UDC 517.562

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# ON TIME DEPENDENT ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

# ПРО НЕСТАЦІОНАРНІ ОРТОГОНАЛЬНІ ПОЛІНОМИ НА ОДИНИЧНОМУ КОЛІ

Two index formulas for operators defined by infinite band matrices are proved. These results may be interpreted as a generalization of the classical theorem of M. G. Krein for orthogonal polynomials. The proofs are based on dichotomy and nonstationary inertia theory.

Доведено дві формули індексу для операторів, визначених матрицями нескінченного порядку. Ці результати можна інтерпретувати як узагальнення класичної теореми М. Г. Крейна про ортогональні поліноми. Доведення базується на дихотомії та нестаціонарній теорії інерції.

**1. Introduction.** In this paper, we prove the following two theorems.

**Theorem 1.** *Let* $R = (R_{ij})_{ij=0}^{\infty}$ *be a self-adjoint block matrix with blocks* $R_{ij}$ *of order* $r$ *such that the submatrices* $L_n = (R_{ij})_{ij=n}^{n+m}$, $n = 0, 1, \dots$, *are invertible with*

$$\sup_{n=0,1,\dots} (\| L_n \|, \| L_n^{-1} \|) < \infty, \tag{1}$$

*and*

$$(L_n^{-1})_{m,m} \geq \varepsilon I_r, \quad n = 0, 1, \dots, \tag{2}$$

*for some positive number* $\varepsilon$, *where* $(L_n^{-1})_{m,m}$ *is the* $r \times r$ *block in the right lower corner of* $L_n^{-1}$. *For each* $n = 0, 1, \dots$, *let* $a_{n,n}, \dots, a_{n+m,n}$ *be* $m+1$ *matrices of the order* $r$, *which solve the following system of matrix equations:*

$$\sum_{k=0}^{m} R_{n+h,\,n+k} a_{n+k,\,n} = \delta_{h,m} I_r, \quad h = 0, \dots, m, \tag{3}$$

*and let* $a_{ij} = 0$ *if* $j > i \geq 0$ *or* $i - m > j \geq 0$. *Then the operator* $G = (a_{ij})_{ij=0}^{\infty}$ *determines a Fredholm operator in* $l_r^2$, *and*

$$\operatorname{index} G = -\nu_+ ((R_{ij})_{ij=n}^{n+m-1}), \tag{4}$$

*for sufficiently large* $n$.

In this theorem, the following notation is used. For a finite self-adjoint matrix $A$, we denote by $\nu_+(A)$ the number of positive eigenvalues of $A$, counting multiplicities. We also denote by $l_r^2$ the Hilbert space of all square summable sequences $(x_n)_{n=0}^{\infty}$ with $x_n \in \mathbb{C}^r$, $n = 0, 1, \dots$, and let $l_r^2(\mathbb{Z})$ be the Hilbert space of all square summable sequences $(x_n)_{n=-\infty}^{\infty}$ with $x_n \in \mathbb{C}^r$, $n = 0, \pm 1, \dots$.

In this paper, we also prove the following result for the bilateral case.

**Theorem 2.** *Let* $R = (R_{ij})_{ij=-\infty}^{\infty}$ *be a self-adjoint block matrix with blocks* $R_{ij}$ *of the order* $r$ *such that the submatrices* $L_n = (R_{ij})_{ij=n}^{n+m}$, $n = 0, \pm 1, \dots$, *are invertible with*

$$\sup_{n=0,\pm 1,\dots} (\| L_n \|, \| L_n^{-1} \|) < \infty,$$

*and*

$$(L_n^{-1})_{m,m} \geq \varepsilon I_r, \quad n = 0, \pm 1, \dots,$$

*for some positive number* $\varepsilon$, *where* $(L_n^{-1})_{m,m}$ *is the* $r \times r$ *block in the right lower corner of* $L_n^{-1}$. *For each* $n = 0, \pm 1, \dots$, *let* $a_{n,n}, \dots, a_{n+m,n}$ *be* $m+1$ *matrices of the order* $r$, *which solve the following system of matrix equations:*

$$\sum_{k=0}^{m} R_{n+h,\, n+k} a_{n+k\, n} = \delta_{h,m} I_r, \quad h = 0, \dots, m,$$

*and let* $a_{ij} = 0$ *if* $i < j$ *or* $i > j + m$. *Then the matrix* $A = (a_{ij})_{ij=-\infty}^{\infty}$ *defines a Fredholm operator in* $l_r^2(\mathbb{Z})$. *Moreover,* $\operatorname{Ker} A = \{0\}$ *and there exist nonnegative integers* $N, p$, *and* $q$ *such that*

$$\nu_+((R_{ij})_{ij=n}^{n+m-1}) = q, \quad \nu_+((R_{ij})_{ij=-n}^{-n+m-1}) = p, \quad n = N, N+1, \dots,$$

*and*

$$\operatorname{index} A = p - q.$$

These two results are a generalization of a classical theorem of M. G. Krein on the location of zeros of the orthogonal polynomials on the unit circle $\mathbb{T}$. To show this, assume that $\mu$ is a bounded real measure on the unit circle. Denote by

$$R_n = \int_{\mathbf{T}} e^{-in\theta} d\mu, \quad n = 0, \pm 1, \dots,$$

the Fourier coefficients of $\mu$, and assume that $(R_{i-j})_{ij=0}^{m}$ is invertible for some positive integer $m$. Let $a_0, \dots, a_m$ be complex numbers solving the following system of equations

$$\sum_{k=0}^{m} R_{h-k} a_k = \delta_{h,m}, \quad h = 0, \dots, m.$$

Define a polynomial $p_m$ by

$$p_m(z) = \sum_{k=0}^{m} a_k z^k.$$

Then the equalities

$$\int_{\mathbf{T}} p_m(e^{i\theta}) \overline{e^{i\theta h}} \, d\mu = \sum_{k=0}^{m} R_{h-k} a_k = \delta_{h,m}, \quad h = 0, \dots, m,$$

indicate that $p_m(z)$ is the $m$th orthogonal polynomial relative to $\mu$.

Denote $L = (R_{i-j})_{ij=0}^{m}$ and $L' = (R_{i-j})_{ij=0}^{m-1}$. Assume that both $L$ and $L'$ are invertible. We set $L^{-1} = (\gamma_{i,j})_{ij=0}^{m}$ and note that since $\gamma_{m,m} = \det L' / \det L$, $\gamma_{m,m}$ is nonzero. First, assume that $\gamma_{m,m} > 0$. In this case, we may apply Theorem 1 to the matrix $R = (R_{i-j})_{ij=0}^{\infty}$. It follows that the Toeplitz operator $G = (a_{i-j})_{ij=0}^{\infty}$ is a Fredholm operator and the index $G = -\nu_+(L')$. Here, $a_0, \dots, a_m$ are defined as above and $a_k = 0$ for $k < 0$ and $k > m$. Since $G$ is a Toeplitz operator whose symbol is $p_m(z)$, we find that $p_m(z)$ does not vanish on the unit circle and the number of zeros of $p_m(z)$ inside the unit disc is equal to $\nu_+(L')$. On the other hand, in the case where $\gamma_{m,m} < 0$, we can apply the previous consideration to the measure $-\mu$. In this case, we also find that $p_m(z)$ does not vanish on the unit circle and the number of zeros of $p_m(z)$ inside the unit disc is equal to $\nu_+(-L')$. Interpreting the inertia of $L' = (R_{i-j})_{ij=0}^{m-1}$ in terms of the Jacobi rule, we obtain the Krein's result, presented below under the stronger hypothesis used in [11].

