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RESEARCH ARTICLE

# Nonstandard additively finite triangulated categories of Calabi-Yau dimension one in characteristic 3

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Abstract. We prove that there exist nonstandard $K$-linear triangulated categories with finitely many indecomposable objects and Calabi-Yau dimension one over an arbitrary algebraically closed field $K$ of characteristic 3, using deformed preprojective algebras of generalized Dynkin type.

Throughout the paper $K$ denotes an algebraically closed field. By a triangulated category we mean a small $K$-linear triangulated category $\mathcal{T}$ with split idempotents and finite dimensional morphism spaces. Recall that a triangulated category $\mathcal{T}$ admits an autoequivalence $\mathrm{T} : \mathcal{T} \to \mathcal{T}$ (translation of $\mathcal{T}$) and a collection of morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathrm{T}\, X$ (trangulation of $\mathcal{T}$) satisfying axioms (TR1) – (TR4) (see [9]). Important examples of triangulated categories of algebraic nature are provided by the derived categories $D^{\mathrm{b}}(\operatorname{mod} A)$ of bounded complexes of finite dimensional modules over finite dimensional $K$-algebras $A$ and the stable module categories $\underline{\operatorname{mod}}\, \Lambda$ of finite dimensional selfinjective (Frobenius) $K$-algebras $\Lambda$. For basic background on triangulated categories we refer to [9] and [11].

Following [6], [12], a Serre functor of a triangulated category $\mathcal{T}$ is an autoequivalence $\nu : \mathcal{T} \to \mathcal{T}$ together with natural isomorphisms $D \operatorname{Hom}_{\mathcal{T}}(X, ?) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}}(?, \nu X)$ for all objects $X$ of $\mathcal{T}$, where $D$ is

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the duality $\mathrm{Hom}_K(-,K)$. A triangulated category $\mathcal{T}$ is called Calabi-Yau if there exists an integer $d \geq 1$ such that the iteration $\mathrm{T}^d$ of the shift functor T of $\mathcal{T}$ is a Serre functor of $\mathcal{T}$. Then $\mathcal{T}$ is called $d$-Calabi-Yau and the smallest $d$ with this property is the Calabi-Yau dimension of $\mathcal{T}$. We refer to [7] for a complete description of Calabi-Yau stable module categories of tame symmetric algebras and their Calabi-Yau dimensions. Examples of triangulated categories of Calabi-Yau dimension 2 are provided by the stable module categories of the preprojective algebras of generalized Dynkin type (see [3], [4], [8]).

In this paper, we are concerned with the structure of additively finite triangulated categories, that is, triangulated categories with finitely many isomorpism classes of indecomposable objects. It is known that every additively finite triangulated category has a Serre functor (see [1, Theorem 1.1.1]). Fundamental examples of additively finite triangulated categories are provided by the stable module categories $\underline{\mathrm{mod}}\,\Lambda$ of all selfinjective algebras of finite representation type (we refer to [13] for description of these categories). Moreover, the authors gave in [5] necessary and sufficent conditions for these categories to be Calabi-Yau. In general, it follows from [14] that if $\mathcal{T}$ is an additively finite triangulated $K$-category then the Auslander-Reiten quiver $\Gamma_{\mathcal{T}}$ of $\mathcal{T}$ is of the form $\mathbb{Z}\Delta/G$, where $\Delta$ is a Dynkin quiver of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, and $G$ is a weakly admissible group of automorphisms of the translation quiver $\mathbb{Z}\Delta$. Moreover, such a triangulated category $\mathcal{T}$ is called standard if $\mathcal{T}$ is $K$-linearly equivalent to an orbit category $D^{\mathrm{b}}(\mathrm{mod}\,K\Delta)/H$, where $K\Delta$ is the path algebra of $\Delta$ and $H$ is an automorphism group of $D^{\mathrm{b}}(\mathrm{mod}\,K\Delta)$. In [1], C. Amiot proved that, for most groups $G$, the triangulated categories $\mathcal{T}$ with $\Gamma_{\mathcal{T}} \cong \mathbb{Z}\Delta/G$ are standard. Moreover, C. Amiot gave in [1] sufficient conditions for the category $\mathrm{proj}\,\Lambda$ of finite dimensional projective modules over a selfinjective algebra $\Lambda$ to be a triangulated category. Then, invoking the main result of [4], C. Amiot proved in [1] that the class of 1-Calabi-Yau additively finite triangulated categories $\mathcal{T}$ coincides with the class of the categories $\mathrm{proj}\,P^f(\Delta)$ over deformed preprojective algebras $P^f(\Delta)$ of generalized Dynkin types $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{L}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, described bellow. This allowed to construct nonstandard 1-Calabi-Yau additively finite triangulated categories over an arbitrary algebraically closed field $K$ of characteristic 2.

