \cdot a \in \mathbb{N}_{\mathscr{F},\min} \times \{0\}$ for every $x \in \mathbb{N}_{\mathscr{F},\min} \times \{0\}$. Since for every open neighbourhood $U(a)$ of the point a in S the set $U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$ is infinite the semilattice operation in E implies that for every $x \in (\mathbb{N}_{\mathscr{F},\min} \times \{0\}) \setminus \{(\mathscr{F}, 0)\}$ the set $x \cdot U(a)$ is infinite and hence we have that $x \cdot a \notin N \times \{0\} = (\mathbb{N}_{\mathscr{F},\min} \times \{0\}) \setminus \{(\mathscr{F}, 0)\}$. Therefore we obtain that $x \cdot a = (\mathscr{F}, 0)$. Now, since in $\mathbb{N}_{\mathscr{F},\min}$ the sets $F \cup \{\mathscr{F}\}$, $F \in \mathscr{F}$, form a neighbourhood base at the unique non-isolated point $\mathscr{F}$, we conclude that $x \cdot U(a) \nsubseteq (F \cup \{\mathscr{F}\}) \times \{0\}$, which contradicts the separate continuity of the semilattice operation on S. Hence we get that $S \setminus E = \varnothing$. This implies that the topological semilattice E is H-closed in the class of semitopological semilattices.

Now, by Theorem 3 of [3] the topological semilattice E is not absolutely H-closed in the class of topological semilattices, and hence E is not absolutely H-closed in the class of semitopological semilattices. □

Remark 3.16. Corollary 3.2 implies that the topological semilattice E determined in Theorem 3.15 is an example a topological inverse semigroup which is H-closed but is not absolutely H-closed in the class of semitopological semigroups with continuous inversion.

Remark 3.17. Proposition 3.6 and Theorem 3.7 imply that Theorem 2 of [3] describes all H-closed semilattices in the class of semitopological semilattices which contain the discrete semilattice $(\mathbb{N}, \min)$ or the discrete semilattice $(\mathbb{N}, \max)$ as a dense subsemilattice.

4. On the closure of topological Brandt λ-extensions in a semitopological inverse semigroup with continuous inversion

In this section we study the preserving of H-closedness and absolute H-closedness by topological Brandt λ^0-extensions and orthogonal sums of semitopological semigroups.

Theorem 4.1. *Let S be a Hausdorff semitopological inverse monoid with zero and continuous inversion. Then the following conditions are equivalent:*

(i) S is absolutely H-closed in the class of semitopological inverse semigroups with continuous inversion;

(ii) there exists a cardinal $\lambda \geqslant 2$ such that every topological Brandt λ^0-extension of S is absolutely H-closed in the class of semitopological inverse semigroups with continuous inversion;

(iii) for each cardinal $\lambda \geqslant 2$ every topological Brandt λ^0-extension of S is absolutely H-closed in the class of semitopological inverse semigroups with continuous inversion.

Proof. $(i) \Rightarrow (iii)$. Suppose that the semigroup S is absolutely H-closed in the class of semitopological inverse semigroups with continuous inversion. We fix an arbitrary cardinal $\lambda \geqslant 2$. Let $B^0_\lambda(S)$ be a topological Brandt λ^0-extension of S in the class of semitopological inverse semigroups with continuous inversion, T be a semitopological inverse semigroup with continuous inversion and $h\colon B^0_\lambda(S) \to T$ be a continuous homomorphism.

First we observe that by Proposition 2.3 of [24], either h is an annihilating homomorphism or the image $(B^0_\lambda(S))h$ is isomorphic to the Brandt λ^0-extension $B^0_\lambda((S_{\alpha,\alpha})h)$ of the semigroup $(S_{\alpha,\alpha})h$ for some $\alpha \in \lambda$. If h is an annihilating homomorphism then $(S_{\alpha,\alpha})h$ is a singleton, and therefore we have that $(S_{\alpha,\alpha})h$ is a closed subset of T. Hence, later we assume that h is a non-annihilating homomorphism.

