Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 4. **(2007).** pp. 45 – 58

Exponent matrices and topological equivalence of maps

Volodymyr Fedorenko, Volodymyr Kirichenko and Makar Plakhotnyk

Abstract. Conjugate classes of continuous maps of the interval [0, 1] into itself, whose iterations form a finite group are described. For each of possible groups of iterations one to one correspondence between conjugate classes of maps and equivalent classes of (0, 1)-exponent matrices of special form is constructed. Easy way of finding the quiver of the map in terms of the set of its extrema is found.

Introduction. Recall that a right Noetherian semiperfect semiprime ring A is semimaximal if the endomorphism ring of every indecomposable finitely generated projective A-module is a discrete valuation ring. It is known [2, chapter 14] that semiperfect semiprime ring Λ is semimaximal if and only if it is isomorphic to the finite direct product of prime semimaximal rings. Every semimaximal ring is semidistributive two-sided Noetherian ring with nonzero Jacobson radical. Prime Noetherian semiperfect semidistributive ring with nonzero Jacobson radical is called tiled order.

Every tiled order Λ has the classical ring of fractions $M_n(D)$ which is the ring of all square matrices of order n over the division ring D. Denote by e_{ij} the matrix units of $M_n(D)$ where $1 \leqslant i, j \leqslant n$.

It is known [2, Chapter 14] that every tiled order is isomorphic to a tiled order of the following form $\Lambda = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$ where $\mathcal{O}$ is a discrete valuation ring with the prime element π, and $\mathcal{E} = (\alpha_{ij})$ is integer matrix of order $n \geqslant 1$ with zeros on the diagonal, for whose elements the inequality $\alpha_{ij} + \alpha_{jk} - \alpha_{ik} \geqslant 0$ takes place for all i, j, k, $1 \leqslant i, j, k \leqslant n$. These

2000 Mathematics Subject Classification: *05C50, 37C15, 37C25.*
Key words and phrases: *exponent matrix, finite orbits, topological equivalence.*

inequalities are called ring inequalities and matrix $\mathcal{E}$ is called exponent matrix.

A tiled order can be considered as a pair $(\mathcal{O}, \mathcal{E})$. Semiperfect ring Λ with Jacobson radical R is called reduced if the quotient ring Λ/R is a direct product of division rings [2]. Tiled order is reduced if and only if its exponent matrix has no symmetrical zeros. Such exponent matrix is called reduced.

A tiled order Λ is called a Gorenstein tiled order if $inj.dim_{\Lambda}\Lambda_{\Lambda} = 1$.

A reduced tiled order $\Lambda = (\mathcal{O}, \mathcal{E}(\Lambda) = (\alpha_{ij}))$ is Gorenstein if and only if there exists a permutation σ without fixed points such that for any numbers i, j the equality $\alpha_{ij} + \alpha_{j\sigma(i)} = a_{i\sigma(i)}$ take place.

A reduced exponent matrix such that for all i, j these equalities take place is called Gorenstein matrix.

Theorem 1. *[3, Theorem 7.1.1] Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a reduced prime Noetherian SPSD-ring Λ with the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij}) \in \mathbb{Z}$. Then $inj.dim_{\Lambda}\Lambda_{\Lambda} = 1$ if and only if matrix $\mathcal{E}(\Lambda)$ is Gorenstein. In this case $inj.dim_{\Lambda\Lambda}\Lambda = 1$.*

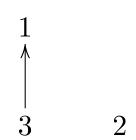
It is known that there in one to one correspondence between finite posets and (0, 1)-reduced exponent matrices of order n. This equivalence can be realized in the next way. Let $\Theta = (\theta_1, \ldots, \theta_n)$ be a finite poset of n elements. The equality $\alpha_{ij} = 0$ should be equivalent to inequality $\theta_i \geqslant \theta_j$. Ring inequalities for all indices give that this correspondence will be really one to one correspondence.

Let $F = \{f_1, \ldots, f_n\}$ be a finite linearly ordered set of n elements. Let ψ be real function defined on F. This function ψ induces one more relation $\leqslant$ on the set F by the rule $f_i \leqslant f_j$ if and only if $\psi(f_i) \leqslant \psi(f_j)$. The set F with this relation will be called doubly ordered set. For the doubly ordered set F correspond exponent matrix can be defined by the same rule as for a poset.

Let U_n be matrix of order n all whose elements are equal to 1 and E_n be identity matrix. Then for any order n exponent (0, 1) matrix $\mathcal{E}$, consider the matrix $\widetilde{\mathcal{E}} = U_n - \mathcal{E} - E_n$. This formula defines an adjacency matrix of the diagram of the finite partially ordered set Θ [3, p. 277].

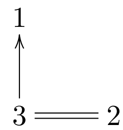
Note that for any poset its correspond matrix $\widetilde{\mathcal{E}}$ should not be necessarily an exponent matrix.

Example 1. Consider the poset of 3 elements, whose diagram is the next



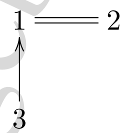
According to the definition, its exponent matrix is $\mathcal{E} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\widetilde{\mathcal{E}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Matrix $\widetilde{\mathcal{E}}$ is not exponent matrix because the inequality $\widetilde{\alpha}_{32} + \widetilde{\alpha}_{21} \geq \widetilde{\alpha}_{31}$ does not take place.

Example 2. Consider the doubly ordered set of 3 elements, whose diagram is the next



According to the definition, its exponent matrix is $\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\widetilde{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Matrix $\widetilde{\mathcal{E}}$ is an exponent matrix.