The main aim of this paper is to show existence of nonstandard 1-Calabi-Yau additively finite triangulated categories over an arbitrary algebraically closed field $K$ of characteristic 3.

We also note that the nonstandard stable module categories of selfinjective algebras of finite type there exist only in characteristic 2 (see [13]).

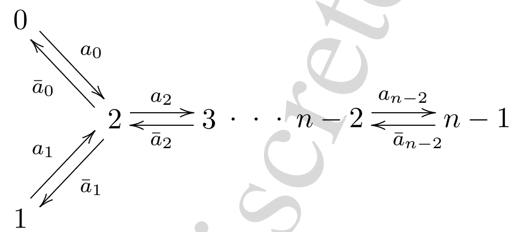
We recall now the deformed preprojective algebras of generalized Dynkin type introduced in [4].

Let $\Delta$ be a generalized Dynkin graph of type $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$, $\mathbb{E}_n (n = 6, 7, 8)$ or $\mathbb{L}_n (n \geq 1)$. Let $Q_\Delta$ be the following associated quiver:

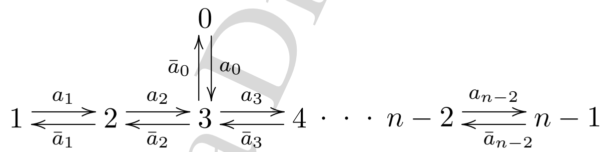
$\Delta = \mathbb{A}_n :$ $(n \geq 1)$

$$0 \underset{\bar{a}_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{\bar{a}_1}{\overset{a_1}{\rightleftarrows}} 2 \ \cdots \ n-2 \underset{\bar{a}_{n-2}}{\overset{a_{n-2}}{\rightleftarrows}} n-1$$

$\Delta = \mathbb{D}_n :$ $(n \geq 4)$




$\Delta = \mathbb{E}_n :$ $(n = 6, 7, 8)$




$\Delta = \mathbb{L}_n :$ $(n \geq 1)$

$$\varepsilon = \bar{\varepsilon} \circlearrowright 0 \underset{\bar{a}_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{\bar{a}_1}{\overset{a_1}{\rightleftarrows}} 2 \ \cdots \ n-2 \underset{\bar{a}_{n-2}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \, .$$

The preprojective algebra $P(\Delta)$ associated to the graph $\Delta$ is the bound quiver algebra $KQ_\Delta / I_\Delta$, where $KQ_\Delta$ is the path algebra of $Q_\Delta$ and $I_\Delta$ is the ideal in $KQ_\Delta$ generated by the relations of the form

$$\sum_{a, sa = i} a\bar{a}, \quad i \text{ vertices of } Q_\Delta,$$

where $sa$ denotes the source of an arrow $a$ of $Q_\Delta$. The preprojective algebra $P(\Delta)$ is a finite dimensional selfinjective algebra and its Nakayama permutation is identity for $\Delta = \mathbb{A}_1, \mathbb{D}_{2n}, \mathbb{E}_7, \mathbb{E}_8$ and $\mathbb{L}_n$, and of order 2 in all other cases. Further, consider the associated algebra $R(\Delta)$ as follows

$$\begin{aligned} R(\mathbb{A}_n) &= K; \\ R(\mathbb{D}_n) &= K\langle x, y\rangle / (x^2, y^2, (x+y)^{n-2}); \\ R(\mathbb{E}_n) &= K\langle x, y\rangle / (x^2, y^3, (x+y)^{n-3}); \\ R(\mathbb{L}_n) &= K[x]/(x^{2n}). \end{aligned}$$

