Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 3. **(2007).** pp. 59 – 66

There isn't much duality in radical theory

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Communicated by M. Ya. Komarnytskyj

Abstract. The definitions of radical and semi-simple classes are in a natural sense dual to each other. However, statements dual in the same sense to theorems of radical theory tend to be false. Some insights may nevertheless be gained from consideration of duality, and we illustrate this with a link between additive radicals and semi-simple radical classes.

1. Introduction

Radical classes $\mathcal{R}$ are characterized by the following conditions which are converse to each other.

(R1) If $A \in \mathcal{R}$ then for each non-zero homomorphic image A'' of A there exists a non-zero $C \in \mathcal{R}$ with $C \triangleleft A''$.

(R2) If A is such that for every non-zero homomorphic image A'' there there is a non-zero $C \in \mathcal{R}$ with $C \triangleleft A''$, then $A \in \mathcal{R}$.

Semi-simple classes $\mathcal{S}$ are characterized by the following mutually converse properties.

(S1) If $B \in \mathcal{S}$, then if $0 \neq I \triangleleft B$, I has a non-zero homomorphic image $I'' \in \mathcal{S}$.

(S2) If B is such that whenever $0 \neq I \triangleleft B$, I has a non-zero homomorphic image $I'' \in \mathcal{S}$, then $B \in \mathcal{S}$.

It would be neater to replace (S2) by the condition "$\mathcal{S}$ is hereditary", but semi-simple classes in some contexts do not have this property. We can always replace (R1) by the condition "$\mathcal{R}$ is homomorphically closed", but (R1) is frequently used because of the resultant symmetry in the

2000 Mathematics Subject Classification: *16N80, 16S90, 18E40.*
Key words and phrases: *radical class, semi-simple class, duality.*

characterizations of the two types of class. This is suggestive of a "duality" in radical theory resulting from the following interchanges

$$\text{radical class} \longleftrightarrow \text{semi-simple class}$$
$$\text{homomorphic image} \longleftrightarrow \text{ideal.}$$

As being an ideal is not transitive, unlike being a homomorphic image, one might replace "ideal" by "accessible subring", or just possibly "subring". We shall consider these variations in the course of our discussion, but our main contention is that even in universal classes with very well-behaved radical theories, generally speaking this duality does not extend far beyond the characterizations of radical and semi-simple classes. We are able to give a couple of exceptions, however. We shall consider a number of results, formulate their duals and (generally) show that the latter are false.

To keep the discussion manageable we shall take associative rings as our usual universal class and cite results from other classes for comparison.

2. Results with false duals

2.1. *Every homomorphically closed semi-simple class is a radical class.*

The first appearance of this result was probably in [20]. Since both radical and semi-simple classes are closed under extensions, it is a consequence of a result of Kogalovskii that classes closed under subdirect products and homomorphic images are varieties. A dual of 2.1. would assert that every hereditary (or perhaps strongly hereditary) radical class is a semi-simple class. Both statements are obviously false. The failure of duality here is widespread: 2.1. is true everywhere in orthodox radical theory, but its dual is rarely true even in varieties of modules.

For a semi-simple class $\mathcal{S}$ let $\hat{\mathcal{S}}$ denote the class $\{A : A/I \in \mathcal{S} \forall I \triangleleft A\}$; we call the members of $\hat{\mathcal{S}}$ *strongly semi-simple* [2].

2.2. *If $\mathcal{S}$ corresponds to a hereditary radical class, then $\hat{\mathcal{S}}$ is a radical class.*

For a radical class $\mathcal{R}$, let

$$\tilde{\mathcal{R}} = \{A : \textit{every accessible subring of } A \textit{ is in } \mathcal{R}\}.$$

This may be considered a dual of $\hat{\mathcal{S}}$. If $\mathcal{R}$ has a homomorphically closed semi-simple class, then $\mathcal{R}$ is hereditary, i.e. $\mathcal{R} = \tilde{\mathcal{R}}$. But $\mathcal{R}$ is supernilpotent, and so (except when it is the class of all rings) not a semi-simple radical class. In particular it is not a semi-simple class, so *the dual of 2.2. is false.*

2.3. *For every radical class $\mathcal{R}$ the class $\tilde{\mathcal{R}}$ is also radical.*