Example 3. Consider the doubly ordered set of 3 elements, whose diagram is the next



According to the definition, its exponent matrix is $\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\widetilde{\mathcal{E}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Matrix $\widetilde{\mathcal{E}}$ is an exponent matrix.

In our work we prove that for any doubly ordered set its correspond matrix $\widetilde{\mathcal{E}}$ is always an exponent matrix.

Maps f and g, which map interval $[0, 1]$ into itself, are named topological equivalent if they are conjugate with invertible map of this interval into itself.

Iterations of each each map f form a semigroup. It is known [5, 4] that if this semigroup is finite group then it is either trivial or is the group C_2, i.e. the group of order two. More then that, there exist numbers p, q, $0 \leqslant p \leqslant q \leqslant 1$, dependent on f, such that for every $x \in [0, 1]$ the inclusion $f(x) \in [p, q]$ takes place. Iterations group is trivial if and only if for any $x \in [p, q]$ the equality $f(x) = x$ takes place and this group is C_2 if and only if f decrease on $[p, q]$ and this interval is invariant under the action of this map. Up to the end of our paper all maps of interval will be considered as continuous whose iteration semigroup is finite group.

In this paper for the map f and its iterations group G we construct an (0, 1)-exponent matrix, which is a matrix of some doubly ordered set and has some additional properties, which can be easily described as in terms of matrices as in terms of doubly ordered sets. Any class of topological equivalent maps corresponds to the pair of matrices, each of them is a matrix of some doubly ordered set and one of them can be obtained from another by an easy rule. And conversely for any pair of matrices of doubly ordered sets such that one of them can be obtained from another by the rule mentioned above and such that satisfy some easily formulated restrictions, the conjugate class of the map f with iterations group G can be uniquely restored.

We also determine more explicit definition of the quiver of exponent matrix if it is given that it is a matrix of the equivalence class of continuous map.

Remind the notion of equivalence of exponent matrices, which appeared in their algebraic introduction.

Exponent matrices A, B are called equivalent, if one may be obtained from another by the following transformations:

1) adding an integer to all elements of some row with simultaneous subtracting it from the elements of the column with the same number.

2) simultaneous interchanging of two rows and equally numbered columns,

or by compositions of such transformations.

Lemma 1. *Under the transformations of the first type Gorenstein matrix goes to Gorenstein one with the same correspond permutation [1].*

If one consider (0, 1)-matrices $H_n = (h_{ij})$ such that $h_{ij} = 1$ if and only if $i > j$ and

$$G_{2m} = \left(\begin{array}{|c|c|} H_m & H_m^{(1)} \\ \hline H_m^{(1)} & H_m \end{array} \right)$$

where $H_m^{(1)} = E + H_m$, then it will be easy to see, that H_n is Gorenstein matrix with permutation $\sigma(H_n) = (n, n-1, \ldots 2, 1)$, and G_{2m} is Gorenstein one with permutation $\sigma(G_{2m}) = \prod_{i=1}^{m} (i, m+i)$.

Lemma 2. *Gorenstein* (0, 1) *matrix is equivalent to either matrix H_n or to G_{2m} [1].*

In our work we will show that Gorenstein matrix which is a matrix of some doubly order, not all whose elements are equal, is equivalent to H_n.

1. Doubly orders. For every doubly order $\{f_1, \ldots, f_n\}$ construct a (0, 1)-matrix $\mathcal{E} = (\alpha_{ij})$ of order n in the next way:

$$\alpha_{ij} = \begin{cases} 0 & f_i \geqslant f_j, \\ 1 & f_i < f_j. \end{cases}$$

Matrix $\mathcal{E}$ is exponent matrix without symmetrical 1. Really as it is (0, 1)-matrix, then inequality $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ can be violated only in the case when $\alpha_{ik} = 1$ and in the same time $\alpha_{ij} = \alpha_{jk} = 0$. From the construction of the matrix, equalities $\alpha_{ij} = \alpha_{jk} = 0$ mean that $f_i \geqslant f_j$ and $f_j \geqslant f_k$, which is $f_i \geqslant f_k$, whence $\alpha_{ik} = 0$, but not $\alpha_{ik} = 1$. Absence of symmetrical 1 is obvious.

Lemma 3. *Considering exponent matrix without symmetrical* 1*, constructed by the doubly order* $\{f_1, f_2\}$ *it is possible to restore unambiguously the correspondence "bigger-smaller" between symbols* f_1 *and* f_2*.*

Proof. Exponent (0, 1)-matrices of order 2 which have no symmetrical 1 can be found explicitly, and so we get

$$\mathcal{E}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \; \mathcal{E}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \; \mathcal{E}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For the matrix $\mathcal{E}_1$ obtain $f_1 \geqslant f_2$ and $f_2 \geqslant f_1$, whence $f_1 = f_2$.
For the matrix $\mathcal{E}_2$ obtain $f_1 \geqslant f_2$ and $f_2 < f_1$, whence $f_2 < f_1$.
For the matrix $\mathcal{E}_3$ obtain $f_1 < f_2$ and $f_2 \geqslant f_1$, whence $f_1 < f_2$. □

Using lemma 3, considering any exponent matrix $\mathcal{E}$, which has no symmetrical 1, we can construct a map A from the pair sets (i, j) to the set of letter expressions {"$f_i < f_j$", "$f_i = f_j$", "$f_i > f_j$"}, which will compare symbols f_i and f_j, with using the matrix

$$\begin{pmatrix} 0 & \alpha_{ij} \\ \alpha_{ji} & 0 \end{pmatrix},$$

where α_{ij} and α_{ji} are elements of the matrix $\mathcal{E}$. Determine some properties of the map $A(i, j)$.