Next we show that for any $\gamma, \delta \in \lambda$ the set $(S_{\gamma,\delta})h$ is closed in the space T. By Definition 1.1 there exists $\alpha \in \lambda$ such that $(S_{\alpha,\alpha})h$ is a closed subset of T. We define the maps $\varphi_h, \psi_h\colon T \to T$ by the formulae $(x)\varphi_h = (\alpha, 1_S, \gamma)h\cdot(x)h\cdot(\delta, 1_S, \alpha)h$ and $(x)\psi_h = (\gamma, 1_S, \alpha)h\cdot(x)h\cdot(\alpha, 1_S, \delta)h$. Then the maps φ_h and ψ_h are continuous because left and right translations in T and homomorphism $h\colon B^0_\lambda(S) \to T$ are continuous maps. Thus, the full preimage $A = ((S_{\alpha,\alpha})h)\varphi_h^{-1}$ is a closed subset of T. Then the restriction map $(\varphi_h\circ\psi_h)|_A\colon A\to(S_{\gamma,\delta})h$ is a retraction, and therefore the set $(S_{\gamma,\delta})h$ is a retract of A. This implies that $(S_{\gamma,\delta})h$ is a closed subset of T.

Suppose to the contrary that $(B^0_\lambda(S))h$ is not a closed subsemigroup of T. By Lemma II.1.10 of [27], $(B^0_\lambda(S))h$ is an inverse subsemigroup of T. Since by Proposition 2.2 the closure of an inverse subsemigroup $(B^0_\lambda(S))h$ in a semitopological inverse semigroup T with continuous inversion is an inverse semigroup, without loss of generality we may assume that $(B^0_\lambda(S))h$ is a dense proper inverse subsemigroup of T.

We fix an arbitrary $x\in\operatorname{cl}_T((B^0_\lambda(S))h)\setminus(B^0_\lambda(S))h$. Then only one of the following cases holds:

a) x is an idempotent of the semigroup T;

b) x is a non-idempotent element of T.

Suppose that case *a*) holds. By the previous part of the proof we have that every open neighbourhood $U(x)$ of the point x in the topological space T intersects infinitely sets of the form $(S_{\alpha,\beta})h$, $\alpha,\beta\in\lambda$. By Proposition 2.3 of [24], $(B^0_\lambda(S))h$ is isomorphic to the Brandt λ^0-extension $B^0_\lambda((S_{\alpha,\alpha})h)$ of the semigroup $(S_{\alpha,\alpha})h$ for some $\alpha\in\lambda$, and since $(B^0_\lambda(S))h$ is a dense subsemigroup of semitopological semigroup T, the zero 0 of the semigroup $(B^0_\lambda(S))h$ is zero of T (see [20, Lemma 23]). Then the semigroup operation of $B^0_\lambda((S_{\alpha,\alpha})h)$ implies that either $0\in(\alpha,e,\alpha)h\cdot U(x)$ or $0\in U(x)\cdot(\alpha,e,\alpha)h$ for every non-zero idempotent (α,e,α) of $B^0_\lambda(S)$, $e\in E(S)$, $\alpha\in\lambda$. Now by the Hausdorffness of the space T and the separate continuity of the semigroup operation of T we have that either $(\alpha,e,\alpha)h\cdot x=0$ or $x\cdot(\alpha,e,\alpha)h=0$ for every non-zero idempotent (α,e,α) of $B^0_\lambda(S)$, $e\in E(S)$, $\alpha\in\lambda$. Since in an inverse semigroup any two idempotents commute we conclude that $(\alpha,e,\alpha)h\cdot x=x\cdot(\alpha,e,\alpha)h=0$ for every non-zero idempotent (α,e,α) of the semigroup $B^0_\lambda(S)$.

We fix an arbitrary non-zero element $(\alpha,s,\beta)h$ of the semigroup $(B^0_\lambda(S))h$, where $\alpha,\beta\in\lambda$ and $s\in S^*$. Then by the previous part of the proof we obtain that

$$\begin{aligned}x\cdot(\alpha,s,\beta)h=x\cdot(\alpha,ss^{-1}s,\beta)h&=x\cdot((\alpha,ss^{-1},\alpha)(\alpha,s,\beta))h\\&=x\cdot(\alpha,ss^{-1},\alpha)h\cdot(\alpha,s,\beta)h=0\cdot(\alpha,s,\beta)h=0\end{aligned}$$

and

$$\begin{aligned}(\alpha,s,\beta)h\cdot x=(\alpha,ss^{-1}s,\beta)h\cdot x&=((\alpha,s,\beta)(\beta,s^{-1}s,\beta))h\cdot x\\&=(\alpha,s,\beta)h\cdot(\beta,s^{-1}s,\beta)h\cdot x=(\alpha,s,\beta)h\cdot 0=0.\end{aligned}$$

This implies that for every open neighbourhood $U(x)$ of the point x in the space T we have that $0 \in x \cdot U(x)$ and $0 \in U(x) \cdot x$. Then by Hausdorffness of the space T and the separate continuity of the semigroup operation in T we get that $x \cdot x = 0$, and hence $x = 0$. This implies that $E(T) = E((B^0_\lambda(S))h)$.