**Theorem** (M. G. Krein [11]). *Let* $T = (R_{i-j})_{ij=0}^{m}$ *be a self-adjoint Toeplitz matrix such that* $\det(R_{i-j})_{ij=0}^{k} \neq 0$, $k = 0, \dots, m$. *Denote* $d_k = \det(R_{i-j})_{ij=0}^{k}$. *Let* $P$ *and* $V$ *be, respectively, the number of permanences and the number of variations of sign in the sequence* $1, d_0, d_1, \dots, d_{m-1}$, *and let* $p(z) = = a_m z^m + \dots + a_0$ *be the polynomial whose coefficients satisfy*

$$T\begin{pmatrix} a_0 \\ \vdots \\ a_{m-1} \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

*Then* $p(z)$ *does not vanish on the unit circle. Moreover, if* $d_m d_{m-1} > 0$ *(respectively,* $d_m d_{m-1} < 0$, *), then* $p(z)$ *has* $P$ *(respectively,* $V$*) zeros inside the unit disc and* $V$ *(respectively,* $P$*) zeros outside the unit disc, counting multiplicities.*

The complete proofs and the further refinements of Krein's theorem can be found in [1, 8, 10], as well as the other related results.

Theorems 1 and 2 given above are also the generalization of Theorem 1.1 in [4], which is stated below.

**Theorem.** Let $R = (R_{ij})_{ij=-\infty}^{\infty}$ *be a self-adjoint block matrix whose entries* $R_{ij}$ *are* $r \times r$ *complex matrices with the following properties:*

(i) $R_{ij} = 0$ *if* $|i-j| > m$, *where* $m$ *is a positive integer, and* $\sup_{ij}\| R_{ij}\| < \infty$;

(ii) *the matrices* $(R_{ij})_{ij=n}^{n+m}$ and $(R_{ij})_{ij=n}^{n+m-1}$, $n = 0, \pm 1, \dots$, *are invertible and*

$$\sup_n \left(\| [(R_{ij})_{ij=n}^{n+m}]^{-1} \|, \; \| [(R_{ij})_{ij=n}^{n+m-1}]^{-1} \|\right) < \infty; \tag{5}$$

(iii) *the number of negative eigenvalues of the matrices* $(R_{ij})_{ij=n}^{n+m-1}$, $n = 0, \pm 1, \dots$, *does not depend on* $n$ *and*

$$([(R_{ij})_{ij=n}^{n+m}]^{-1})_{m,m} > 0, \quad n = 0, \pm 1, \dots,$$

*where* $([(R_{ij})_{ij=n}^{n+m}]^{-1})_{m,m}$ *is the* $r \times r$ *block in the right lower corner of* $[(R_{ij})_{ij=n}^{n+m}]^{-1}$.

*For every integer* $n$, *let* $a_{n,n}, \dots, a_{n+m,n}$ *be* $m+1$ *matrices of the order* $r$ *solving the following system of matrix equations:*

$$\sum_{k=0}^{m} R_{n+h,n+k} a_{n+k,n} = \delta_{0,h} I_r, \quad h = 0, \dots, m, \tag{6}$$

*and let* $a_{ij} = 0$ *if* $i < j$ *or* $i > j+m$.

*Then the matrix* $A = (a_{ij})_{ij=-\infty}^{\infty}$ *defines an invertible operator in* $l_r^2(\mathbb{Z})$, *the matrix* $G = (a_{ij})_{ij=0}^{\infty}$ *defines a Fredholm operator in* $l_r^2$, *and the index of* $G$ *is* *equal to the negative of the number of negative eigenvalues of* $(R_{ij})_{ij=n}^{n+m-1}$ *for any* $n = 0, \pm 1, \dots$, *counting multiplicities.*

This assertion differs from Theorems 1 and 2 by the assumption that the inertia of the matrices $(R_{ij})_{ij=n}^{n+m-1}$ does not depend on $n$ and the fact that we consider the first column of the matrices inverse to $(R_{ij})_{ij=n}^{n+m}$ (in Theorems 1 and 2, we do not impose any restrictions on the inertia of $(R_{ij})_{ij=n}^{n+m-1}$ and deal with the last columns of the matrices inverse to $(R_{ij})_{ij=n}^{n+m}$). The proofs of Theorems 1 and 2 do not coincide with the proof of this assertion in [4] but exploit the same ideas.

Let us explain how this assertion can be deduced from Theorems 1 and 2. We set $M_n = (R_{ij})_{ij=n}^{n+m}$ and $K_n = (R_{ij})_{ij=n}^{n+m-1}$, $n = 0, \pm 1, \dots$. Since $M_n$ and $K_{n+1}$ are invertible, it follows from the Schur factorization of the matrix

$$M_n = \begin{pmatrix} R_{n,n} & * \\ * & K_{n+1} \end{pmatrix}$$

that $(M_n^{-1})_{0,0}$ is invertible as well. Furthermore, inequality (5) results in

$$\sup_n \| [(M_n^{-1})_{0,0}]^{-1} \|) < \infty. \tag{7}$$

In addition, the Schur factorization also implies that

$$\nu_+(K_n) + \nu_+((M_n^{-1})_{m,m}) = \nu_+(M_n) = \nu_+((M_n^{-1})_{0,0}) + \nu_+(K_{n+1}). \tag{8}$$

To show this, we note that, for an invertible block self-adjoint matrix $\Lambda = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, where $A$ is invertible, the factorization

$$\Lambda = \begin{pmatrix} I & 0 \\ B^* A^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^* A^{-1} B \end{pmatrix} \begin{pmatrix} I & A^{-1} B \\ 0 & I \end{pmatrix}$$

and the Sylvester law of inertia yield $\nu_+(\Lambda) = \nu_+(A) + \nu_+(C - B^* A^{-1} B)$. Denoting $\Lambda^{-1} = \begin{pmatrix} A_1 & B_1 \\ B_1^* & C_1 \end{pmatrix}$, we obtain $C_1^{-1} = C - B^* A^{-1} B$, and hence,

$$\nu_+(\Lambda) = \nu_+(A) + \nu_+(C_1). \tag{9}$$

Similarly, if $C$ is invertible, we get

$$\nu_+(\Lambda) = \nu_+(A_1) + \nu_+(C). \tag{10}$$

By using (9) with $\Lambda = M_n$, $A = K_n$, and $C_1 = (M_n^{-1})_{m,m}$, we obtain the first equality in (8), while (10) with $\Lambda = M_n$, $A_1 = (M_n^{-1})_{0,0}$, and $C = K_{n+1}$ implies the second equality in (8).

By condition (iii), we have $\nu_+(K_n) = \nu_+(K_{n+1})$, and $(M_n^{-1})_{m,m} > 0$. Hence, (8) leads to $(M_n^{-1})_{0,0} > 0$, and therefore, inequality (7) implies

$$(M_n^{-1})_{0,0} \geq \varepsilon I_r, \quad n = 0, \pm 1, \dots, \tag{11}$$

for some positive number $\varepsilon$.

By replacing the indices $n, h$, and $k$ in equality (6) by $-n-m$, $m-h$, and $m-k$, respectively, we obtain

$$\sum_{k=0}^{m} R_{-n-h,-n-k}a_{-n-k,-n-m} = \delta_{m,h}I_r, \quad h = 0, \dots, m.$$

We now set

$$R'_{ij} = R_{-i-j}, \quad a'_{ij} = a_{-i,-j-m}, \quad i,j = 0, \pm 1, \dots .$$

Then the equalities given above imply that

$$\sum_{k=0}^{m} R'_{n+h,n+k}\, a'_{n+k,n} = \delta_{m,h}I_r, \quad h = 0, \dots, m. \tag{12}$$

Let us apply Theorems 1 and 2 to these equalities. First, we note that if either $i < j$ or $i > j+m$, then $-i > -j-m+m$ or $-i < -j-m$, and therefore, $a'_{i,j} = a_{-i,-j-m} = 0$. Denote $L'_n = (R'_{ij})_{ij=n}^{n+m}$. Then

$$L'_n = (R_{-i,-j})_{ij=n}^{n+m} = J\,[(R_{ij})_{ij=-n-m}^{-n}]\,J = J M_{-n-m} J,$$

where $J = (\delta_{i,m-j} I_r)_{ij=0}^{m}$. This equality and condition (ii) imply that $L'_n$ is invertible with $\sup_n (\| L'_n \|, \| L'^{-1}_n \|) < \infty$. In addition, $L'^{-1}_n = J M^{-1}_{-n-m} J$, whence, by virtue of (11), $(L'^{-1}_n)_{m,m} = (M^{-1}_{-n-m})_{0,0} \geq \varepsilon I_r$. Thus, Theorems 1 and 2 are applicable to the matrices $(R'_{ij})_{ij=0}^{\infty}$ and $(R'_{ij})_{ij=-\infty}^{\infty}$ and equalities (12). Note that, by condition (iii), there exists an integer $\nu_+$ such that

$$\nu_+\left((R_{ij})_{ij=n}^{n+m-1}\right) = \nu_+, \quad n = 0, \pm 1, \dots . \tag{13}$$