(See [16]. In fact $\tilde{\mathcal{R}} = \{A : I \triangleleft J \triangleleft A \Rightarrow I \in \mathcal{R}\}$. For associative rings, even the class $\{A : I \triangleleft A \Rightarrow I \in \mathcal{R}$ is a radical class [16].) Now for a semi-simple class $\mathcal{S}$, $\hat{\mathcal{S}}$ need not be a semi-simple class: being homomorphically closed, it needs to be a semi-simple radical class to be semi-simple. Thus *the dual of 2.3. is false*, even if we substitute "ideal" for "accessible subring" in 2.3. Using "subring" we get another undualisable result.

2.4. *For every radical class $\mathcal{R}$ the class*

$$\{A : \text{all subrings of } A \textit{ are in } \mathcal{R}\}$$

is a radical class. [18]

The following result does not seem to have appeared in print. The proof makes use of fairly standard arguments (see Proposition 7.4, pp 101-102 of [7]), but we include it for the sake of completeness. The result also follows from Theorem 3.5, p.171 of [7].

2.5. *The homomorphic closure $\bar{\mathcal{S}}$ of a semi-simple class S is the variety generated by S.*

Proof. We first show that $\bar{\mathcal{S}}$ is a variety. If $B_\lambda \in \hat{\mathcal{S}}$ for each $\lambda \in \Lambda$ then for each λ there exists an $A_\lambda \in \mathcal{S}$ and a surjective homomorphism $f_\lambda : A_\lambda \to B_\lambda$. These induce a surjective homomorphism $\prod A_\lambda \to \prod B_\lambda$, so $\prod B_\lambda \in \bar{\mathcal{S}}$. Now let B be in $\bar{\mathcal{S}}$ have a subring C. Let $g : A \to B$ be a surjective homomorphism with $A \in \mathcal{S}$, and let D be the inverse image of C under g. For $A = A_1 = A_2 = \dots$ consider the ring

$$R = \{(a_n) \in \prod A_n : \exists m \in \mathbb{Z}^+, d \in D; a_n = d \forall n \geq m\}.$$

Clearly R is a subdirect product of the A_n, so $R \in \mathcal{S}$. We get a surjective homomorphism from R to C by prescribing $(a_1, a_2, \dots, a_{m-1}, d, d, \dots) \mapsto g(d)$. Thus C is in $\hat{\mathcal{S}}$.

Now $\mathcal{S} \subseteq \bar{\mathcal{S}}$ so $\bar{\mathcal{S}}$ contains the variety generated by $\mathcal{S}$. But $\mathcal{S} = P(\mathcal{S}) \subseteq SP\{\mathcal{S})$, so $\bar{\mathcal{S}} = H(\mathcal{S}) \subseteq HSP(\mathcal{S})$ and by Tarski's Theorem [5], $HSP(\mathcal{S})$ is the variety generated by $\mathcal{S}$. □

Hué and Szász [15] showed that the homomorphic closures of some semi-simple classes contain all rings. We note that 1.5. is valid in every variety of multioperator groups. (The same proof works.) A special case of 2.5. is the result that homomorphically closed semi-simple classes are varieties. There are plenty of examples of semi-simple classes $\mathcal{S}$ for which $\bar{\mathcal{S}}$ is neither $\mathcal{S}$ nor the class of all rings. In [12] it was shown that for every variety $\mathcal{V}$ of rings the class $\breve{\mathcal{V}}$ of *semiprime* rings in $\mathcal{V}$ is a semi-simple class, so $\bar{\breve{\mathcal{V}}} \subseteq \mathcal{V}$.

It is not so clear what we might ask of duality in the case of 2.5., as it doesn't have a "radical conclusion". On the other hand, a variety is a rather self-dual thing. The class dual to $\bar{\mathcal{S}}$ is, for a radical class $\mathcal{R}$, the *hereditary closure*, $\mathcal{R}_h = \{A : A$ *is an accessible subring of a ring in* $\mathcal{R}\}$. The nature of $\mathcal{R}_h$ is rather unclear. However, there seems little prospect of dualizing 2.5. as it is possible that $\mathcal{R}$ is closed under neither subrings nor direct products.