Suppose that case b) holds. If $xx^{-1} = 0$, then $x = xx^{-1}x = 0 \cdot x = 0$, and similarly if $x^{-1}x = 0$, then $x = xx^{-1}x = x \cdot 0 = 0$. This implies that $xx^{-1}, x^{-1}x \in E((B^0_\lambda(S))h) \setminus \{0\}$.

Then by Lemma I.7.10 of [27] there exist idempotents (α, e, α), $(\beta, f, \beta) \in B^0_\lambda(S)$ such that $xx^{-1} = (\alpha, e, \alpha)h$ and $x^{-1}x = (\beta, f, \beta)h$, where $e, f \in (E(S))^*$ and $\alpha, \beta \in \lambda$. Then we have that $x \cdot (\beta, f, \beta)h = (\alpha, e, \alpha)h \cdot x = x$. Since $x \in \mathrm{cl}_T((B^0_\lambda(S))h) \setminus (B^0_\lambda(S))h$, every open neighbourhood $U(x)$ of the point x in the space T intersects infinitely many sets $(S_{\gamma,\delta})h$, $\gamma, \delta \in \lambda$, and hence we obtain that either $U(x) \cdot (\beta, f, \beta)h \ni 0$ or $(\alpha, e, \alpha)h \cdot U(x) \ni 0$. Then the Hausdorffness of the space T and the separate continuity of the semigroup operation on T imply that $x \cdot (\beta, f, \beta)h = 0$ or $(\alpha, e, \alpha)h \cdot x = 0$. If $x \cdot (\beta, f, \beta)h = 0$ then $x = x \cdot xx^{-1} = x \cdot (\beta, f, \beta)h = 0$ and if $(\alpha, e, \alpha)h \cdot x = 0$ then $x = xx^{-1}x = (\alpha, e, \alpha)h \cdot x = 0$. All these two cases imply that $x = 0$, and hence we get that $T = (B^0_\lambda(S))h$, which completes the proof of our theorem.

The implication $(iii) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (i)$. Suppose to the contrary: there exists semigroup S such that S is not absolutely H-closed semigroup S in the class of semitopological inverse semigroups with continuous inversion and condition (ii) holds for S. Then there exists a semitopological inverse semigroup T with continuous inversion and continuous homomorphism $h\colon S \to T$ such that $(S)h$ is non-closed subset of T. Now, by Proposition 2.2 without loss of generality we may assume that $(S)h$ is a proper dense inverse subsemigroup of T.

Next, for the cardinal λ we define topologies τ^B_T and τ^B_S on Brandt λ^0-extensions $B^0_\lambda(T)$ and $B^0_\lambda(S)$, respectively, in the following way. We put

$$\mathscr{B}^T_{(\alpha,t,\beta)} = \{(U(t))_{\alpha,\beta}\colon 0 \notin U(t) \in \mathscr{B}_T(t)\}$$
$$\text{and} \quad \mathscr{B}^s_{(\alpha,s,\beta)} = \{(U(s))_{\alpha,\beta}\colon 0 \notin U(s) \in \mathscr{B}_S(s)\}$$

are bases of topologies τ^B_T and τ^B_S at non-zero elements $(\alpha, t, \beta) \in B^0_\lambda(T)$ and $(\alpha, s, \beta) \in B^0_\lambda(S)$, respectively, $\alpha, \beta \in \lambda$, where $\mathscr{B}_T(t)$ and $\mathscr{B}_S(s)$ are bases of topologies of spaces T and S at non-zero elements $t \in T$ and $s \in S$, respectively. Also, if $\mathscr{B}_T(0_T)$ and $\mathscr{B}_S(0_S)$ are bases at zeros $0_T \in T$