Then

$$\nu_+\left((R'_{ij})_{ij=n}^{n+m-1}\right) = \nu_+\left((R_{-i,-j})_{ij=n}^{n+m-1}\right) = \nu_+\left((R_{i,j})_{ij=-n-m+1}^{-n}\right), \; n = 0, \pm 1, \dots .$$

Consequently, Theorem 1 guarantees that the matrix $G' = (a'_{ij})_{ij=0}^{\infty}$ determines a Fredholm operator in $l_r^2$ with

$$\operatorname{index} G' = -\nu_+.$$

Similarly, Theorem 2 shows that the matrix $A' = (a'_{ij})_{ij=-\infty}^{\infty}$ defines a Fredholm operator in $l_r^2(\mathbb{Z})$ with $\operatorname{Ker} A' = \{0\}$ and the index $A' = 0$. Thus, $A'$ is invertible. The invertibility of $A' = (a'_{ij})_{ij=-\infty}^{\infty} = (a_{-i,-j-m})_{ij=-\infty}^{\infty}$ implies that $(a_{i,j-m})_{ij=-\infty}^{\infty}$ is invertible, too. Hence, $A = (a_{ij})_{ij=-\infty}^{\infty}$ is also invertible.

Denote $G'' = (a'_{ij})_{ij=-\infty}^{-1}$. Then, $G'' \oplus G'$ is equal to a finite-dimensional perturbation of $A'$. Since $A'$ is invertible and $G'$ is a Fredholm operator whose index is equal to $-\nu_+$, we conclude that $G''$ is a Fredholm operator whose index is equal to $\nu_+$. By rearranging the indices, we find that $(a'_{-i,-j})_{ij=1}^{\infty} = (a_{i,j-m})_{ij=1}^{\infty}$ is a Fredholm operator with the index $\nu_+$. Hence, the operator $G''' = (a_{i,j-m})_{ij=0}^{\infty}$ is also a Fredholm operator with the index $\nu_+$. Denote by $S = (\delta_{i,j+1} I_r)_{ij=0}^{\infty}$ the block forward shift. Then, $G = G''' S^m$, and therefore, $G$ is a Fredholm operator with the index $G = \nu_+ + m(\operatorname{index} S) = \nu_+ - mr$. On the other hand, it follows form (13) that $\nu_+ - mr$

is equal to the negative of the number of negative eigenvalues of $(R_{ij})_{ij=n}^{n+m-1}$ for any $n = 0, \pm 1, \ldots$, counting multiplicities. This implies the required result.

The proofs of Theorems 1 and 2 are obtained by using the general theorems on block weighted shifts, dichotomy, and the nonstationary Stein inequalities. Preliminary results appear in Section 2. The proof of Theorem 1 is given in Section 3, where we use a special construction of the nonstationary Stein inequalities. In Section 4, we present the proof of Theorem 2.

**2. Preliminary Results for Band Operators. Dichotomy. Nonstationary Stein Inequalities.** In this section, we present some general results for the band operators and focus our attention on their connection with the nonstationary Stein inequalities. These results are used in the next section when proving Theorem 1.

We first consider Lemma 3.1 in [3] that guarantees the linearization of the band operators. For convenience, we formulate this lemma below.

Let $G_1$ and $G_2$ be bounded operators in the Hilbert spaces $H_1$ and $H_2$. We say that $G_1$ and $G_2$ are *equivalent* if $S_1 G_1 = G_2 S_2$, where $S_1, S_2 : H_1 \to H_2$ are bounded one-to-one operators mapping $H_1$ onto $H_2$.

**Lemma 1.** *Let $G = (a_{ij})_{ij=0}^{\infty}$ be a bounded operator in $l_r^2$, whose entries $a_{ij}$ are $r \times r$ matrices with $a_{ij} = 0$ if $i > j+m$ of $i < j$. Define two sequences $(A_n)_{n=0}^{\infty}$ and $(B_n)_{n=0}^{\infty}$ of $m \times m$ block matrices by the relations*

$$A_n = \begin{pmatrix} I_r & 0 & \ldots & 0 & 0 \\ 0 & I_r & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I_r & 0 \\ 0 & 0 & \ldots & 0 & a_{n+m,\,n} \end{pmatrix} \quad and \quad B_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & a_{n,\,n} \\ -I_r & 0 & \ldots & 0 & a_{n+1,\,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & a_{n+m-2,\,n} \\ 0 & 0 & \ldots & -I_r & a_{n+m-1,\,n} \end{pmatrix}$$

*for $n = 0, 1, \ldots$. Then the operator $G_1$ acting in $(l_r^2)^m$ and defined by*

$$G_1 = \begin{pmatrix} G & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I & 0 \\ 0 & 0 & \ldots & 0 & I \end{pmatrix},$$

*where $I$ denotes the identity operator in $l_r^2$, is equivalent to the operator $T = (\delta_{ij} B_j + \delta_{i-1,j} A_j)_{ij=0}^{\infty}$ acting in $l_{rm}^2$.*

This result, in fact, reduces the study of invertibility and the Fredholm properties of the band operators to the case of block weighted shifts. The next theorem establish a basic connection between the properties of the block weighted shifts and the nonstationary Stein inequalities. Generally speaking, the latter are systems of inequalities of the form $X_n - A_n^* X_{n+1} A_n > 0$, $n = 0, 1, \ldots$. The results connecting the Fredholm properties of the block weighted shifts with these inequalities can be found in [5]. Here, we consider slightly different inequalities. For a finite self-adjoint matrix $X$, we denote by $\nu_-(X)$ the number of negative eigenvalues of $X$, counting multiplicities, and set $\nu_0(X) = \dim \operatorname{Ker} X$.

**Theorem 3.** *Let $\{X_n\}_{n=0}^{\infty}$ and $\{A_n\}_{n=0}^{\infty}$ be two bounded sequences of $r \times r$ matrices with self-adjoint $X_n$. Assume that the following inequalities hold:*

$$X_{n+1} - A_n X_n A_n^* \geq 0, \quad n = 0, 1, \ldots, \tag{14}$$

*and*

$$X_{n+m} - A_{n+m-1} \dots A_n X_n A_n^* \dots A_{n+m-1}^* \geq \delta I_r, \quad n = 0, 1, \dots, \tag{15}$$

*where* $\delta > 0$ *and* $m$ *is a positive integer. Let* $\Gamma$ *be the operator in* $l_r^2$ *defined as follows:* $\Gamma = (\delta_{i,j+1} A_j)_{ij=0}^{\infty}$. *Then* $I - \Gamma$ *is a Fredholm operator in* $l_r^2$, *and there exists a positive integer* $N$ *such that*

$$\operatorname{index}(I - \Gamma) = -\nu_-(X_n), \quad i = N, N+1, \dots, \tag{16}$$

*and*

$$\nu_0(X_n) = 0, \quad i = N, N+1, \dots. \tag{17}$$

First, we define the concept of dichotomy, which is underlying in the proof of this result. Consider a system of the form

$$x_{n+1} = A_n X_n, \quad n = 0, 1, \dots, \tag{18}$$

where $\{A_n\}_{n=0}^{\infty}$ is a sequence of complex $r \times r$ matrices and $x_n \in \mathbb{C}^r$, $n = 0, 1, \dots$. A bounded sequence of projections $\{P_n\}_{n=0}^{\infty}$ in $\mathbb{C}^r$ with constant rank, is called a *dichotomy* for system (18) if the commutation relations

$$P_{n+1} A_n = A_n P_n, \quad n = 0, 1, \dots, \tag{19}$$

hold and there exist two positive numbers $a < 1$ and $M$, such that

$$\| A_{n+k-1} \dots A_n P_n x \| \leq M a^k \| P_n x \| \tag{20}$$

and

$$\| A_{n+k-1} \dots A_n (I_r - P_n) x \| \geq (M a^k)^{-1} \| (I_r - P_n) x \|, \tag{21}$$

for $x \in \mathbb{C}^r$, $n = 0, 1, \dots$, and $k = 1, 2, \dots$. The constant number Rank $P_n$, $n = 0, 1, \dots$, is called the rank of the dichotomy. This concept appears in [2, 6, 7, 9].

