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On subgroups of finite exponent in groups

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Abstract. We investigate properties of groups with subgroups of finite exponent and prove that a non-perfect group G of infinite exponent with all proper subgroups of finite exponent has the following properties:

(1) G is an indecomposable p-group,
(2) if the derived subgroup G' is non-perfect, then G/G'' is a group of Heineken-Mohamed type.

We also prove that a non-perfect indecomposable group G with the non-perfect locally nilpotent derived subgroup G' is a locally finite p-group.

1. Introduction

A group G is called *locally graded* if every its non-trivial finitely generated subgroup contains a proper subgroup of finite index. If the derived subgroup G' is proper in G, then G is called *non-perfect*, and is called *perfect* otherwise. Recall that a group with the maximal condition on subgroups is called *Noetherian.* An infinite group with all proper quotients to be finite is called *just infinite* (see e.g. [7] and [13]). If A and B are subgroups of G and $A \lhd B$, then the quotient B/A is *a section* of G. If any non-trivial section of G is non-perfect, then G is called *absolutely imperfect.*

We prove the following

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Proposition 1.1. *A group G of finite exponent satisfies the following properties:*

(1) *if G is a locally graded group, then it is finite or non-simple locally finite,*

(2) *if G is an absolutely imperfect group, then it is locally finite.*

Recall that a group G in which any two proper subgroups generate a proper subgroup is called *indecomposable*.

Proposition 1.2. *Let G be a non-perfect indecomposable group. If the derived subgroup G' of it is a non-perfect locally nilpotent (in particular, hypercentral) group, then G is a locally finite p-group.*

A. Arikan and H. Smith [1] have investigated groups with all proper subgroups of finite exponent and, in particular, have proved that a non-perfect group of infinite exponent with proper subgroups of finite exponent is countable and semi-radicable (i.e., $G = G^n$ for any positive integer n). Our next result is

Theorem 1.3. *Let G be a non-perfect group of infinite exponent group with all proper subgroups of finite exponent. Then G has the following properties:*

(1) *G is an indecomposable p-group,*

(2) *if the derived subgroup G' is non-perfect, then G/G'' is a group of Heineken-Mohamed type.*

Remember that a group with all proper subgroups to be nilpotent and subnormal is called *a group of Heineken-Mohamed type* [8]. Any group of Heineken-Mohamed type is indecomposable and absolutely imperfect.

Throughout this paper p will always denote a prime, $\mathbb{C}_{p^\infty}$ the quasi-cyclic p-group. For a group G, G', G'' will indicate the terms of derived series of G and G^n the subgroup of G generated by the nth powers of all elements in G, $G^{\mathcal{F}}$ the finite residual of G (i.e., the intersection of all normal subgroups of finite index in G).

Any unexplained terminology is standard as in [10] and [11].

2. Preliminary results

A group G with an descending chain $\{H_n\}_{n=1}^{\infty}$ of normal subgroups H_n of finite index in G such that

$$\bigcap_{n=1}^{\infty} H_n = 1$$

is called *residually finite*. From the solution of the restricted Burnside problem it follows the following

Theorem A. *A residually finite group of finite exponent is finite.*

If $H \neq 1$ is a non-trivial normal subgroup of G, then the quotient group G/H is called *proper*. If every proper quotient group of G is non-perfect, then we say that G is *imperfect*.

Lemma 2.1. *Let G be an imperfect group. If all its proper normal subgroups are locally finite and all its proper quotient groups are of finite exponent, then G is a locally finite group.*

Proof. By H_0 we denote the subgroup of G generated by all its proper normal subgroups. Then H_0 is locally finite. If $H_0 \neq G$, then

$$(G/H_0)' \neq G/H_0$$

and therefore $G' \leqslant H_0$. Since G/H_0 is a simple abelian group, we deduce that G is a locally finite group. □

Lemma 2.2. *Let G be a finitely generated just infinite group without non-trivial abelian subnormal subgroups. If $G^{\mathcal{F}}$ not contains proper subgroups of finite index, then it is a finite direct product of simple groups.*