Example. Let $\mathcal{R}$ be the lower radical class defined by the ring $\mathbb{G}$ of Gaussian integers. Then $\mathbb{Z} \subseteq \mathbb{G} \in \mathcal{R} \subseteq \mathcal{R}_h$, but as it has an identity, $\mathbb{Z}$ could only be in $\mathcal{R}_h$ if it were in $\mathcal{R}$. If $f : \mathbb{G} \to \mathbb{Z}$ is a homomorphism, then $f(1) = 1$ or 0. But if $f(1) = 1$ then $f(i)^2 = f(i^2) = f(-1) = -f(1) = -1$ and this can't happen in $\mathbb{Z}$, so we must have $f(1) = 0$, whence $f = 0$. Hence $\mathbb{Z} \notin \mathcal{R}$, so it isn't in $\mathcal{R}_h$, which accordingly is not closed under subrings. Let $A_p = \mathbb{G}/p\mathbb{G}$ for every prime p. Then $A_p \in \mathcal{R} \subseteq \mathcal{R}_h$ for each p. Since $\prod A_p$ has an identity it is not in $\mathcal{R}_h$ if it is not in $\mathcal{R}$. Suppose, therefore, it *is* in $\mathcal{R}$. Then also $\prod A_p / \bigoplus A_p \in \mathcal{R}$. But additively the latter ring is torsion-free and divisible, while as a member of $\mathcal{R}$ it has an ideal and therefore a direct summand which is a homomorphic image of $\mathbb{G}$ and therefore additively torsion or reduced. From this contradiction we conclude that $\mathcal{R}_h$ is not closed under direct products.

Whether $\mathcal{R}_h$ is homomorphically closed, in the case of our example or generally, is not known.

Problem. Describe or find closure properties of the hereditary closure of a radical class.

2.6. *If a class* $\mathcal{M}$ *is hereditary, so is its lower radical class* $L(\mathcal{M})$.

This was first proved by Hoffman and Leavitt [14] and is true in all varieties of multioperator groups. There are situations where the result is false, however: hausdorff topological associative rings (Arnautov [3]), compact hausdorff abelian groups ([11], pp.76-77). Note that the compact hausdorff abelian groups form an abelian category.

The dual of 2.6. would be the claim that homomorphically closed classes generate homomorphically closed semi-simple classes (i.e. semisimple radical classes in the case of rings (and more generally)), but this happens rarely. Consider, for example the semi-simple classes generated by simple rings.

3. A little duality

A dramatic contrast with associative rings is provided by modules over perfect rings. In what follows **"module" means "unital left module"**.

It was first shown by Alin and Armendariz [1] that every hereditary radical class of modules over a *right* perfect ring is closed under direct products and is therefore a semi-simple class. If $\mathcal{R}$ is any radical class of modules over a right perfect ring, then $\tilde{\mathcal{R}}$ is a hereditary radical class and hence a semi-simple class. Thus *for modules over a right perfect ring, the duals of* 2.1. *and* 2.2. *are valid, as are* 2.1. *and* 2.2. *themselves.*

It was proved in Dickson's thesis [8] (and a proof is given in [9]) that a semi-simple class of modules over a *left* perfect ring is homomorphically closed if and only if its radical class is closed under projective covers. The dual of 2.6 ([9], Corollary 4) follows from this: *the dual of* 2.6 *is valid for modules over a left perfect ring, as is* 2.6. *itself.*

Thus for modules over *left and right* perfect (e.g. artinian) rings, 2.1., 2.2., 2.6. and their duals all hold. The category of modules over left and right perfect rings is particularly symmetric in its radical theory. A notable feature is that each semi-simple radical class is generated both as a radical class and as a semi-simple class by the simple modules it contains. [9] Maybe there is further duality there.

Problem. Investigate radical duality for modules over left and right perfect rings.

For fully ordered abelian groups both 2.6. and its dual are true, but neither 2.1. nor its dual is true. [10] Chehata and Wiegandt [6] showed that both 2.1. and its dual are false also for fully ordered groups (not necessarily abelian). These two universal classes are not varieties and some of the results of §2 are false, so some of the "evidence" from §2 is not relevant to the duality question here.

4. New results based on duality

Though radical theory for associative rings doesn't have much duality, nevertheless we can use duality to get interesting questions and the ways in which duality fails may produce interesting results.