1. *If* $A(m, t) =$ "$f_m = f_t$" *and in the same time* $A(t, k) =$ "$f_t = f_k$", *then* $A(m, k) =$ "$f_m = f_k$".

In other words, if $\alpha_{mt} = \alpha_{tm} = 0$ together with $\alpha_{tk} = \alpha_{kt} = 0$, then $\alpha_{mk} = \alpha_{km} = 0$. From the inequality $\alpha_{mt} + \alpha_{tk} \geqslant \alpha_{mk}$ and equalities

$\alpha_{mt} = \alpha_{tk} = 0$ obtain that $\alpha_{mk} = 0$. From the inequality $\alpha_{kt} + \alpha_{tm} \geqslant \alpha_{km}$ and equalities $\alpha_{kt} = \alpha_{tm} = 0$ obtain that $\alpha_{km} = 0$, which is necessary.

2. *If $A(m, t) =$ "$f_m = f_t$" and in the same time $A(t, k) =$ "$f_t < f_k$", then $A(m, k) =$ "$f_m < f_k$".*

In other words, if $\alpha_{mt} = \alpha_{tm} = 0$ and $\alpha_{tk} = 1$, then $\alpha_{mk} = 1$. consider the ring inequality $\alpha_{tm} + \alpha_{mk} \geqslant \alpha_{tk}$. As $\alpha_{tm} = 0$, and $\alpha_{tk} = 1$, then $\alpha_{mk} = 1$, which is necessary.

3. *If $A(m, t) =$ "$f_m = f_t$" and in the same time $A(t, k) =$ "$f_t > f_k$", then $A(m, k) =$ "$f_m > f_k$".*

In other words, if $\alpha_{mt} = \alpha_{tm} = 0$ and $\alpha_{kt} = 1$, then $\alpha_{km} = 1$. Consider ting inequality $\alpha_{km} + \alpha_{mt} \geqslant \alpha_{kt}$. As $\alpha_{mt} = 0$ and $\alpha_{kt} = 1$, then $\alpha_{km} = 1$.

4. *If $A(m, t) =$ "$f_m > f_t$" and in the same time $A(t, k) =$ "$f_t > f_k$", then $A(m, k) =$ "$f_m > f_k$".*

In other words, if $\alpha_{tm} = \alpha_{kt} = 1$, then $\alpha_{km} = 1$. As matrix $\mathcal{E}$ has no symmetrical 1, then the equality $\alpha_{kt} = 1$ holds $\alpha_{tk} = 0$. Consider the inequality $\alpha_{tk} + \alpha_{km} \geqslant \alpha_{tm}$. As $\alpha_{tm} = 1$ and $\alpha_{tk} = 0$, then $\alpha_{km} = 1$.

Lemma 4. *Considering exponent matrix without symmetrical 1, constructed by the doubly order $\{f_1, \dots f_n\}$ it is possible to restore unambiguously the correspondence "bigger-smaller" between elements of this doubly order.*

Proof. Prove this lemma by induction for n. Induction base for $n = 2$ holds from lemma 3.

Now, let symbols $f_1, \dots, f_{n-1}$ be renumbered in such a way that $f_{i_1} \leqslant f_{i_2} \leqslant \dots \leqslant f_{i_{n-1}}$. By the map $A(n, i_k)$, $k = 1, 2, \dots$ compare consequently f_n with f_{i_1}, f_{i_2} etc. till get the inequality $f_n < f_{i_1}$, or quality $f_n = f_{i_k}$, or pair of inequalities $f_{i_k} < f_n < f_{i_k+1}$, or inequality $f_n > f_{i_{n-1}}$. Doing in such a way, we will find the place of the symbol f_n in the set of symbols $f_{i_1}, \dots, f_{i_{n-1}}$. Unambiguousness of ordering holds from the construction and consistence holds from the properties $1, \dots, 4$ of the map A. □

Lemma 5. *For any exponent $(0, 1)$-matrix $\mathcal{E} = (\alpha_{ij})$ without symmetrical 1, matrix $\widetilde{\mathcal{E}} = (\widetilde{\alpha}_{ij})$ is reduced exponent $(0, 1)$-matrix.*

Proof. $\widetilde{\mathcal{E}}$ is reduced because of definitions. Check ring inequalities for this matrix.

As inequality $\widetilde{\alpha}_{ij} + \widetilde{\alpha}_{jk} \geqslant \widetilde{\alpha}_{ik}$ can be violated only in the case when $\widetilde{\alpha}_{ij} = \widetilde{\alpha}_{jk} = 0$ and $\widetilde{\alpha}_{ik} = 1$. As according to lemma 4 there is the unique doubly ordered set $\{f_1, \dots f_n\}$ with $\widetilde{\mathcal{E}}$ as correspond matrix then inequalities $\widetilde{\alpha}_{ij} = \widetilde{\alpha}_{jk} = 0$ means that $f_i \leqslant f_j \leqslant f_k$, whence $f_i \leqslant f_k$ and $\widetilde{\alpha}_{ik} = 0$. □

Theorem 2. *Let $\mathcal{E} = (\alpha_{ij})$ be such reduced exponent $(0, 1)$-matrix that matrix $\widetilde{\mathcal{E}}$ is also exponent. There is one to one correspondence between such matrices and doubly orders.*

Let h be invertible map, p_1, p_2, q_1 and q_2 be numbers such that $f([0, 1]) = [p_1, q_1]$, and $g([0, 1]) = [p_2, q_2]$. Let $a_1, \dots, a_n$ and $b_1, \dots b_m$ be extrema of maps f and g correspondingly. Points 0, 1, ends of each maximal interval of constantness of the map and each end of maximal interval of fixed points also consider as extrema.