A dichotomy can be also defined for bilateral systems

$$x_{n+1} = A_n x_n, \quad n = 0, \pm 1, \dots, \tag{22}$$

and for systems indexed by negative integers

$$x_{n+1} = A_n x_n, \quad n = 0, -1, \dots. \tag{23}$$

A dichotomy for (22) (respectively, for (23)) is a bounded sequence of projections $\{P_n\}_{n=-\infty}^{\infty}$ (respectively, $\{P_n\}_{n=-\infty}^{1}$) with constant rank satisfying conditions (19)–(21) with the obvious change of indices.

Finally, we say that system (22) admits an asymptotic dichotomy in the positive (negative) direction, if there exists an integer $N$ such that the system $x_{n+1} = A_{n+N} x_n$, $n = 0, 1, \dots$ ($n = 0, -1, \dots$) admits a dichotomy.

We refer to [5] for the results concerning dichotomies and their connection with the nonstationary Stein equations and the block weighted shifts.

We now prove Theorem 3. Denote the inertia of a finite self-adjoint matrix $X$ by $\operatorname{In}(X) = (\nu_+, \nu_0, \nu_-)$, where $\nu_+ = \nu_+(X)$, $\nu_0 = \dim \operatorname{Ker} X$, and $\nu_- = \nu_-(X)$.

***Proof of Theorem* 3.** Denote

$$Y_k = X_{km}, \quad k = 0, 1, \dots, \tag{24}$$

and

$$B_k = A_{km+m-1} \dots A_{km}, \quad k = 0, 1, \dots. \tag{25}$$

Then (15) leads to

$$Y_{k+1} - B_k Y_k B_k^* \geq \delta I_r, \quad k = 0, 1, \dots . \tag{26}$$

To transform the latter system to the ordinary form, we set

$$C_k = B_{-k}^* \quad \text{and} \quad Z_k = Y_{-k+1}, \quad k = 0, -1, \dots . \tag{27}$$

Hence, (26) turns into

$$Z_k - C_k^* Z_{k+1} C_k \geq \delta I_r, \quad k = 0, -1, \dots .$$

By Corollary 2.6 in [5], it follows from these inequalities that there exists a positive integer $k_0$, and nonnegative integers $\nu_+$ and $\nu_-$ such that

$$\mathrm{In}(Z_k) = (\nu_+, 0, \nu_-), \quad k = -k_0, -k_0 - 1, \dots . \tag{28}$$

Let us apply Theorem 3.7 of [5] to the system

$$Z_k - C_k^* Z_{k+1} C_k \geq \delta I_r, \quad k = -k_0 - 1, -k_0 - 2, \dots .$$

Since the sequence $\{Z_k\}_{k=-\infty}^{-k_0}$ is of constant inertia $(\nu_+, 0, \nu_-)$, we conclude that the system

$$x_{k+1} = C_k x_k, \quad k = -k_0 - 1, -k_0 - 2, \dots ,$$

admits a dichotomy of rank $\nu_+$.

Now recall that $C_k = B_{-k}^*$, $k = 0, -1, \dots$. Hence, Proposition 3.3 in [5] shows that the system

$$x_{k+1} = B_k x_k, \quad k = k_0 + 1, k_0 + 2, \dots ,$$

admits a dichotomy of rank $\nu_+$.

Thus, the system

$$x_{k+1} = B_k x_k, \quad k = 0, 1, \dots , \tag{29}$$

admits an asymptotic dichotomy of rank $\nu_+$.

In view of the definition (25) of $B_k$ and the boundedness of $(A_n)_{n=0}^{\infty}$, one can apply Proposition 1 presented below to system (29). This implies that the system

$$x_{n+1} = A_n x_n, \quad n = 0, 1, \dots ,$$

admits an asymptotic dichotomy of rank $\nu_+$.

Theorem 2.11 in [5] implies that $I - \Gamma$ is a Fredholm operator with

$$\mathrm{index}\,(I - \Gamma) = \nu_+ - r = -\nu_- . \tag{30}$$

It follows from (28) and the definitions (27) and (24) of $Z_k$ and $Y_k$ that

$$\mathrm{In}(X_{km}) = (\nu_+, 0, \nu_-), \quad k = k_0 + 1, \ k_0 + 2, \dots .$$

Assume that $s \in \{0, \dots, m-1\}$. Let us apply the argument presented above to the sequences $\{X_{n+s}\}_{n=0}^{\infty}$ and $\{A_{n+s}\}_{n=0}^{\infty}$. We find that $I - \Gamma_s$, where $\Gamma_s = (\delta_{i,j+1} A_{j+s})_{ij=0}^{\infty}$, is a Fredholm operator. Furthermore, there exist integers $k_s$, $\nu_{+,s}$, and $\nu_{-,s}$ such that

$$\mathrm{index}\,(I - \Gamma_s) = -\nu_{-,s} \tag{31}$$

and

$$\mathrm{In}\,(X_{km+s}) = (\nu_{+,s}, 0, \nu_{-,s}), \quad k = k_s + 1, k_s + 2, \dots . \tag{32}$$

However, it is clear that index $(I - \Gamma)$ = index $(I - \Gamma_s)$. Thus, (30) and (31) imply that $\nu_- = \nu_{-,s}$. Hence, $\nu_{+,s} = r - \nu_{-,s} = r - \nu_- = \nu_+$. Therefore, (32) yields $\mathrm{In}(X_{km+s}) = (\nu_+, 0, \nu_-)$, $k = k_s + 1, k_s + 2, \dots$, for $s = 0, \dots, m-1$.

This implies

$$\mathrm{In}\,(X_n) = (\nu_+, 0, \nu_-), \quad n = N, N+1, \dots ,$$

where $N = m(2 + \max_{0 \le s \le m-1} k_s)$. These equalities and (30) lead to (16) and (17).

**Proposition 1.** *Let* $\{A_n\}_{n=0}^{\infty}$ *be a bounded sequence of* $r \times r$ *matrices. Denote*

$$B_k = A_{km+m-1} \dots A_{km}, \quad k = 0, 1, \dots , \tag{33}$$

*where* $m$ *is a positive integer. If the system*

$$x_{k+1} = B_k x_k, \quad k = 0, 1, \dots , \tag{34}$$

*admits an (asymptotic) dichotomy of rank* $q$, *then the system*

$$x_{n+1} = A_n x_n, \quad n = 0, 1, \dots , \tag{35}$$

*also admits an (asymptotic) dichotomy of the same rank* $q$.

***Proof.*** The statement concerning the asymptotic dichotomies follows from the statement concerning the dichotomies after an obvious change of indices. Hence, we only prove the latter. Moreover, the statement is trivial for $m = 1$. Therefore, we consider $m \ge 2$.