and $0_S \in S$ then we define

$$\mathscr{B}_0^T = \Big\{\{0\} \cup \bigcup_{\alpha,\beta\in\lambda} (U(0_T))^*_{\alpha,\beta} \colon U(0_T) \in \mathscr{B}_T(0_T)\Big\}$$

$$\text{and} \quad \mathscr{B}_0^S = \Big\{\{0\} \cup \bigcup_{\alpha,\beta\in\lambda} (U(0_S))^*_{\alpha,\beta} \colon U(0_S) \in \mathscr{B}_S(0_S)\Big\}$$

to be the bases of topologies τ_T^B and τ_S^B at zeros $0 \in B_\lambda^0(T)$ and $0 \in B_\lambda^0(S)$, respectively.

Simple verifications show that if T and S are semitopological inverse semigroups with continuous inversion, then so are $(B_\lambda^0(T), \tau_T^B)$ and $(B_\lambda^0(S), \tau_S^B)$. Also the continuity of homomorphism $h\colon S \to T$ implies that the map $h_B\colon B_\lambda^0(S) \to B_\lambda^0(T)$ defined by the formulae

$$(\alpha, s, \beta)h_B = \begin{cases} (\alpha, (s)h, \beta), & \text{if } (s)h \neq 0_T; \\ 0, & \text{otherwise,} \end{cases}$$

$s \in S^*$, $\alpha, \beta \in \lambda$, and $(0)h_B = 0$ is continuous. Also, by Theorem 3.10 of [24] so defines map $h_B\colon B_\lambda^0(S) \to B_\lambda^0(T)$ is a homomorphism. The definition of the topology τ_T^B on $B_\lambda^0(T)$ implies that the homomorphic image $(B_\lambda^0(S))h_B$ is a dense proper subsemigroup of the semitopological inverse semigroup $(B_\lambda^0(T), \tau_T^B)$ with continuous inversion, which contradicts to statement *(ii)*. The obtained contradiction implies the requested implication. □

Now, if we put h is a topological isomorphic embedding of semitopological semigroups with continuous inversions in the proof of Theorem 4.1, then we get the proof of the following theorem:

Theorem 4.2. *Let S be a Hausdorff semitopological inverse monoid with zero and continuous inversion. Then the following conditions are equivalent:*

- *(i) S is H-closed in the class of semitopological inverse semigroups with continuous inversion;*
- *(ii) there exists a cardinal $\lambda \geqslant 2$ such that every topological Brandt λ^0-extension of S is H-closed in the class of semitopological inverse semigroups with continuous inversion;*
- *(iii) for each cardinal $\lambda \geqslant 2$ every topological Brandt λ^0-extension of S is H-closed in the class of semitopological inverse semigroups with continuous inversion.*

Theorem 4.1 implies Corollary 4.3 which generalizes Corollary 20 from [17].

Corollary 4.3. *For any cardinal $\lambda \geqslant 2$ the semigroup of $\lambda\times\lambda$-units B_λ is algebraically h-complete in the class of semitopological inverse semigroups with continuous inversion.*

Also, Theorems 4.1 and 4.2 imply the following corollary:

Corollary 4.4. *For an inverse monoid S with zero the following conditions are equivalent:*

(i) *S is algebraically complete (algebraically h-complete) in the class of semitopological inverse semigroups with continuous inversion;*

(ii) *there exists a cardinal $\lambda \geqslant 2$ such the Brandt λ^0-extension of S is algebraically complete (algebraically h-complete) in the class of semitopological inverse semigroups with continuous inversion;*

(iii) *for each cardinal $\lambda \geqslant 2$ the Brandt λ^0-extension of S is algebraically complete (algebraically h-complete) in the class of semitopological inverse semigroups with continuous inversion.*

Theorems 4.5, 4.6 and 4.7 give a method of the construction of absolutely H-closed and H-closed semigroups in the class of semitopological inverse semigroups with continuous inversion.

Theorem 4.5. *Let $S=\bigcup_{\alpha\in\mathscr{A}} S_\alpha$ be a semitopological inverse semigroup with continuous inversion such that*

(i) *S_α is an absolutely H-closed semigroup in the class of semitopological inverse semigroups with continuous inversion for any $\alpha\in\mathscr{A}$; and*

(ii) *there exists an ideal T of S which is absolutely H-closed in the class of semitopological inverse semigroups with continuous inversion such that $S_\alpha\cdot S_\beta\subseteq T$ for all $\alpha\neq\beta$, $\alpha,\beta\in\mathscr{A}$.*

Then S is an absolutely H-closed semigroup in the class of semitopological inverse semigroups with continuous inversion.