Proof. By Corollary 4.5 of [13], every subnormal subgroup S of G such that

$$S \leqslant G^{\mathcal{F}}$$

is a direct factor of a subnormal subgroup of finite index in G. This gives that

$$G^{\mathcal{F}} = S \times D$$

for some $D \lhd S^G$ and therefore $S \lhd G$. As a consequence, $G^{\mathcal{F}}$ is a T-group (i.e., normality of subgroups in $G^{\mathcal{F}}$ is a transitive relation). By Theorem 5.2 of [12], $G^{\mathcal{F}}$ is a direct product of finitely many simple groups. □

Lemma 2.3. *If G is a finitely generated (respectively Noetherian) group of finite exponent, then it has a simple section (respectively a simple homomorphic image) or is finite.*

Proof. Suppose that G is infinite. By Proposition 3 of [7], G has a just infinite homomorphic image B. By Corollary 3.8 of [13], B has no non-trivial finite subnormal subgroups and so (as a torsion group) it not contains non-trivial abelian subnormal subgroups. Assume that B is not simple. Then

$$B^{\mathcal{F}} \neq B.$$

If $B^{\mathcal{F}} = 1$, then B is a residually finite group and, by Theorem A, B is locally finite (and therefore finite) group, a contradiction. Hence $B^{\mathcal{F}}$ is non-trivial and so it not contains proper subgroups of finite index. The rest it follows from Lemma 2.2. □

Corollary 2.4. *A residually finite Noetherian group of finite exponent is finite.*

Corollary 2.5. *An absolutely imperfect finitely generated group of finite exponent is finite.*

Lemma 2.6 ([5, Lemma 4])**.** *Every simple locally finite group of finite exponent is finite.*

Corollary 2.7. *Let G be a locally finite group, H the subgroup generated by all proper normal subgroups of G. If G is of finite exponent, then it is finite or G is non-simple and H is a subgroup of finite index in G.*

Proof. Indeed, if $H = 1$, then G is finite by Lemma 2.7. Assume that H is non-trivial. If H is proper in G, then the quotient group G/H is simple and consequently finite by Lemma 2.6. □

Proof of Proposition 1.1. Let H be any finitely generated subgroup of G.

a) Assume that G is a locally graded group. Then H contains a proper subgroup of finite index and so $H^{\mathcal{F}}$ is a proper subgroups in H. Since the quotient group $H/H^{\mathcal{F}}$ is residually finite, it is locally finite (and therefore finite) by Theorem A. The subgroup $H^{\mathcal{F}}$ is finitely generated and therefore it contains a non-trivial subgroup of finite index (that leads to a contradiction) or H is finite in view of Theorem A. Thus G is a locally finite group. From Corollary 2.7 it holds that G is finite or non-simple.

b) If G is absolutely imperfect, then the assertion holds in view of Lemma 2.3 and Corollary 2.5. □

Lemma 2.8. *Let G be a residually finite group. Then G contains an infinite abelian subgroup if and only if it has an infinite subgroup of finite exponent.*

Proof. ($\Rightarrow$) By contrary. Assume that G has an infinite abelian subgroup A and every subgroup of finite exponent is finite in G. Let B be a basic subgroup of A (see [6, §33]). Since B is a direct product of cyclic subgroups and $B_1 = \{b \in B \mid b^p = 1\}$ is finite, we deduce that B is finite and, by Theorem 27.5 of [6],

$$A = B \times D$$

is a direct product, where D is a divisible group. In view of the residually finiteness, $D = 1$, a contradiction.

($\Leftarrow$) Let H be an infinite subgroup of finite index in G. By Theorem A, H is locally finite and, by the Kargapolov-Ph. Hall-Kulatilika Theorem (see e.g. [11, Theorem 14.3.7]), it contains an infinite abelian subgroup. □

A quasicyclic 2-group $\mathbb{C}_{2^\infty}$ is an abelian group of infinite exponent with finite proper subgroups of finite exponent. As was proved by O. Kegel (see e.g. [11, Exercises 14.4(4)]), a non-abelian 2-group of infinite exponent contains an infinite abelian subgroups (and so a non-abelian 2-group of infinite exponent contains an infinite subgroup of finite exponent). For infinite p-groups ($p > 2$) of infinite exponent a problem of the existence of an infinite subgroup of finite exponent is open.