For a radical class $\mathcal{R}$ and a ring A, let

$$A(\mathcal{R}) = \bigcap\{I : I \triangleleft A \text{ and } A/I \in \mathcal{R}\}.$$

We'll consider the condition

$$A/A(\mathcal{R}) \in \mathcal{R} \text{ for all } A \tag{*}$$

The following will be familiar to many.

Proposition 4.1. *A radical class $\mathcal{R}$ satisfies* (*) *if and only if it is closed under subdirect products.*

Proof. If $\mathcal{R}$ is closed under subdirect products, then if $A/I \in \mathcal{R}$ we have $(A/A(\mathcal{R})/(I/(A(\mathcal{R})) \cong A/I \in \mathcal{R}$ and $\bigcap\{I/A(\mathcal{R}) : A/I \in \mathcal{R}\} = 0$ so $A/A(\mathcal{R}) \in \mathcal{R}$. Conversely, if (*) holds and a ring A has ideals $I_\lambda, \lambda \in \Lambda$ with $\bigcap I_\lambda = 0$ and each $A/I_\lambda \in \mathcal{R}$, then $A(\mathcal{R}) \subseteq \bigcap I_\lambda = 0$ so $A \cong A/0 = A/A(\mathcal{R}) \in \mathcal{R}$. Thus $\mathcal{R}$ is closed under subdirect products. □

Condition (*) is more familiarly associated with semi-simple classes than radical classes. A dual of (*) for a semi-simple class $\mathcal{S}$ (though the condition is more usually associated with radical classes) is

$$\mathcal{S}(A) := \sum \{I \triangleleft A : I \in \mathcal{S}\} \in \mathcal{S} \forall A \qquad (\dagger)$$

Proposition 4.2. *A semi-simple class $\mathcal{S}$ satisfies (†) if and only if*
(i) whenever $I, J \triangleleft A$ and $I, J \in \mathcal{S}$ we have $I + J \in \mathcal{S}$ and
(ii) unions of chains of ideals from $\mathcal{S}$ are in $\mathcal{S}$.

Proof. If $\mathcal{S}(A) \in \mathcal{S}$ then every sum of ideals in $\mathcal{S}$, as an ideal of $\mathcal{S}(A)$, is in $\mathcal{S}$. This gives us (i) and (ii) as special cases. Conversely, if (i) and (ii) are satisfied then by Zorn's Lemma and (ii) a given A has an ideal M which is maximal with respect to membership of $\mathcal{S}$. Then by (i) for every ideal L of A which is in $\mathcal{S}$ we have $L + M \in \mathcal{S}$, so from the maximality of M we get $L \subseteq M$. Hence $\mathcal{S}(A) = M \in \mathcal{S}$. □

It is well known that a radical class satisfies (*) if and only if it is a semi-simple class. Of course semi-simple radical classes satisfy (†) but a semi-simple class with this property need not be radical, as we shall see.

Proposition 4.3. *If $\mathcal{R}$ is a hereditary radical class whose semi-simple class $\mathcal{S}$ satisfies (†), then for all ideals I, J of any ring A we have $\mathcal{R}(I) + \mathcal{R}(J) = \mathcal{R}(I + J)$.*

Proof. As $I, J \triangleleft I + J$ we have $\mathcal{R}(I), \mathcal{R}(J) \subseteq \mathcal{R}(I + J)$ so $\mathcal{R}(I) + \mathcal{R}(J) \subseteq \mathcal{R}(I + J)$. Now

$$\mathcal{R}(I) = I \cap \mathcal{R}(I) \subseteq I \cap (\mathcal{R}(I) + \mathcal{R}(J) \subseteq I \cap \mathcal{R}(I + J) = \mathcal{R}(I)$$

so $\mathcal{R}(I) = I \cap (\mathcal{R}(I) + \mathcal{R}(J))$, and likewise $\mathcal{R}(J) = J \cap (\mathcal{R}(I) + \mathcal{R}(J))$. Then we have

$$(I + (\mathcal{R}(I) + \mathcal{R}(J)))/(\mathcal{R}(I) + \mathcal{R}(J)) \cong I/I \cap (\mathcal{R}(I) + \mathcal{R}(J)) = I/\mathcal{R}(I) \in \mathcal{S}$$