Proof. Suppose to the contrary that there exists a semitopological inverse semigroup K with continuous inversion and continuous homomorphism $h\colon S\to K$ such that the image $(S)h$ is not a closed subsemigroup of K. By Lemma II.1.10 of [27], $(S)h$ is an inverse subsemigroup of K. Since by Proposition 2.2 the closure $\operatorname{cl}_K((S)h)$ of an inverse subsemigroup $(S)h$ in a semitopological inverse semigroup K with continuous inversion is an inverse semigroup, without loss of generality we may assume that $(S)h$ is a dense proper inverse subsemigroup of K.

We observe that the assumption of the theorem states that T is an ideal of S. This implies that $(T)h$ is an ideal in $(S)h$. Then by Proposition I.1.8(*iii*) of [30] the closure of an ideal of a semitopological semigroup is again an ideal, and hence we get that $(T)h$ is a closed ideal of the semigroup K.

We fix an arbitrary $x \in K \setminus (S)h$. Then only one of the following cases holds:

a) x is an idempotent of the semigroup K;

b) x is a non-idempotent element of K.

First we show that $x \cdot y, y \cdot x \in (T)h$ for every $y \in (S)h$. We fix an arbitrary open neighbourhood $U(x)$ of the point x in the space K. Since $U(x)$ intersects infinitely many subsemigroups of K from the family $\{(S_\alpha)h \colon \alpha \in \mathscr{A}\}$ we conclude that $U(x) \cdot y \cap (T)h \neq \varnothing$ and $y \cdot U(x) \cap (T)h \neq \varnothing$ for every $y \in (S)h$. Then the separate continuity of the semigroup operation in K implies that any open neighbourhoods $W(x \cdot y)$ and $W(y \cdot x)$ of the points $x \cdot y$ and $y \cdot x$ in K, respectively, intersect the ideal $(T)h$. This implies that $x \cdot y, y \cdot x \in \mathrm{cl}_K((T)h)$. Since the ideal $(T)h$ is closed in K we conclude that $x \cdot y, y \cdot x \in (T)h$.

Suppose that case *a*) holds. Then there exists an open neighbourhood $U(x)$ of the point x in the space K such that $U(x) \cap (T)h = \varnothing$ and the neighbourhood $U(x)$ intersects infinitely many semigroups from the family $\{(S_\alpha)h \colon \alpha \in \mathscr{A}\}$. By the separate continuity of the semigroup operation in K we have that for every open neighbourhood $U(x)$ of the point x in K such that $U(x) \cap (T)h = \varnothing$ there exists an open neighbourhood $V(x)$ of x in K such that $x \cdot V(x) \subseteq U(x)$ and $V(x) \cdot x \subseteq U(x)$. Now, the previous part of proof implies that $x \cdot V(x) \cap (T)h \neq \varnothing$ and $V(x) \cdot x \cap (T)h \neq \varnothing$, which contradict the assumption $U(x) \cap (T)h = \varnothing$. The obtained contradiction implies that $E((S)h) = E(K)$.

Suppose that case *b*) holds. Then there exist idempotents e and f in $(S)h$ such that $xx^{-1} = e$ and $x^{-1}x = f$. We observe that $e, f \notin (T)h$. Indeed, if $e \in (T)h$ or $f \in (T)h$, then we have that

$$x = xx^{-1}x = ex \in (T)h \qquad \text{and} \qquad x = xx^{-1}x = xf \in (T)h,$$

because $(T)h$ is an ideal of the semigroup K. Since $x \in \mathrm{cl}_K((S)h)$, every open neighbourhood of the point x in K intersects infinitely many semigroups from the family $\{(S_\alpha)h\colon \alpha\in\mathscr{A}\}$, and hence we get that

$$(U(x)\cdot f)\cap (T)h\neq\varnothing \qquad \text{and} \qquad (e\cdot U(x))\cap (T)h\neq\varnothing.$$

Then the Hausdorffness of K and the separate continuity of the semigroup operation in K imply that $x=xx^{-1}x=x\cdot f=e\cdot x\in (T)h$. This contradicts the assumption that $x\neq (T)h$. The obtained contradiction implies the statement of our theorem. □

The proof of Theorem 4.6 is similar to the proof of Theorem 4.5.