Moreover, choose the exceptional vertex of the quiver $Q_\Delta$ of $P(\Delta)$ as follows

0, for $\Delta = \mathbb{A}_n$ or $\mathbb{L}_n$;
2, for $\Delta = \mathbb{D}_n$;
3, for $\Delta = \mathbb{E}_n$.

We note that if $e$ is the idempotent of $P(\Delta)$ corresponding to the exceptional vertex then $R(\Delta)$ is isomorphic to $eP(\Delta)e$, and hence is local, finite dimensional and selfinjective. Let $f$ be an element of the square $\operatorname{rad}^2 R(\Delta)$ of the Jacobson radical of $R(\Delta)$. Then the deformed preprojective algebra $P^f(\Delta)$ of generalized Dynkin type $\Delta$ is the bound quiver algebra $KQ_\Delta/I^f_\Delta$, where $I^f_\Delta$ is the ideal in the path algebra $KQ_\Delta$ of $Q_\Delta$ generated by the relations of the form

$$\sum_{a,sa=i} a\bar{a}, \quad \text{for each nonexceptional vertex } i \text{ of } Q_\Delta,$$

and

$$\begin{array}{ll} a_0\bar{a}_0, & \text{for } \Delta = \mathbb{A}_n; \\ \bar{a}_0a_0 + \bar{a}_1a_1 + a_2\bar{a}_2 + f(\bar{a}_0a_0, \bar{a}_1a_1),\ (\bar{a}_0a_0 + \bar{a}_1a_1)^{n-2}, & \text{for } \Delta = \mathbb{D}_n; \\ \bar{a}_0a_0 + \bar{a}_2a_2 + a_3\bar{a}_3 + f(\bar{a}_0a_0, \bar{a}_2a_2),\ (\bar{a}_0a_0 + \bar{a}_2a_2)^{n-3}, & \text{for } \Delta = \mathbb{E}_n; \\ \varepsilon^2 + a_0\bar{a}_0 + \varepsilon f(\varepsilon),\ \varepsilon^{2n}, & \text{for } \Delta = \mathbb{L}_n. \end{array}$$

Therefore, $P^f(\Delta)$ is obtained from $P(\Delta)$ by deforming the relation at the exceptional vertex of $Q_\Delta$, and $P^f(\Delta) = P(\Delta)$ if $f = 0$. Moreover, $P^f(\Delta)$ is a selfinjective algebra with $\dim_K P^f(\Delta) = \dim_K P(\Delta)$ and the Cartan matrices of $P^f(\Delta)$ and $P(\Delta)$ coincide.

It is shown in [4, Theorem 1.3] that, for an algebraically closed field $K$ of characteristic 2 and a generalized Dynkin graph $\Delta$ other than $\mathbb{A}_n$ and $\mathbb{L}_1$, there exists a deformed preprojective algebra $P^f(\Delta)$ over $K$ which is not isomorphic to the preprojective algebra $P(\Delta)$. In such a case, $\operatorname{proj} P^f(\Delta)$ is a nonstandard 1-Calabi-Yau additively finite triangular category (see [1, Theorem 9.3.3]).

The following theorem is the main result of the paper.