Assume that (34) admits a dichotomy of rank $q$. It follows from Proposition 6.1 and Theorem 4.2 in [2] that there exists a bounded sequence of self-adjoint $r \times r$ matrices $\{X_{km}\}_{k=0}^{\infty}$ of constant inertia $(q, 0, r-q)$, and a positive number $\varepsilon$ such that

$$X_{km} - B_k^* X_{(k+1)m} B_k \ge \varepsilon I_r, \quad k = 0, 1, \dots . \tag{36}$$

Let

$$L = \sup_{n=0,1,\dots} \|A_n\| + 1.$$

Define

$$\delta = \varepsilon / (2(m-1) L^{2m}).$$

For every $k = 0, 1, \dots$, we define the matrices $X_{km+m-1}, X_{km+m-2}, \dots, X_{km+1}$ recursively via

$$X_{km+s} = A_{km+s}^* X_{km+s+1} A_{km+s} + \delta I_r, \quad s = m-1, \dots, 1. \tag{37}$$

Then $(X_n)_{n=0}^{\infty}$ is a bounded sequence with

$$X_{km+1} = A_{km+1}^* A_{km+2}^* \dots A_{km+m-1}^* X_{(k+1)m} A_{km+m-1} \dots A_{km+2} A_{km+1} + R,$$

where

$$R = \delta \left( I_r + \sum_{s=1}^{m-2} A_{km+1}^* \dots A_{km+s}^* A_{km+s} \dots A_{km+1} \right)$$

satisfies

$$\|R\| < \delta(m-1)L^{2(m-2)}.$$

Hence,

$$X_{km} - A_{km}^* X_{km+1} A_{km} = X_{km} - A_{km}^* \ldots A_{km+m-1}^* X_{(k+1)m} A_{km+m-1} \ldots A_{km} + R_1, \tag{38}$$

where $R_1 = -A_{km}^* X_{km+1} A_{km}$ satisfies

$$\|R_1\| \le \delta(m-1)L^{2m} = \varepsilon/2.$$

Taking the definition of $B_k$ and (36) into account, we obtain from (38)

$$X_{km} - A_{km}^* X_{km+1} A_{km} \ge X_{km} - B_k^* X_{(k+1)m} B_k - \varepsilon I_r/2 \ge \varepsilon I_r/2, \quad k = 0, 1, \ldots .$$

Combining this with (37), we get

$$X_n - A_n^* X_{n+1} A_n \ge \delta I_r, \quad n = 0, 1, \ldots, \tag{39}$$

where we have taken into account that $\delta \le \varepsilon/2$.

We now show that

$$\operatorname{In}(X_n) = (q, 0, r-q), \quad n = 0, 1, \ldots . \tag{40}$$

Recall that (40) holds for $n = km$, $k = 0, 1, \ldots$, by construction. For a natural number $n$, let $k$ be such that $km > n$. It easily follows from (39) that

$$X_n - A_n^* \ldots A_{km-1}^* X_{km} A_{km-1} \ldots A_n \ge \delta I_r,$$

and, similarly,

$$X_0 - A_0^* \ldots A_{n-1}^* X_n A_{n-1} \ldots A_0 \ge \delta I_r,$$

Let $\operatorname{In}(X_n) = (\nu_{+,n}, \nu_{0,n}, \nu_{-,n})$. By Lemma 2.5 in [5] and $\operatorname{In}(X_0) = \operatorname{In}(X_{km}) = (q, 0, r-q)$, these inequalities imply that

$$\nu_{+,n} \ge q, \quad q \ge \nu_{+,n} + \nu_{0,n}.$$

Thus,

$$\operatorname{In}(X_n) = (\nu_{+,n}, \nu_{0,n}, \nu_{-,n}) = (q, 0, r-q).$$

Finally, we apply Theorem 6.4 in [2] to inequalities (39). Since $(X_n)_{n=0}^{\infty}$ is a bounded sequence of constant inertia $(q, 0, r-q)$, system (35) admits a dichotomy of rank $q$.

**3. Proof of the Main Result.** In this section, we prove Theorem 1. We use the results obtained in the previous section and Lemma 2 below, which shows how the nonstationary Stein inequalities can be derived in the special situation under consideration. In the next statement and in the proof, we use the following notation: Let $m$ be a a positive integer and let $R = (R_{ij})_{ij=0}^{2m-1}$ be a self-adjoint block matrix with the blocks $R_{ij}$ of order $r$ satisfying the equality $R_{ij} = 0$ for $|i-j| > m$. Denote certain submatrices of $R$ as follows:

$$L_n = (R_{ij})_{ij=n}^{n+m}, \quad n = 0, \ldots, m-1, \tag{41}$$

$$K_n = (R_{ij})_{ij=n}^{n+m-1}, \quad n = 0, \ldots, m, \tag{42}$$

and

$$H_n = (R_{ij})_{i=n+1, j=n}^{n+m, n+m-1}, \quad n = 0, \ldots, m-1.$$

**Lemma 2.** *For every positive number* $\rho$, *there exist a positive number* $\delta$ *with the following property: Assume that the matrices* $L_n$, $n = 0, \ldots, m-1$, *and* $K_n$, $n = 0, \ldots, m$, *defined above are invertible with*

$$\max_{n=0, \ldots, m-1} (\| L_n \|, \| L_n^{-1} \|) \le \rho, \tag{43}$$

$$\max_{n=0, \ldots, m} \| K_n^{-1} \| \le \rho, \tag{44}$$

*and that*

$$(L_n^{-1})_{m,m} \ge 0, \quad n = 0, \ldots, m-1. \tag{45}$$

*Then the following inequalities hold:*

$$K_{n+1} - H_n K_n^{-1} H_n^* > 0, \quad n = 0, \ldots, m-1, \tag{46}$$

*and*

$$K_m - (H_{m-1} K_{m-1}^{-1}) \ldots (H_1 K_1^{-1}) H_0 K_0^{-1} H_0^* (H_1 K_1^{-1})^* \ldots (H_{m-1} K_{m-1}^{-1})^* \ge \delta I_r. \tag{47}$$

***Proof.*** Fix $\rho > 0$. Denote by $\Omega$ the set of finite self-adjoint matrices $R = (R_{ij})_{ij=0}^{2m-1}$ with blocks $R_{ij}$ of order $r$, which satisfy the equality $R_{ij} = 0$ for $|i-j| > m$ and are such that the matrices $L_n$, $n = 0, \ldots, m-1$, and $K_n$, $n = 0, \ldots, m$, given by (41), (42), are invertible and satisfy (43) – (45).

The space $\Omega$ endowed with the usual topology is compact and the matrix on the left-hand side of (47) is a continuous function of $R \in \Omega$. Thus, in order to prove the existence of $\delta > 0$ satisfying (47) for all $R \in \Omega$, it suffices to show that, for each $R \in \Omega$, the following inequality holds:

$$K_m - (H_{m-1} K_{m-1}^{-1}) \ldots (H_1 K_1^{-1}) H_0 K_0^{-1} H_0^* (H_1 K_1^{-1})^* \ldots (H_{m-1} K_{m-1})^* > 0. \tag{48}$$

We now prove (46) and (48) for $R \in \Omega$. The invertibility of $K_n$ and (45) imply that

$$(L_n^{-1})_{m,m} > 0, \quad n = 0, \ldots, m-1. \tag{49}$$

Denote

$$Q_n = H_n K_n^{-1}, \quad n = 0, \ldots, m-1. \tag{50}$$

Note that the $i$th row of $H_n$ is equal to the $(i+1)$th row of $K_n$. Thus, it follows from $H_n = Q_n K_n$ that $Q_n$ has the following form:

$$Q_n = \begin{pmatrix} 0 & I_r & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & I_r \\ * & * & \ldots & * & * \end{pmatrix}, \quad n = 0, \ldots, m-1.$$

Consequently, $H_n K_n^{-1} H_n^* = Q_n H_n^*$ has the following form:

$$H_n K_n^{-1} H_n^* = \begin{pmatrix} R_{n+1,\,n+1} & \dots & R_{n+1,\,n+m} \\ R_{n+2,\,n+1} & \dots & R_{n+2,\,n+m} \\ \dots & \dots & \dots \\ R_{n+m-1,\,n+1} & \dots & R_{n+m-1,\,n+m} \\ * & \dots & * \end{pmatrix}, \quad n = 0, \dots, m-1. \quad (51)$$

Denote

$$E_n = K_{n+1} - H_n K_n^{-1} H_n^*, \quad n = 0, \dots, m-1. \quad (52)$$

Since both $K_{n+1}$ and $H_n K_n^{-1} H_n^*$ are self-adjoint and coincide in their $m-1$ upper rows, by virtue of (51), we conclude that $E_n$ has the following form:

$$E_n = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \mu_n \end{pmatrix}, \quad n = 0, \dots, m-1. \quad (53)$$

Equality (52) also shows that

$$\mu_n = R_{n+m,n+m} - (R_{n+m,n} \dots R_{n+m,\,n+m-1}) K_n^{-1} \begin{pmatrix} R_{n,\,n+m} \\ \vdots \\ R_{n+m-1,\,n+m} \end{pmatrix}, \quad n = 0, \dots, m-1.$$

Thus, $\mu_n$ is the Schur complement of $K_n$ in $L_n$. Consequently,

$$\mu_n = \left( (L_n^{-1})_{m,m} \right)^{-1}, \quad n = 0, \dots, m-1.$$

Hence, (49) implies that

$$\mu_n > 0, \quad n = 0, \dots, m-1. \quad (54)$$

This inequality and (53) lead to

$$E_n \geq 0, \quad n = 0, \dots, m-1. \quad (55)$$

Hence, (52) yields (46).