2. Increase conjugate map for idempotent maps. Let f and g be idempotent increasing maps. Everywhere instead situations where this has to be proven consider the equality $g = h^{-1}(f(h))$ to take place.

Lemma 6. *The equalities $h(p_2) = p_1$ and $h(q_2) = q_1$ take place.*

Proof. Let $x_0 \in Fix(g)$. If $h(x_0) > q_1$ then $f(h(x_0)) < h(x_0)$, because $f(h(x_0)) \in f([0, 1]) = [p_1, q_1]$, and $h(x_0) > q_1$. As the map h increase, then h^{-1} also increase, whence $h^{-1}(f(h(x_0))) < h^{-1}(h(x_0))$. But the last equality means that $g(x_0) < x_0$, which contradicts to fact that x_0 is the fixed point of g. The case when $h(x_0) < p_1$ should be considered in the same way.

So, the image of the interval $[p_2, q_2]$ under acting of h is contained in $[p_1, q_1]$. As the equality $g = h^{-1}(f(h))$ holds $f = h(g(h^{-1}))$ then in the same way we can prove that image of the interval $[p_1, q_1]$ under the action of h^{-1} is contained in $[p_2, q_2]$. The last together with invertibleness of h proves lemma. □

Lemma 7. *$m = n$ and for any $i \in [1, n]$ the equality $h(b_i) = a_i$ takes place. Also for every $r, s \in [1, n]$ the inequality $g(b_r) \leqslant g(b_s)$ is equivalent to $f(a_r) \leqslant f(a_s)$.*

Proof. Really, this lemma holds from the fact that composition of increasing monotone functions is still increasing function.

Fix arbitrary number $i \in [1, n-1]$ and consider the monotonicity interval $[a_i, a_{i+1}]$ of the map f. Let f increase at this interval. Show that in this case the map g increase at $[h^{-1}(a_i), h^{-1}(a_{i+1})]$. For arbitrary $x_1, x_2 \in [a_1, a_{i+1}]$, $x_1 < x_2$, obtain $h(x_1) < h(x_2)$; as the map f increase at $[a_i, a_{i+1}]$ then $f(h(x_1)) < f(h(x_2))$, whence we get an inequality $h^{-1}(f(h(x_1))) < h^{-1}(f(h(x_2)))$, which shows that $[h^{-1}(a_i), h^{-1}(a_{i+1})]$ is increasing interval of g. If f decrease at $[a_i, a_{i+1}]$, then proving of decreasing of g on $[h^{-1}(a_i), h^{-1}(a_{i+1})]$ is almost similar.

Consider points $b_i = h^{-1}(a_i)$ for $i = 1, \dots, n$. Proving above gives that these points are extrema of the map g, and for any $i = 1, \dots, n$ the extremum character of b_i is the same as one of a_i.

Let for some $r,\, s \in [1,\, n]$ there is an inequality $g(b_r) \leqslant g(b_s)$ i.e. $h^{-1}(f(h(b_r))) \leqslant h^{-1}(f(h(b_s)))$. Taking h of both sides, using increasing of h, obtain $f(h(b_r)) \leqslant f(h(b_s))$. Using equalities $b_r = h^{-1}(a_r)$ and $b_s = h^{-1}(a_s)$ obtain $f(a_r) \leqslant f(a_s)$ which is necessary. □

Lemma 8. *Let $m = n$, for any $i \in [1,\, n]$ the equality $f(a_i) \leqslant f(a_j)$ is equivalent to $g(b_i) \leqslant g(b_j)$ and p_1 and p_2 have the same places in sets of extrema of these maps. Then maps f and g are conjugate.*

Proof. Let us construct the map h such that equality $g = h^{-1}(f(h))$ take place. Let the graph of h contains points $(g(b_i),\, f(a_i))$ for all i. Note that as $f(a_i) \leqslant f(a_j)$ is equivalent to $g(b_i) \leqslant g(b_j)$ then the restriction for h does not contradict to its monotonicity. More then that as $h(p_2) = p_1$ and $h(q_2) = q_1$ that this restriction is localized at $[p_2,\, q_2]$.

Let the map h be arbitrary increasing map on the interval $[p_2,\, q_2]$ whose graph contained reminded points.

Consider any number $i < n$ such that interval $[a_i,\, a_{i+1}]$ not to be interval $[p_1,\, q_1]$. Consider an arbitrary number $x_0 \in [b_i,\, b_{i+1}]$. Then the condition $g(x_0) = h^{-1}(f(h(x_0)))$ is equivalent to $h(g(x_0)) = f(h(x_0))$. As map h is already defined on the interval $[p_2,\, q_2]$ then the number $h(g(x_0))$ can be found. As $x_0 \in [b_i,\, b_{i+1}]$, then $h(x_0) \in [a_i,\, a_{i+1}]$. As the map f is monotone on the interval $[a_i,\, a_{i+1}]$, then it has locally inverse f_i, which is defined on the interval $(f(a_i),\, f(a_{i+1}))$, or on the interval $(f(a_{i+1}),\, f(a_i))$, dependent on increasing or decreasing of f on $[a_i,\, a_{i+1}]$. So, for the equality $h(g(x_0)) = f(h(x_0))$ to take place it is enough to have $h(x_0) = f_i^{-1}(h(g(x_0)))$.