Problem 2.9. *Is there a group (respectively a p-group or a finitely generated p-group) of infinite exponent with all proper subgroups of finite exponent to be finite?*

3. On groups with proper subgroups of finite exponent

Lemma 3.1 (see [9, Lemma 1.D.4]). *If K is a normal subgroup of the locally finite group such that the quotient group G/K is a countable p-group for some prime p, then there is a p-subgroup P of G with $KP = G$.*

Lemma 3.2 (see [4, Lemma 2.3]). *Let G be a torsion abelian group and $M \neq 0$ be a $\mathbb{Z}[G]$-module which is torsion-free as a group. Then, for any finite set Π of primes, there is a $\mathbb{Z}[G]$-submodule N of M such that the quotient module M/N is torsion as a group and, for all $p \in \Pi$, contains an element of degree p..*

Proof of Proposition 1.2. By Lemma 1 of [2], $G/G' \cong \mathbb{C}_{p^\infty}$ is a quasicyclic p-group for some prime p. Assume that G is not torsion. Without loss of generality suppose that $G'' = 1$. Since the torsion part $\tau(G')$ of the derived subgroup G' is normal in G, we can assume that G' is abelian torsion-free. Let q be a prime and $p \neq q$. Then G' is a

$\mathbb{Z}[G/G']$-module and, by Lemma 3.2, there is a G-invariant subgroup N of G' such that G'/N is a torsion group with a non-trivial p-element. By Lemma 3.1, there exists a p-subgroup $P \leqslant G$ such that

$$G = G'P.$$

Then, by Lemma 3.3, $G = P$, a contradiction. Hence G is a torsion group and therefore a p-group. □

Lemma 3.3. *Let G be a group with every subgroup to be of finite exponent. Then the following hold:*

(1) *if G is of infinite exponent, then*
   (a) *G is perfect, or*
   (b) *G is a non-perfect indecomposable group and its derived subgroup G' not contains proper G-invariant subgroups of finite index,*

(2) *if G is a finitely generated group of infinite exponent, then it is perfect.*

Proof. It is easy to see that G is a torsion group. Suppose that G is a non-perfect group of infinite exponent. Then G/G' is an indecomposable group and, by Lemma 1 of [2], it is a quasicyclic p-group for some prime p. If $G = \langle A, B\rangle$ for some its proper subgroups A, B of finite exponent, then

$$\overline{G} = G/G' = \overline{A} \cdot \overline{B},$$

where $\overline{A}$ and $\overline{B}$ are homomorphic images of A and B respectively. Then we obtain, for example, that $\overline{G} = \overline{B}$. This means that $G = G'B = B$, a contradiction. Hence G is indecomposable.

If H is a G-invariant subgroup of finite index in G', then the quotient group $B = G/H$ has a finite derived subgroup B'. Inasmuch $B' \leqslant Z(B)$, we obtain a contradiction. □

Problem 3.4. *Is there a finitely generated simple group (respectively p-group) of infinite exponent with all proper subgroups of finite exponent?*

Proof of Theorem 1.3. a) Indeed, G is indecomposable by Lemma 3.3 and the quotient group G/G' is a countable group. By Lemma 3.1, there exists a p-subgroup $P \leqslant G$ such that

$$G = G'P.$$

Then, by Lemma 3.3, $G = P$.

b) As proved in (a), G is a p-group. Assume that $G'' = 1$. If K is any proper subgroup of G, then $G'K$ is also proper in G. Since all extensions of a nilpotent p-group of finite exponent by a finite p-group are nilpotent [3],

G is a nilpotent p-group. This means that K is a nilpotent subnormal subgroup of G. Hence G is a Heineken-Mohamed type group. □

Corollary 3.5. *Let G be a non-perfect group of infinite exponent. Then its every proper subgroup is of finite exponent if and only if G is an indecomposable p-group with the derived subgroup G' of finite exponent.*

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