We now prove (48). If $m = 1$, then (53) reduces to $E_0 = \mu_0$. This, together with (52) and (54), leads to $K_1 - H_0 K_0^{-1} H_0^* = E_0 = \mu_0 > 0$. Thus, (48) is true in this case. Assume that $m \geq 2$. First, we prove that

$$E_{m-1} + \sum_{n=0}^{m-2} Q_{m-1} \dots Q_{n+1} E_n Q_{n+1}^* \dots Q_{m-1}^* > 0. \quad (56)$$

We prove (56) by contradiction. Assume that (56) is false. Note that the matrix on the left-hand side of (56) is nonnegative by (55). Hence, there exists a nonzero vector $x = (x_i)_{i=0}^{m-1}$ in $(\mathbb{C}^r)^m$ such that

$$x^* E_{m-1} x + \sum_{n=0}^{m-2} x^* Q_{m-1} \dots Q_{n+1} E_n Q_{n+1}^* \dots Q_{m-1}^* x = 0.$$

By virtue of (55), this gives $x^* E_{m-1} x = 0$ and

$$x^* Q_{m-1} \dots Q_{n+1} E_n Q_{n+1}^* \dots Q_{m-1}^* x = 0, \quad n = 0, \dots, m-2. \quad (57)$$

It follows from $x^* E_{m-1} x$ and the special structure of $E_{m-1}$ given by (53) that

$x^*_{m-1}\mu_{m-1}x_{m-1} = 0$. Thus, it follows from (54) that $x_{m-1} = 0$. Let $s \in \{0, \ldots, m-2\}$ be the unique integer such that $x_s \neq 0$ and $x_{s+1} = \ldots = x_{m-1} = 0$. Note that $s$ exists because $x \neq 0$ and $x_{m-1} = 0$. It follows from the special structure of $Q_n$ given in (50) that

$$Q^*_{s+1} \ldots Q^*_{m-1} x = \begin{pmatrix} * \\ \vdots \\ * \\ x_s \end{pmatrix}.$$

Hence, by taking the special structure of $E_s$ given by (53) into account, we obtain

$$x^* Q_{m-1} \ldots Q_{s+1} E_s Q^*_{s+1} \ldots Q^*_{m-1} x =$$
$$= (Q^*_{s+1} \ldots Q^*_{m-1} x)^* E_s (Q^*_{s+1} \ldots Q^*_{m-1} x) = x^*_s \mu_s x_s.$$

By (57), this leads to $x^*_s \mu_s x_s = 0$ and, hence, (54) implies that $x_s = 0$ arriving at a contradiction. Consequently, (56) is true.

Now note that equality (52) yields

$$K_{n+1} - (H_n K_n^{-1}) K_n (H_n K_n^{-1})^* = E_n, \quad n = 0, \ldots, m-1.$$

Hence, by the definition (50) of $Q_n$, we get

$$K_{n+1} - Q_n K_n Q^*_n = E_n, \quad n = 0, \ldots, m-1. \tag{58}$$

In particular,

$$K_m - Q_{m-1} K_{m-1} Q^*_{m-1} = E_{m-1}. \tag{59}$$

In addition, it follows from (58) that

$$Q_{m-1} \ldots Q_{n+1} K_{n+1} Q^*_{n+1} \ldots Q^*_{m-1} - Q_{m-1} \ldots Q_n K_n Q^*_n \ldots Q^*_{m-1} =$$
$$= Q_{m-1} \ldots Q_{n+1} E_n Q^*_{n+1} \ldots Q^*_{m-1}, \quad n = 0, \ldots, m-2. \tag{60}$$

By summing (59) and (60) for $n = 0, 1, \ldots, m-2$, we obtain

$$K_m - Q_{m-1} \ldots Q_0 K_0 Q^*_0 \ldots Q_{m-1} =$$
$$= E_{m-1} + \sum_{n=0}^{m-2} Q_{m-1} \ldots Q_{n+1} E_n Q^*_{n+1} \ldots Q^*_{m-1}.$$

Therefore, (56) gives

$$K_m - Q_{m-1} \ldots Q_0 K_0 Q^*_0 \ldots Q^*_{m-1} > 0.$$

Recalling the definition (50) of $Q_n$, we find

$$K_m - (H_{m-1} K^{-1}_{m-1}) \ldots (H_0 K_0^{-1}) K_0 (H_0 K_0^{-1})^* \ldots (H_{m-1} K_{m-1})^* > 0.$$

Hence, (48) is valid.

***Proof of Theorem*** **1.** We use the notation, which generalizes the notation of the previous lemma. We set $L_n = (R_{ij})_{ij=n}^{n+m}$, $K_n = (R_{ij})_{ij=n}^{n+m-1}$, and $H_n = (R_{ij})_{i=n+1, j=n}^{n+m, n+m-1}$, $n = 0, 1, \ldots$. Without loss of generality, we can assume that $R_{ij} = 0$ for $|i-j| > m$.

Note that the Schur complement of $(L_n^{-1})_{m,m}$ in $L_n$ is $K_n$. Thus, inequalities (1) and (2) imply that $K_n$ is invertible $(n = 0, 1, \dots)$, with

$$\sup_{n=0,\,1,\,\dots} \| K_n^{-1} \| < \infty.$$

Denote

$$\rho = \sup_{n=0,\,1,\,\dots} (\| L_n \|, \| L_n^{-1} \|, \| K_n^{-1} \|).$$

By the previous inequality and (1), we get $\rho < \infty$. Thus, we can apply Lemma 2 to the block matrix $(R_{ij})_{ij=n}^{n+2m-1}$ for $n = 0, 1, \dots$. We have

$$K_{n+1} - H_n K_n^{-1} H_n^* \geq 0, \quad n = 0, 1, \dots,$$

and

$$K_{n+m} - (H_{n+m-1} K_{n+m-1}^{-1}) \dots (H_{n+1} K_{n+1}^{-1}) H_n K_n^{-1} H_n^* (H_{n+1} K_{n+1}^{-1})^* \dots$$
$$\dots (H_{n+m-1} K_{n+m-1}^{-1})^* \geq \delta I_r, \quad n = 0, 1, \dots,$$

where $\delta$ is positive and independent of $n$. Hence,

$$K_{n+1} - (H_n K_n^{-1}) K_n (H_n K_n^{-1})^* \geq 0, \quad n = 0, 1, \dots, \tag{61}$$

and

$$K_{n+m} - (H_{n+m-1} K_{n+m-1}^{-1}) \dots (H_n K_n^{-1}) K_n (H_n K_n^{-1})^* \dots$$
$$\dots (H_{n+m-1} K_{n+m-1}^{-1})^* \geq \delta I_r, \quad n = 0, 1, \dots. \tag{62}$$