For any $i = 1, \ldots, n$ define the map h on the interval $[b_i,\, b_{i+1}]$ by the formula $h = f_i^{-1}(h(g))$. From the constructing obtain that the map h defines conjugation of maps f and g. □

Using f construct the matrix $\widetilde{\mathcal{E}} = (\widetilde{\alpha}_{ij})$ of order n by the rule

$$\widetilde{\alpha}_{ij} = \begin{cases} 1 & f(a_i) \geqslant f(a_j) \text{ and } \quad i \neq j; \\ 0 & f(a_i) < f(a_j) \text{ or } \quad\;\; i = j. \end{cases}$$

Denote it $\widetilde{\mathcal{E}}(f(a_1), \ldots, f(a_n))$. It is (0, 1)-reduced exponent matrix and the matrix $\mathcal{E} = U_n - \widetilde{\mathcal{E}} - E_n$ is an exponent matrix.

As f is idempotent map then there should exist a number i such that $f(a_i) = \min\limits_k f(a_k)$ and $f(a_{i+1}) = \max\limits_k f(a_k)$. This condition can be easily reformulated in terms of matrix $\widetilde{\mathcal{E}}$.

Using the theorem 2, we can formulate the next theorem.

Theorem 3. *Consider the set of pairs $(\mathcal{E}, i_0)$, where $\mathcal{E}$ is exponent matrix and $\widetilde{\mathcal{E}}$ is reduced exponent such that conditions necessary condition mentioned above is satisfied. There is one to one correspondence between such pairs and equivalent classes of maps (conjugated by increasing map), whose iterations group is trivial.*

3. Increase conjugate map for generators of C_2. Let maps f and g be generators of the group C_2 and h be increasing map.

Lemma 9. *Let $g = h^{-1}(f(h))$. Then $h(p_2) = p_1$ and $h(q_2) = q_1$.*

Proof. The inequality $g = h^{-1}(f(h))$ holds $g^2 = h^{-1}(f^2(h))$. Now lemma is a corollary of lemma 6. □

Lemma 10. *Let f and g be two decreasing maps of interval such that $f^2 = id$ and $g^2 = id$. Then they are conjugate by increasing map.*

Proof. Let x_0^f and x_0^g be fixed points of maps f and g correspondingly. Construct new maps f_1 and g_1 by the next rule: $f_1(x) = f(x)$ for $x \leqslant x_0^f$ and $f_1(x) = x$ for $x > x_0^f$; also $g_1(x) = g(x)$ for $x \leqslant x_0^g$ and $g_1(x) = x$ for $x > x_0^g$.

Construct increasing map h_1, which realizes conjugateness of f_1 and g_1. Let the graph of h contains the point (x_0^g, x_0^f) and be defined arbitrary on the interval $[x_0^g, 1]$.

Then for an arbitrary $x \in [x_0^g, 1]$ we have $h(x) \in [x_0^f, 1]$, whence $f_1(h(x)) = h(x)$, which means that $h^{-1}(f_1(h(x))) = h^{-1}(h(x)) = x$, and so $g_1 = h^{-1}(f_1(h))$ on the interval $[x_0^f, 1]$. As g_1 is monotone on the interval $[0, x_0^g]$, then it is enough to take $h = f_1^{-1}(h(g_1))$ on the interval $[0, x_0^g]$, for conjugateness of f_1 and g_1. Here map h in the righthand side is already defined, because $g_1^{-1}([0, x_0^g]) = [x_0^g, 1]$.

Let us show that map h which has been constructed in such a way really realizes the conjugateness of map f and g. the fact that equality $g(x) = h^{-1}(f(h(x)))$ takes place for every $x \in [0, x_0^g]$ holds from construction. As there are compositions of monotone functions at the righthand side of the equality $g_1(x) = h^{-1}(f_1(h(x)))$ then it is possible to write the equality of inverse functions and get the equality $g_1^{-1}(x) = h^{-1}(f_1^{-1}(h(x)))$ for $x \in [0, x_0^g]$. Considering equalities $f^2 = id$ and $g^2 = id$ we have $g_1^{-1}(x) = g(x)$ and $f_1^{-1}(h(x)) = f(h(x))$ for $x \in [x_0^g, 1]$, which means that the equality $g = h^{-1}(f(h))$ s correct at the whole interval $[0, 1]$. The last finishes the proof of the lemma. □

Lemma 11. *Let $g = h^{-1}(f(h))$. Then for arbitrary $i \in [1, n]$ the equality $h(b_i) = a_i$ take place. Also the inequality $g(b_r) \leqslant g(b_s)$ is equivalent to $f(a_r) \leqslant f(a_s)$ for all $r, s \in [1, n]$.*

Proof. The proof of this lemma is exactly the same as of lemma 7. □

For any functions w with extrema $v_1, \dots, v_n$ denote

$$\mathcal{A}'(w) = \widetilde{\mathcal{E}}(w(v_1), w^2(v_1), \dots, w(v_n), w^2(v_n)).$$