Theorem 4.6. *Let $S=\bigcup_{\alpha\in\mathscr{A}} S_\alpha$ be a semitopological inverse semigroup with continuous inversion such that*

(i) S_α is an H-closed semigroup in the class of semitopological inverse semigroups with continuous inversion for any $\alpha\in\mathscr{A}$; and

(ii) there exists an ideal T of S which is H-closed in the class of semitopological inverse semigroups with continuous inversion such that $S_\alpha\cdot S_\beta\subseteq T$ for all $\alpha\neq\beta$, $\alpha,\beta\in\mathscr{A}$.

Then S is an H-closed semigroup in the class of semitopological inverse semigroups with continuous inversion.

Theorem 4.7. *Let a semitopological semigroup S with continuous inversion be the orthogonal sum of a family $\{S_\alpha\colon \alpha\in\mathscr{I}\}$ of semitopological inverse semigroups with zeros. Then S is an (absolutely) H-closed semigroup in the class of semitopological inverse semigroups with continuous inversion if and only if so is any element of the family $\{S_\alpha\colon \alpha\in\mathscr{I}\}$.*

Proof. First we observe that if S is a semitopological semigroup with continuous inversion then so is every semigroup from the family $\{S_\alpha\colon \alpha\in\mathscr{I}\}$.

The implication $(\Leftarrow)$ follows from Theorems 4.5 and 4.6.

First we shall prove the implication $(\Rightarrow)$ in the case of absolute H-closedness.

Suppose to the contrary that there exists an absolute H-closed semigroup S in the class of semitopological inverse semigroups with continuous inversion which is an orthogonal sum of a family $\{S_\alpha\colon \alpha\in\mathscr{I}\}$ of semitopological inverse semigroups and there exists a semigroup S_{α_0} in this family such that S_{α_0} is not absolute H-closed in the class of semitopological inverse semigroups with continuous inversion. Then there exists a semitopological inverse semigroup K with continuous inversion and continuous homomorphism $h\colon S_{\alpha_0}\to K$ such that the image $(S_{\alpha_0})h$ is not a closed subsemigroup of K. By Lemma II.1.10 of [27], $(S_{\alpha_0})h$ is an inverse subsemigroup of K. Since by Proposition 2.2 the closure $\mathrm{cl}_K((S_{\alpha_0})h)$ of an inverse subsemigroup $(S_{\alpha_0})h$ in a semitopological inverse semigroup K with continuous inversion is an inverse semigroup, without loss of generality we may assume that $(S_{\alpha_0})h$ is a dense proper inverse subsemigroup of K. Also, the semigroup K has zero because $(S_{\alpha_0})h$ contains zero.

We define a map $f\colon S \to K$ by the formula

$$(x)f = \begin{cases} 0_K, & \text{if } x \in S \setminus S^*_{\alpha_0};\\ (x)h, & \text{if } x \in S^*_{\alpha_0}, \end{cases}$$

where 0_K is zero of the semigroup K. Simple verifications show that so defined map f is a continuous homomorphism, but the image $(S)f = (S_{\alpha_0})h$ is a dense proper subsemigroup of K. This contradicts the assumption that the semigroup S is absolutely H-closed semigroup in the class of semitopological inverse semigroups with continuous inversion.

Now, we suppose that there exists an H-closed semigroup S in the class of semitopological inverse semigroups with continuous inversion which is an orthogonal sum of a family $\{S_\alpha\colon \alpha \in \mathscr{I}\}$ of semitopological inverse semigroups and there exists a semigroup S_{α_0} in this family such that S_{α_0} is not H-closed in the class of semitopological inverse semigroups with continuous inversion. Then there exists a semitopological inverse semigroup K with continuous inversion such that S_{α_0} is not a closed subsemigroup of K. Since by Proposition 2.2 the closure $\mathrm{cl}_K(S_{\alpha_0})$ of an inverse subsemigroup S_{α_0} in a semitopological inverse semigroup K with continuous inversion is an inverse semigroup, without loss of generality we may assume that S_{α_0} is a dense proper inverse subsemigroup of K.