We now apply Theorem 3 to the set (61) and (62) of nonstationary Stein inequalities. Let $\Gamma = (\delta_{i,j+1} H_j K_j^{-1})_{ij=0}^{\infty}$ be an operator acting in $l_{rm}^2$. By Theorem 3, $I - \Gamma$ is a Fredholm operator and there exists a positive integer $N$ such that index $(I - \Gamma) = -\nu_-(K_n)$, $n = N, N+1, \dots$. Hence,

$$I - \Gamma^* = (\delta_{ij} I_r - \delta_{i+1\,j} K_i^{-1} H_i^*)_{ij=0}^{\infty} \tag{63}$$

is a Fredholm operator with

$$\text{index}\,(I - \Gamma^*) = \nu_-(K_n), \; n = N, N+1, \dots. \tag{64}$$

We now define $m \times m$ block matrices $A_n$ and $B_n$, $n = 0, 1, \dots$, by

$$A_n = \begin{pmatrix} I_r & 0 & \dots & 0 & 0 \\ 0 & I_r & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_r & 0 \\ 0 & 0 & \dots & 0 & a_{n+m,\,n} \end{pmatrix}; \quad B_n = \begin{pmatrix} 0 & 0 & \dots & 0 & a_{n,\,n} \\ -I_r & 0 & \dots & 0 & a_{n+1,\,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{n+m-2,\,n} \\ 0 & 0 & \dots & -I_r & a_{n+m-1,\,n} \end{pmatrix}, \tag{65}$$

where $a_{ij}$ are as in the statement of the theorem. Note that $(a_{n+k,n})_{k=0}^{m}$ is the last column of the inverse of $L_n$. Hence, $(A_n)_{n=0}^{\infty}$ and $(B_n)_{n=0}^{\infty}$ are bounded sequences. Furthermore, note that $a_{n+m,n} = (L_n^{-1})_{m,m}$. Thus, inequality (2) demonstrates that

$a_{n+m,n}$ is invertible with $\sup_{n=0,1,\ldots} \| a_{n+m,n}^{-1} \| < \infty$. Finally, we get

$$\sup_{n=0,1,\ldots} (\|A_n\|, \|B_n\|, \|A_n^{-1}\|) < \infty. \tag{66}$$

In addition, the following equalities hold

$$H_n^* A_n = \begin{pmatrix} R_{n,n+1} & \cdots & R_{n,n+m-1} & R_{n,n+m} a_{n+m,n} \\ \vdots & \vdots & \vdots & \vdots \\ R_{n+m-1,n+1} & \cdots & R_{n+m-1,n+m-1} & R_{n+m-1,n+m} a_{n+m,n} \end{pmatrix}$$

and

$$K_n B_n = \begin{pmatrix} -R_{n,n+1} & \cdots & -R_{n,n+m-1} & \sum_{k=0}^{m-1} R_{n,n+k} a_{n+k,n} \\ \vdots & \vdots & \vdots & \vdots \\ -R_{n+m-1,n+1} & \cdots & -R_{n+m-1,n+m-1} & \sum_{k=0}^{m-1} R_{n+m-1,n+k} a_{n+k,n} \end{pmatrix},$$

for $n = 0, 1, \ldots$. Taking equations (3) defining $a_{n+k,n}$ into account, we obtain

$$H_n^* A_n + K_n B_n = 0, \quad n = 0, 1, \ldots.$$

Consequently, $-K_n^{-1} H_n^* = B_n A_n^{-1}$, $n = 0, 1, \ldots$. Hence, equality (63) shows that

$$I - \Gamma^* = (\delta_{ij} I_r + \delta_{i+1,j} B_i A_i^{-1})_{ij=0}^{\infty}. \tag{67}$$

Define an operator $S$ in $l_{rm}^2$ by

$$S = (\delta_{i,j+1} A_j)_{ij=0}^{\infty}. \tag{68}$$

In follows from the invertibility of the matrices $A_j$ and inequality (66) that $S$ is a Fredholm operator in $l_{rm}^2$ with $\operatorname{index} S = -rm$. This equality and (64) imply that $T = (I - \Gamma^*) S$ is a Fredholm operator with

$$\operatorname{index} T = \nu_-(K_n) - rm, \quad n = N, N+1, \ldots.$$

However, $K_n$ is invertible for $n = 0, 1, \ldots$, and, therefore, $\nu_+(K_n) + \nu_-(K_n) = rm$. Thus,

$$\operatorname{index} T = -\nu_+(K_n), \quad n = N, N+1, \ldots. \tag{69}$$

In addition, we obtain from (67) and (68) that

$$T = (I - \Gamma^*) S = (\delta_{ij} B_j + \delta_{i-1,j} A_j)_{ij=0}^{\infty}.$$

Finally, we apply Lemma 1 to the operator $G = (a_{ij})_{ij=0}^{\infty}$ defined in the statement of Theorem 3. We conclude that the operator $G_1 = G \oplus \ldots \oplus I$ is equivalent to $T$. Hence, $G$ is a Fredholm operator and (69) leads to

$$\operatorname{index} G = -\nu_+(K_n), \quad n = N, N+1, \ldots.$$

Therefore, (4) is valid for $n \geq N$.

**4. Bilateral Case.** In this section, we present the proof of Theorem 2. Since this proof is similar to the proof of Theorem 1, only the principal changes are indicated.

***Proof of Theorem* 2.** As in the proof of Theorem 1, we denote

$$\begin{cases} K_n = (R_{ij})_{ij=n}^{n+m-1}; \\ H_n = (R_{ij})_{i=n+1,\, j=n}^{n+m,\, n+m-1} \end{cases}$$

for $n = 0, \pm 1, \ldots$. We have the inequalities

$$\sup_{n=0,\, \pm 1,\, \ldots} (\| L_n \|, \| L_n^{-1} \|, \| K_n^{-1} \|) < \infty.$$

By using Lemma 2, we obtain as in (61) and (62)

$$K_{n+1} - (H_n K_n^{-1}) K_n (H_n K_n^{-1})^* \geq 0, \tag{70}$$

$$K_{n+m} - (H_{n+m-1} K_{n+m-1}^{-1}) \ldots (H_n K_n^{-1}) K_n (H_n K_n^{-1})^* \ldots$$
$$\ldots (H_{n+m-1} K_{n+m-1}^{-1})^* \geq \varepsilon_1 I_r \tag{71}$$

for $n = 0, \pm 1, \ldots$, where $\varepsilon_1 > 0$. Denote

$$\Gamma = (\delta_{i,j+1} H_j K_j^{-1})_{ij=0}^{\infty}.$$

It follows from inequalities (70) and (71) and Theorem 4 below that $I - \Gamma$ is a Fredholm operator with

$$\operatorname{Im}(I - \Gamma) = l_r^2(\mathbb{Z}) \tag{72}$$

and there exist nonnegative integers $N, q,$ and $p$ such that

$$\nu_+(K_n) = q, \quad \nu_+(K_{-n}) = p, \quad n = N, N+1 \ldots, \tag{73}$$

and

$$\operatorname{index}(I - \Gamma) = q - p. \tag{74}$$

As in equality (67) in the proof of Theorem 3, we get

$$I - \Gamma^* = (\delta_{ij} I_r + \delta_{i+1,j} B_i A_i^{-1})_{ij=-\infty}^{\infty},$$

where $A_n$ and $B_n$ are given by (65) for $n = 0, \pm 1, \ldots$. Furthermore, we also have

$$\sup_{n=0,\, \pm 1,\, \ldots} (\| A_n \|, \| B_n \|, \| A_n^{-1} \|) < \infty$$

as in (66). Hence, the operator

$$S = (\delta_{i,j+1} A_j)_{ij=-\infty}^{\infty}$$

is invertible. Therefore, by (72) and (74),

$$T = (I - \Gamma^*) S = (\delta_{ij} B_j + \delta_{i,j+1} A_j)_{ij=-\infty}^{\infty}$$

is a Fredholm operator such that

$$\operatorname{Ker} T = S^{-1}(\operatorname{Ker}(I - \Gamma^*)) = S^{-1}((\operatorname{Im}(I - \Gamma))^{\perp}) = \{0\}$$

and

$$\operatorname{index} T = -\operatorname{index}(I - \Gamma) = p - q.$$

Finally, Lemma 3.1 in [3] shows that the operator $A \oplus I \oplus \ldots \oplus I$ is equivalent to $T$. Hence, $A$ is a Fredholm operator, $\operatorname{Ker} A = \{0\}$, and $\operatorname{index} A = p - q$. Together with (73), this proves the theorem.

We now give the bilateral analog of Theorem 3.