Lemma 12. *Let the equality $g = h^{-1}(f(h))$ to take place for maps f and g. Then matrices $\mathcal{A}'(f)$ and $\mathcal{A}'(g)$ coincide.*

Proof. Plug the value $x = g(b_i)$ into equality $g(x) = h^{-1}(f(h(x)))$ and get $g^2(b_i) = h^{-1}(f(h(g(b_i))))$. As according to lemma 8 the graph of the map h contains points $(g(b_i),\ f(a_i))$ for all $i \in [1,\ n]$, then the obtained equality is equivalent to $g^2(b_i) = h^{-1}(f(f(a_i)))$. Plugging left hand and righthand side of obtained equality into function h get an equality $h(g^2(b_i)) = f^2(a_i)$, which means that the graph of function h contains points $(g^2(b_i),\ f^2(a_i))$ for all $i \in [1,\ n]$. From this fact, from increasing of h and from the fact that points $(g(b_i),\ f(a_i))$ for all $i \in [1,\ n]$ belong to the graph of h, obtain the correctness of lemma. □

Lemma 13. *Maps f and g are conjugate if and only if $m = n$, numbers of extrema which are ends of intervals $f([0,\ 1])$ and $g([0,\ 1])$, and matrices $\mathcal{A}'(f)$ and $\mathcal{A}'(g)$ coincide.*

Proof. It is proven in the previous lemma that matrices of conjugated maps are equal. So the only think we have to prove is that inverse proposition is still correct.

Let for maps f and g matrices $\mathcal{A}'(f)$ and $\mathcal{A}'(g)$ coincide. Let's show that these maps are conjugate by increasing function.

From he equality $\mathcal{A}'(f) = \mathcal{A}'(g)$ obtain that there exists continuous map $h_1 : [p_2,\ q_2] \longrightarrow [p_1,\ q_1]$ which realizes conjugateness of maps f and g on their periodic points and whose graph contains points $(g(b_i),\ f(a_i))$ and $(g^2(b_i),\ f^2(a_i))$ for all $i \in [1, \dots n]$. This map is constructed analogously by one from lemma 10.

The map h such that $g = h^{-1}(f(h))$ on the set $[0,\ 1] \setminus [p_2,\ q_2]$ should be found in the same way as in the proof of the lemma 8. □

As for the map f the equality $f = f^3$ take place then there should exist a number i such that $f(a_i) = f^2(a_i)$, $f(a_{i-1}) = \max\limits_k f(a_k)$ and $a_{i+1} = \min\limits_k f(a_k)$. Also $f(a_{i-1}) = f^2(a_{i+1})$, $f(a_{i+1}) = f^2(a_{i-1})$ and for all i, j such that $f(a_i) \leqslant f(a_j)$ the inequality $f^2(a_i) \leqslant f^2(a_j)$ takes place. This condition can be easily reformulated in terms of matrix $\mathcal{E}'(f)$, which gives us the next theorem.

Theorem 4. *Consider the set of pairs $(\mathcal{E}, i)$, where $\mathcal{E}$ is exponent matrix such that matrix $\widetilde{\mathcal{E}}$ is reduced exponent such that the necessary condition mentioned above is satisfied. There is one to one correspondence between such pairs and equivalent classes of maps (conjugated by increasing map), whose iterations group is nontrivial.*

4. Decreasing conjugate map. Let us investigate the changing of exponent matrix of the map under conjugating by decreasing map.

As any arbitrary decreasing map h can be presented in the form $h(x) = 1 - h_1(x)$ where $h_1(x) = 1 - h(x)$ in increasing map, then conjugating by the map h is a composition of conjugating by the map $1 - x$ and increasing one. As it is proven that increasing map does not change the matrix of the map then is is enough to investigate this changing if we have conjugation by the map $1 - x$.

The action of conjugating by the map $1 - x$ on the graph f we can understand as consequent symmetrical reflecting in the line $x = 1/2$ and symmetrical reflecting in the line $y = 1/2$. The first means acting by permutation $(1, n)(2, n-1)\cdots$ or $(1, n-1)(2, n)(3, n-3)(4, n-2)\cdots$ on lines and columns of the matrix $\mathcal{E}'$ dependently on if the iterations group is trivial or nontrivial.

Symmetrical reflecting of the graph in the line $y = 1/2$ moves each inequality $f^{\alpha_i}(a_i) \leqslant f^{\alpha_j}(a_j)$, where $\alpha_i, \alpha_j \in \{1, 2\}$ to $f^{\alpha_j}(a_j) \leqslant f^{\alpha_i}(a_i)$. For the exponent matrices it means that this mappings $A \longrightarrow B$ is defined in the next way:

1) for any numbers i, j equality $a_{ij} = a_{ji} = 1$, holds $b_{ij} = b_{ji} = 1$;

2) for any different i, j the equality $a_{ij} = 0$ holds $b_{ij} = 1$ and $b_{ji} = 0$;

3) $b_{ii} = 0$ for all i.

Here for both trivial and nontrivial iterations group case the number i_0 from theorems 3 and 4 goes to $n - i_0 + 1$.

So, any class of conjugate maps corresponds to pair of exponent matrices of doubly ordered set each of them are image of another in the self invertible action, mentioned just above. These matrices should have obvious additional properties, dependent on the iteration group to make correspond doubly ordered set to be a set of extrema of necessary map.