with a similar condition for J. But this means that $(I + J)/\mathcal{R}(I) + \mathcal{R}(J)) =$

$$(I + \mathcal{R}(I) + \mathcal{R}(J))/(\mathcal{R}(I) + \mathcal{R}(J)) + (J + \mathcal{R}(I) + \mathcal{R}(J))/(\mathcal{R}(I) + \mathcal{R}(J)) \in \mathcal{S},$$

so that $\mathcal{R}(I + J) \subseteq \mathcal{R}(I) + \mathcal{R}(J)$. □

Note that while most of our proofs work for varieties of multioperator groups (at least), this one dos not, as we have made use of the *ADS* property. A radical class satisfying the conclusion of Proposition 4.3 is called *additive* [19]. A detailed study of additive radicals was made by Beidar and Trokanová-Salavová [4].

If $\mathcal{R}$ is additive with semi-simple class $\mathcal{S}$ and if $I, J \triangleleft A$ and $I, J \in \mathcal{S}$, then $\mathcal{R}(I+J) = \mathcal{R}(I) + \mathcal{R}(J) = 0$ so $I + J \in \mathcal{S}$ and $\mathcal{S}$ satisfies (i) of Proposition 4.2. If $\mathcal{R}$ (additive or not) is hereditary, then its semi-simple class $\mathcal{S}$ satisfies (ii) of Proposition 4.2 by the proof of Corollary 1.4 of [13]. Thus we have

Theorem 4.4. *Let $\mathcal{R}$ be a hereditary radical class with semi-simple class $\mathcal{S}$, Then $\mathcal{S}$ satisfies (†) if and only if $\mathcal{R}$ is additive.*

The hereditary additive radicals are described by Theorem 2.2 of [4]. They include the upper radical classes defined by the semi-simple radical classes. This was noted earlier by Szász [19]. There are others, however; for example if $\mathcal{R}$ is a hereditary radical class of (hereditarily) idempotent rings, then it is additive. We can easily show this.

If $I, J \triangleleft A$ and $\mathcal{R}(I) = 0 = \mathcal{R}(J)$, then $\mathcal{R}(I+J)I \subseteq \mathcal{R}(I+J) \cap I = \mathcal{R}(I) = 0$ and similarly $\mathcal{R}(I+J)J = 0$. But then $\mathcal{R}(I+J) = \mathcal{R}(I+J)^2 \subseteq \mathcal{R}(I+J)(I+J) = 0$.

We note again how duality fails. Condition (*), satisfied by all semi-simple classes, forces a radical class to *be* a semi-simple class, but the dual condition (†), satisfied by all radical classes, does not force a semi-simple class to be a radical class. At the same time, considerations of duality lead to a further connection between semi-simple radical classes and additivity.

If a radical class $\mathcal{R}$ has a semi-simple class $\mathcal{S}$ satisfying (†) then for $L \triangleleft A$ with $L \in \mathcal{R}$ we can consider $\mathcal{I} := \{K : K \triangleleft A \& L = \mathcal{R}(K)\}$. Since $\mathcal{R}(L) = L$, $\mathcal{I} \neq \emptyset$. Now for each $K \in \mathcal{I}$ we have $K/L = K/\mathcal{R}(K) \in \mathcal{S}$ whence

$$(\sum \mathcal{I})/L = \sum \{K/L : K \in \mathcal{I}\} \triangleleft \mathcal{S}(R/L) \in \mathcal{S},$$

so $(\sum \mathcal{I})/L \in \mathcal{S}$ and hence $L = \mathcal{R}(\sum \mathcal{I})$. Thus $\sum \mathcal{I}$ is *the largest ideal of A whose radical is L.*

We could also look at a stronger condition on a semi-simple class $\mathcal{S}$: that the join of all subrings from $\mathcal{S}$ should always be in $\mathcal{S}$. If $\mathcal{R}$ is the corresponding radical class, then for every $\mathcal{R}$-subring S of a ring A, there is a largest subring T of A whose radical is S. We can call T the *radicalizer* of S (with respect to $\mathcal{R}$). We shall discuss this matter elsewhere.

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Received by the editors: 04.09.2007
and in final form 04.02.2008.