Next, we put S' be the orthogonal sum of the family $\{S_\alpha\colon \alpha \in \mathscr{I} \setminus \{\alpha_0\}\}$ and the semigroup K. We determine a topology τ on S' in the following way.

First we observe if the orthogonal sum $T = \sum_{i\in\mathscr{J}} T_j$ is an inverse Hausdorff semitopological semigroup, then for every non-zero element $t \in T_j \subset T$ there exists an open neighbourhood $U(t)$ of t in T such that $U(t) \subseteq T^*_j$. Indeed, for every open neighbourhood $W(t) \not\ni 0$ of t in T there exists an open neighbourhood $U(t)$ of t in T such that $tt^{-1} \cdot U(t) \subseteq W(t)$. The neighbourhood $U(t)$ is requested.

We put that the bases of topologies at any point s of $S \setminus S_{\alpha_0}$ and of $S' \setminus K$ coincide in S and in S', respectively. Also the bases at any point s of subspace $K^* \subseteq S'$ coincide with the base at the point s of K^*. The following family determines the base of the topology τ at zero of the semigroup S':

$$\mathscr{B}_0 = \left\{ U \subseteq S' : \begin{array}{l} \text{there exist an element } V \text{ of the base at zero} \\ \text{of the topology of } S \text{ and an element } W \text{ of} \\ \text{the base at zero of the topology of } K \text{ such} \\ \text{that } U \cap S' \setminus K = V \cap S \setminus S_{\alpha_0},\ U \cap K = W \\ \text{and } U \cap S_{\alpha_0} = W \cap S_{\alpha_0} \end{array} \right\}.$$

Simple verifications show that (S', τ) is a Hausdorff semitopological inverse semigroup with continuous inversion and moreover S is a dense proper inverse subsemigroup of (S', τ), which contradicts the assumption of our theorem. The obtained contradiction implies the statement of the theorem. □

Theorem 4.7 implies the following corollary:

Corollary 4.8. *A primitive Hausdorff semitopological inverse semigroup S is (absolutely) H-closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is every its maximal subgroup G with adjoined zero with an induced topology from S.*

Remark 4.9. We observe that the statements of Theorems 4.5, 4.6 and 4.7 hold for H-closed and absolute H-closed semitopological semilattices in the class of semitopological semilattices.

Theorem 4.10. *An infinite semitopological semigroup of $\lambda \times \lambda$-matrix units B_λ id H-closed in the class of semitopological semigroups if and only if the space B_λ is compact.*

Proof. Implication ($\Leftarrow$) is trivial.

($\Rightarrow$). Suppose to the contrary that there exists a Hausdorff non-compact topology τ_B on the semigroup B_λ such that (B_λ, τ_B) is an H-closed semigroup in the class of semitopological semigroups. By Lemma 2 of [18] every non-zero element of B_λ is an isolated point in (B_λ, τ_B). Then there exists an infinite open-and-closed subset $A \subseteq B_\lambda \setminus \{0\}$.

Then we have that at least one of the following cases holds:

1) there exist finitely many $i_1, \ldots, i_n \in \lambda$ such that if $(i, j) \in A$ then $i \in \{i_1, \ldots, i_n\}$;
2) there exist finitely many $j_1, \ldots, j_n \in \lambda$ such that if $(i, j) \in A$ then $i \in \{j_1, \ldots, j_n\}$;
3) cases 1) and 2) don't hold.

Suppose case 1) holds. Then there exists an element $i_0 \in \{i_1, \ldots, i_n\}$ such that the set $\{(i_0, j) \colon j \in \lambda\} \cap A$ is infinite. We denote $A_{i_0} = \{(i_0, j) \in B_\lambda \colon (i_0, j) \in A\}$. It is obvious that A_{i_0} is infinite subset of the semigroup B_λ. By Lemma 2 of [18] every non-zero element of B_λ is an isolated point in (B_λ, τ_B) and hence A_{i_0} is an open-and-closed subset in the topological space (B_λ, τ_B). Since the left shift $l_{(i_0,i)} \colon B_\lambda \to B_\lambda \colon x \mapsto (i_0, i) \cdot x$ is a continuous map for any $i \in \lambda$, $A_i = \{(i, j) \in B_\lambda \colon (i_0, j) \in A\}$ is an infinite open-and-closed subset in (B_λ, τ_B) for every $i \in \lambda$. This implies that the set $B_\lambda \setminus \{A_{i_1} \cup \cdots \cup A_{i_k}\}$ is an open neighbourhood of the zero in (B_λ, τ_B) for every finite subset $\{i_1, \ldots, i_k\} \subset \lambda$.