5. The quiver of a map with finite group of iterations. Let matrix $\mathcal{E} = \widetilde{\mathcal{E}}(b_1, \ldots, b_m) = (\widetilde{\alpha}_{ij})$ be constructed by the rule

$$\widetilde{\alpha}_{ij} = \begin{cases} 1 & b_i \geqslant b_j \text{ and } \quad i \neq j; \\ 0 & b_i < b_j \text{ or } \quad\quad i = j. \end{cases}$$

$\mathcal{E}$ is reduced exponent matrix. Remind how to construct the quiver of such matrix [1]. First, construct the matrix $\mathcal{E}^{(1)} = (\widetilde{\beta}_{ij}) = \mathcal{E} + E$, i.e.

$$\widetilde{\beta}_{ij} = \begin{cases} 1 & b_i \geqslant b_j; \\ 0 & b_i < b_j. \end{cases}$$

Considering matrix $\mathcal{E}^{(1)}$ construct the matrix $\mathcal{E}^{(2)} = (\widetilde{\gamma}_{ij})$ defined by the rule $\widetilde{\gamma}_{ij} = \min\limits_k(\widetilde{\beta}_{ik} + \widetilde{\beta}_{kj})$. Consider matrix $\mathcal{Q} = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ as adjacency matrix of a quiver, which we name a quiver, which corresponds to the map f.

Let us understand, when the quiver has a loop at the vertex i. As $\widetilde{\beta}_{ii} = 1$ then for the loop equalities $\widetilde{\beta}_{ik} = \widetilde{\beta}_{ki} = 1$ have to take place for all k. It is possible but only in the case when $b_1 = \ldots = b_m$. It is obvious that in this case all the vertexes of the quiver have one loop each and there is exactly one arrow from each vertex to each.

Lemma 14. $\widetilde{\gamma}_{ij} = 0$ *if and only if there exists* k*, such that* $b_i < b_k < b_j$*.*

Proof. According to definition of $\widetilde{\gamma}_{ij}$ we get that $\widetilde{\gamma}_{ij} = 0$ if and only if there exists a k such that $\widetilde{\beta}_{ik} = \widetilde{\beta}_{kj} = 0$. From the construction of $\mathcal{E}^{(1)}$ it means that $b_i < b_k < b_j$, which is necessary.

Its obvious that if $b_i < b_k < b_j$ for some k, then $\widetilde{\gamma}_{ij} = 0$. □

Lemma 15. $\widetilde{\gamma}_{ij} = 2$*, if and only if* $b_i = \max\limits_k b_k$*, and* $b_j = \min\limits_k b_k$*.*

Proof. If $\widetilde{\gamma}_{ij} = 2$, then $\widetilde{\beta}_{ik} = \widetilde{\beta}_{kj} = 1$ for all k, in other words $b_i \geqslant b_k \geqslant b_j$ for all k whence $b_i = \max\limits_k b_k$, and $b_j = \min\limits_k b_k$.

It is obvious that if $b_i = \max\limits_k b_k$, and $b_j = \min\limits_k b_k$ then $\widetilde{\gamma}_{ij} = 2$. □

Lemma 16. $\widetilde{\gamma}_{ij} = 1$ *and* $\widetilde{\beta}_{ij} = 0$ *if and only if* $b_j > b_i$ *and there is no any* b_k *between them.*

Proof. Let $\widetilde{\gamma}_{ij} = 1$ and $\widetilde{\beta}_{ij} = 0$. The equality $\widetilde{\gamma}_{ij} = 1$ holds that if for some k the equality $\widetilde{\beta}_{kj} = 0$ takes place then $\widetilde{\beta}_{ik} = 1$. From the constructing of the matrix $\mathcal{E}^{(1)}$ obtain that for all k such that $b_k < b_j$ the inequality $b_k \leqslant b_i$ takes place. This means that $b_j \leqslant b_i$, or $b_j > b_i$, but there is no b_k between them.

As $\widetilde{\beta}_{ij} = 0$ then $b_i < b_j$ which contradicts to the case when $b_j \leqslant b_i$.

Let $b_j > b_i$, but there is no any b_k between them. Show that in this case $\widetilde{\gamma}_{ij} = 1$ and $\widetilde{\beta}_{ij} = 0$. According to lemmas 14 and 15 obtain that equivalent conditions for b_j and b_i to the quality $\widetilde{\gamma}_{ij} = 0$ or $\widetilde{\gamma}_{ij} = 2$ are braked. That is why $\widetilde{\gamma}_{ij} = 1$. As $b_i < b_j$ then $\widetilde{\beta}_{ij} = 0$ which is necessary. □

Using proven lemmas we can rewrite the definition the quiver of f: $q_{ij} = 1$ if and only if $b_i = \max\limits_k b_k$, and $b_j = \min\limits_k b_k$, or $b_j > b_i$ but there is no any b_k between them, where max and min are considered in not strict meaning.

Describe the quiver constructed above. Let $b_{i_1} < b_{i_2} < \ldots < b_{i_k}$ be such numbers that $\{b_{i_1}, \ldots, b_{i_k}\} = \{b_1, \ldots, b_m\}$. From the construction we get that the quiver has cycle $b_{i_1} \to b_{i_2} \to \ldots \to b_{i_k} \to b_{i_1}$.