**Theorem 4.** *Let* $(X_n)_{n=-\infty}^{\infty}$ *and* $(A_n)_{n=-\infty}^{\infty}$ *be two bounded sequences of* $r \times r$ *matrices with self-adjoint* $X_n$. *Consider the operator* $\Gamma = (\delta_{i,j+1} A_j)_{ij=-\infty}^{\infty}$ *in* $l_r^2(\mathbb{Z})$. *Assume that the following equalities hold:*

$$X_{n+1} - A_n X_n A_n^* \geq 0, \quad n = 0, \pm 1, \ldots, \tag{75}$$

*and*

$$X_{n+m} - A_{n+m-1} \ldots A_n X_n A_n^* \ldots A_{n+m-1}^* \geq \varepsilon I_r, \quad n = 0, \pm 1, \ldots, \tag{76}$$

*where* $\varepsilon > 0$ *and* $m$ *is a positive integer. Then* $I - \Gamma$ *is a Fredholm operator with*

$$\operatorname{Im}(I - \Gamma) = l_r^2(\mathbb{Z}), \tag{77}$$

*and there exist nonnegative integers* $N$, $p$, *and* $q$ *such that*

$$\nu_+(X_n) = q, \quad \nu_+(X_{-n}) = q, \quad n = N, N+1, \ldots, \tag{78}$$

*and*

$$\operatorname{index}(I - \Gamma) = q - p. \tag{79}$$

***Proof.*** Denote

$$Y_k = X_{km}, \quad k = 0, \pm 1, \ldots, \tag{80}$$

and

$$B_k = A_{km+m-1} \ldots A_{km}, \quad k = 0, \pm 1, \ldots. \tag{81}$$

Then (76) leads to

$$Y_{k+1} - B_k Y_k B_k^* \geq \varepsilon I_r, \quad k = 0, \pm 1, \ldots.$$

Thus, by setting

$$G_k = B_{-k}^* \quad \text{and} \quad Z_k = Y_{-k+1}, \quad k = 0, \pm 1, \ldots, \tag{82}$$

we obtain

$$Z_k - C_k^* Z_{k+1} C_k \geq \varepsilon I_r, \quad k = 0, \pm 1, \ldots.$$

We apply Theorem 4.2 in [5] to the last set of inequalities. We find that the system

$$x_{k+1} = C_k x_k, \quad k = 0, \pm 1, \ldots \tag{83}$$

admits an asymptotic dichotomies in the positive and negative directions, the rank of which we denote by $p$ and $q$, respectively. Furthermore, there exists a positive integer $k_0$ such that

$$\operatorname{In}(Z_k) = (p, 0, r - p), \quad k = k_0, k_0 + 1, \ldots. \tag{84}$$

and

$$\operatorname{In}(Z_k) = (q, 0, r - q), \quad k = -k_0, -k_0 - 1, \ldots. \tag{85}$$

Since $C_k = B_{-k}^*$ by (82), it follows from Proposition 3.3 in [5] that the system

$$x_{k+1} = B_k x_k, \quad k = 0, \pm 1, \ldots,$$

admits an asymptotic dichotomy of rank $q$ in the positive direction. By Proposition 1, the system

$$x_{n+1} = A_n x_n, \quad n = 0, \pm 1, \ldots, \tag{86}$$

admits an asymptotic dichotomy of rank $q$ in the positive direction.

We now show that (86) admits an asymptotic dichotomy in the negative direction. Let $D_n = A^*_{-n+m-1}$, $n = 0, \pm 1, \ldots$. Then the definitions (81) and (82) of $B_k$ and $C_k$ result in

$$C_k = B^*_{-k} = A^*_{-km} \ldots A^*_{-km+m-1} = D_{km+m-1} \ldots D_{km}.$$

Thus, by applying Proposition 1 to system (83), we find that the system $x_{n+1} = D_n x_n = A^*_{-n+m-1} x_n$, and, therefore, the system

$$x_{n+1} = A^*_{-n} x_n, \quad n = 0, \pm 1, \ldots, \tag{87}$$

admit the asymptotic dichotomies of rank $p$ in the positive direction. Finally, we apply Proposition 3.3 in [5] to system (87). This implies that (86) admits an asymptotic dichotomy of rank $p$ in the negative direction.

We now apply Theorem 4.2 in [5] to system (86). Since (86) admits the asymptotic dichotomies of rank $q$ and $p$ in the positive and negative directions, $I - \Gamma$ is a Fredholm operator with

$$\operatorname{index}(I - \Gamma) = q - p. \tag{88}$$

Recall that, by definitions (80) and (82) and equalities (84) and (85), we have

$$\operatorname{In}(X_{km}) = (q, 0, r-q), \quad k = k_0 + 1,\ k_0 + 2, \ldots, \tag{89}$$

and

$$\operatorname{In}(X_{km}) = (p, 0, r-p), \quad k = -k_0 - 1,\ k_0 - 2, \ldots.$$

Let us show that

$$\operatorname{In}(X_n) = (q, 0, r-q), \quad n = N+1,\ N+2, \ldots, \tag{90}$$

and

$$\operatorname{In}(X_n) = (p, 0, r-p), \quad n = -N-1,\ N-2, \ldots, \tag{91}$$

where $N = (k_0 + 2)m$. Since the proofs of (90) and (91) are similar, we only prove (90). Let $n > N$. It follows from (76) that

$$X_n - A_{n-1} \ldots A_{n-m} X_{n-m} A^*_{n-m} \ldots A^*_{n-1} \geq \varepsilon I_r.$$

Note that $n - m > N - m = (k_0 + 1)m$. By iterating (75), we obtain

$$A_{n-1} \ldots A_{n-m} X_{n-m} A^*_{n-m} \ldots A^*_{n-1} - A_{n-1} \ldots A_{(k_0+1)m} X_{(k_0+1)m} A^*_{(k_0+1)m} \ldots A^*_{n-1} =$$

$$= A_{n-1} \ldots A_{n-m} (X_{n-m} - A_{n-m-1} \ldots A_{(k_0+1)m} X_{(k_0+1)m} A^*_{(k_0+1)m} \ldots$$

$$\ldots A^*_{n-m-1}) A^*_{n-m} \ldots A^*_{n-1} \geq 0.$$

Combining these two inequalities, we get

$$X_n - A_{n-1} \ldots A_{(k_0+1)m} X_{(k_0+1)m} A^*_{(k_0+1)m} \ldots A^*_{n-1} \geq \varepsilon I_r. \tag{92}$$

Denote $\operatorname{In}(X_n) = (\nu_{+,n}, \nu_{0,n}, \nu_{-,n})$. Since $\operatorname{In}(X_{(k_0+1)m}) = (q, 0, r-q)$ by (89), it follows from (92) and Lemma 2.5 in [5] that

$$\nu_{+,n} \geq q. \tag{93}$$

Now let $k'$ with $k'm \geq n+m$. The next equality follows from (75) and (76) in the same way as (92)

$$X_{k'm} - X_{k'm-1} \ldots A_n x_n A_n^* \ldots A_{k'm-1}^* \geq \varepsilon I_r.$$

Since $\operatorname{In}(X_{k'm}) = (q, 0, r-q)$ by (89), Lemma 2.5 in [5] indicates that

$$q \geq \nu_{+,n} + \nu_{0,n}.$$

This inequality and (93) imply $\nu_{0,n} = 0, \nu_{+,n} = q$ and, therefore, $\nu_{-,n} = r - q$. Since $n > N$ is arbitrary and $\operatorname{In}(X_n) = (\nu_{+,n}, \nu_{0,n}, \nu_{-,n})$, this proves (90). Similarly, (91) is valid.

Equalities (90), (91), and (88) imply that (78) and (79) hold.

Finally, we prove (77). Define a bounded self-adjoint operator $X$ in $l_r^2(\mathbb{Z})$ by $X = (\delta_{ij} X_j)_{ij=-\infty}^{\infty}$. Then (76) yields

$$X - \Gamma^m X \Gamma^{*m} \geq \varepsilon I.$$

Assume that $u \in \operatorname{Ker}(I - \Gamma^*)$. Then $u = \Gamma^{*m} u$ and, therefore,

$$0 = \langle Xu, u \rangle - \langle X \Gamma^{*m} u, \Gamma^{*m} u \rangle = \langle (X - \Gamma^m X \Gamma^{*m}) u, u \rangle \geq \varepsilon \| u \|^2.$$

Thus, $u = 0$. This proves that $\operatorname{Ker}(I - \Gamma^*) = \{0\}$. Since $I - \Gamma$ is a Fredholm operator, this implies (77).

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Received 29. 07. 93