Now. for every $i \in \lambda$ we put $a_i \notin B_\lambda$. We extend the semigroup operation from B_λ onto the set $S = B_\lambda \cup \{a_i \colon i \in \lambda\}$ in the following way:

(i) $a_i \cdot a_j = a_i \cdot 0 = 0 \cdot a_i = 0$ for all $i, j \in \lambda$;

(ii) $(s, p) \cdot a_i = \begin{cases} a_s, & \text{if } p = i; \\ 0, & \text{if } p \neq i \end{cases}$ for all $(s, p) \in B_\lambda \setminus \{0\}$ and $i \in \lambda$;

(iii) $a_i \cdot (s, p) = 0$ for all $(s, p) \in B_\lambda \setminus \{0\}$ and $i \in \lambda$.

Simple verifications show that so defines binary operation on S is associative, and hence S is a semigroup.

Next, we define a topology τ_S on the semigroup S in the following way. For every element $x \in B_\lambda$ we put that bases of topologies τ_B and τ_S at the point x coincide. Also, for every $i \in \lambda$ we put

$$\mathscr{B}_S(a_i) = \{\{a_i\} \cup C_i \colon C_i \text{ is a cofinite subset of } A_i\}$$

is a base of the topology τ_S at the point $a_i \in S$. It is obvious that (S, τ_S) is a Hausdorff topological space. The separate continuity of the semigroup operation in (S, τ_S) follows from the cofinality of the set C_i in A_i for each $i \in \lambda$. Therefore we get that the semitopological semigroup (B_λ, τ_B) is a dense proper subsemigroup of (S, τ_S), which contradicts the assumption of the theorem.

In case 2) the proof is similar.

Suppose that cases 1) and 2) don't hold. By induction we construct an infinite sequence $\{(x_i, y_i)\}_{i \in \mathbb{N}}$ in B_λ in the following way. First we fix an arbitrary element $(x, y) \in A$ and denote $(x_1, y_1) = (x, y)$. Suppose that for some positive integer n we construct the finite sequence $\{(x_i, y_i)\}_{i=1,\ldots,n}$. Since the set A is infinite and cases 1) and 2) don't hold, there exists $(x, y) \in A$ such that $x \notin \{y_1, \ldots, y_n\}$ and $y \notin \{x_1, \ldots, x_n\}$. Then we put $(x_{n+1}, y_{n+1}) = (x, y)$.

Let $a \notin B_\lambda$. We put $T = B_\lambda \cup \{a\}$ and extend the semigroup operation from B_λ onto T in the following way:

$$a \cdot x = x \cdot a = a \cdot a = 0, \quad \text{for every } x \in B_\lambda.$$

Next, we define a topology τ_T on the semigroup T in the following way. For every element $x \in B_\lambda$ we put that bases of topologies τ_B and τ_T at the point x coincide. Also, we put

$$\mathscr{B}_T(a) = \{\{a\} \cup C \colon C \text{ is a cofinite subset of the set } \{(x_i, y_i) \colon i \in \mathbb{N}\}\}$$

is a base of the topology τ_T at the point $a \in T$. It is obvious that (T, τ_T) is a Hausdorff topological space, the semigroup operation in (T, τ_T) is separately continuous, and B_λ is a dense subsemigroup of (T, τ_T). This contradicts the assumption of the theorem.

The obtained contradictions imply the statement of our theorem. $\square$

Remark 4.11. By Theorem 2 [18] for every infinite cardinal λ there exists a unique Hausdorff pseudocompact topology τ_c on the semigroup B_λ such that (B_λ, τ_c) is a semitopological semigroup. This topology is compact and it is described in Example 1 of [18].

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Contact information

O. Gutik

Faculty of Mechanics and Mathematics, National University of Lviv, Universytetska 1, Lviv, 79000, Ukraine

E-Mail: o_gutik@franko.lviv.ua, ovgutik@yahoo.com

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