Let for some numbers j, t the equality $b_j = b_{i_t}$ takes place. From the construction of the quiver obtain that vertexes b_j are b_{i_t} connected with all others arrows in the same way. More then that, there is an arrow which connects them if and only if $b_1 = \ldots = b_m$. So, we have got that the quiver of the map f is inflation of the cycle, which s defined in the next way:

Definition 1. *1. The quiver Γ_2 is an inflation of the vertex v of the quiver Γ_1, if the former is obtained from the last by adding the vertex $\widetilde{v}$, which is connected with all another vertexes in the seme way as it was for v in Γ_1. Vertexes v and $\widetilde{v}$ are connected with arrow if and only if there was a loop at v in Γ_1. In this case $\widetilde{v}$ also has a loop.*

2. The quiver Γ_2 is a inflation of Γ_1 if it can be obtained from the last by some number of sequent inflations of vertexes.

6. Gorenstein matrix of the map. Like in previous section let matrix $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(b_1, \ldots, b_m) = (\widetilde{\alpha}_{ij})$ be constructed by the rule

$$\widetilde{\alpha}_{ij} = \begin{cases} 1 & b_i \geqslant b_j \text{ and } \quad i \neq j; \\ 0 & b_i < b_j \text{ or } \quad\;\; i = j. \end{cases}$$

As it was mentioned matrix $\widetilde{\mathcal{E}}$ is reduced exponent matrix. Let us study conditions under which this matrix will be Gorenstein matrix.

Lemma 17. *If matrix $\widetilde{\mathcal{E}}$ is Gorenstein with permutation σ then all numbers $b_1, \ldots, b_m$ are different.*

Proof. Let for different numbers i, j, the equality $b_i = b_j$ takes place. Then $\widetilde{\alpha}_{ij} = 1$. Next, as $b_i = b_j$ then independently on the permutation σ, the inequality $b_i < b_{\sigma(i)}$ and $b_j < b_{\sigma(i)}$ take place or no simultaneously, which means that $\widetilde{\alpha}_{ij} + \widetilde{\alpha}_{j\sigma(i)} = \widetilde{\alpha}_{i\sigma(i)} + 1$, and means that matrix $\widetilde{\mathcal{E}}$ is not Gorenstein. □

Theorem 5. *If all numbers $b_1, \ldots, b_m$ are different then matrix $\widetilde{\mathcal{E}}$ is Gorenstein.*

Proof. Using lemma 1 we may assume that $b_1 < b_2 < \dots b_m$. Consider the permutation $\sigma = (m, m-1, \dots, 1)$ and show that matrix $\widetilde{\mathcal{E}}$ is Gorenstein with σ as correspond permutation.

Consider the equality $\widetilde{\alpha}_{ij} + \widetilde{\alpha}_{j\sigma(i)} = \widetilde{\alpha}_{i\sigma(i)}$. Let $i > 1$. then this equality can be rewritten as $\widetilde{\alpha}_{ij} + \widetilde{\alpha}_{j,i-1} = \widetilde{\alpha}_{i,i-1}$. As $b_i > b_{i-1}$ then $\widetilde{\alpha}_{i,i-1} = 1$. If $j \neq i$ and $j \neq i-1$ then, obviously, $\widetilde{\alpha}_{ij} + \widetilde{\alpha}_{j,i-1} = 1$, which is necessary. Otherwise if $j = i$ then $\widetilde{\alpha}_{ij} = 0$ and $\widetilde{\alpha}_{j,i-1} = \widetilde{\alpha}_{i,i-1}$ and if $j = i-1$ then $\widetilde{\alpha}_{j,i-1} = 0$ and $\widetilde{\alpha}_{i,j} = \widetilde{\alpha}_{i,i-1}$ which is necessary.

Now let $i = 1$. Then for any j the equality $\widetilde{\alpha}_{i,j} = \widetilde{\alpha}_{i,\sigma(i)} = 0$ takes place because b_1 is minimal element of the set $\{b_1, \dots, b_m\}$. As $\sigma(1) = m$ then for any j the equality $\widetilde{\alpha}_{j,\sigma(i)} = 0$ takes place, which means that matrix $\widetilde{\mathcal{E}}$ is really Gorenstein matrix. □

Note that according to the proved theorem and lemma 1 if the matrix of a doubly ordered set is Gorenstein matrix then this matrix is H_n.

References

[1] Dokuchaev M.A., Kirichenko V.V., Zelensky A.V. and Zhuravlev V.N., *Gorenstein Matrices*, Algebra and discrete mathematics. – 2005 No 1. pp. 8–29.

[2] M. Hazewinkel, N. Gubareni and V.V. Kirichenko, *Algebras, Rings and Modules*, MIA, 575, Vol. 1, Kluwer Academic Publishers, 2004, 380p.

[3] M. Hazewinkel, N. Gubareni and V.V. Kirichenko, *Algebras, Rings and Modules*, MIA, 586, Vol. 2, Springer, 2007, 400p.

[4] Pelukh G.P., Sharkovskii A.N., *Introduction to the theory of functional equations*, Kyiv: Naukova Dumka, 1974, 119 p.

[5] Plakhotnyk M.V., Fedorenko V.V. and Fedorenko Y.V. *One-dimensional dynamical systems, whose cardinalities of orbits are uniformly bounded*, Bulletin of the University of Kyiv, No 4. pp. 119–128.

Contact information

V. Fedorenko Department of dynamical systems of the Mathematical institute NASU Tereshchenkivska str., 3, Kyiv, Ukraine.

V. Kirichenko, M. Plakhotnyk Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko Univ., Volodymyrska str., 64, 01033 Kyiv, Ukraine.

Received by the editors: 25.02.2008
and in final form 10.04.2008.