**Theorem.** *Let $K$ be an algebraically closed field of characteristic 3, $\Delta$ a Dynkin graph $\mathbb{E}_n (n = 6, 7, 8)$ and $f = y^2x + (x^2, y^3, (x+y)^{n-3}) \in R(\Delta)$. Then*

*(i)* *$P^f(\Delta)$ is not isomorphic to $P(\Delta)$.*

*(ii)* $\operatorname{proj} P^f(\Delta)$ *is a nonstandard* 1*-Calabi-Yau additively finite triangulated category.*

*Proof.* (i) Let $K$ be an algebraically closed field of characteristic 3, $\Delta$ a Dynkin graph $\mathbb{E}_n(n = 6, 7, 8)$ and $f = y^2x + (x^2, y^3, (x+y)^{n-3}) \in R(\Delta)$. We will show that $P^f(\Delta)$ is not isomorphic to $P(\Delta)$.

Suppose that $\varphi : P^f(\Delta) \to P(\Delta)$ is an algebra isomorphism. Then $\varphi$ is determined by the elements of the form

$$\begin{aligned}
\varphi(a_0) &= \alpha_0^{(0)} a_0 + \alpha_0^{(0)}\alpha_1^{(0)} a_0\bar{a}_2a_2 + \alpha_0^{(0)}\alpha_2^{(0)} a_0\bar{a}_2a_2\bar{a}_0a_0 + \\
&\quad + \alpha_0^{(0)}\alpha_3^{(0)} a_0\bar{a}_2a_2\bar{a}_2a_2 + \dots \\
\varphi(\bar{a}_0) &= \bar{\alpha}_0^{(0)} \bar{a}_0 + \bar{\alpha}_0^{(0)}\bar{\alpha}_1^{(0)} \bar{a}_2a_2\bar{a}_0 + \bar{\alpha}_0^{(0)}\bar{\alpha}_2^{(0)} \bar{a}_0a_0\bar{a}_2a_2\bar{a}_0 + \\
&\quad + \bar{\alpha}_0^{(0)}\bar{\alpha}_3^{(0)} \bar{a}_2a_2\bar{a}_2a_2\bar{a}_0 + \dots \\
\varphi(a_1) &= \alpha_0^{(1)} a_1 + \alpha_0^{(1)}\alpha_1^{(1)} a_1a_2\bar{a}_0a_0\bar{a}_2 + \dots \\
\varphi(\bar{a}_1) &= \bar{\alpha}_0^{(1)} \bar{a}_1 + \bar{\alpha}_0^{(1)}\bar{\alpha}_1^{(1)} a_2\bar{a}_0a_0\bar{a}_2\bar{a}_1 + \dots \\
\varphi(a_2) &= \alpha_0^{(2)} a_2 + \alpha_0^{(2)}\alpha_1^{(2)} a_2\bar{a}_0a_0 + \alpha_0^{(2)}\alpha_2^{(2)} a_2\bar{a}_2a_2 + \\
&\quad + \alpha_0^{(2)}\alpha_3^{(2)} a_2\bar{a}_0a_0\bar{a}_2a_2 + \alpha_0^{(2)}\alpha_4^{(2)} a_2\bar{a}_2a_2\bar{a}_0a_0 + \dots \\
\varphi(\bar{a}_2) &= \bar{\alpha}_0^{(2)} \bar{a}_2 + \bar{\alpha}_0^{(2)}\bar{\alpha}_1^{(2)} \bar{a}_0a_0\bar{a}_2 + \bar{\alpha}_0^{(2)}\bar{\alpha}_2^{(2)} \bar{a}_2a_2\bar{a}_2 + \\
&\quad + \bar{\alpha}_0^{(2)}\bar{\alpha}_3^{(2)} \bar{a}_0a_0\bar{a}_2a_2\bar{a}_2 + \bar{\alpha}_0^{(2)}\bar{\alpha}_4^{(2)} \bar{a}_2a_2\bar{a}_0a_0\bar{a}_2 + \dots \\
\varphi(a_3) &= \alpha_0^{(3)} a_3 + \alpha_0^{(3)}\alpha_1^{(3)} \bar{a}_0a_0a_3 + \alpha_0^{(3)}\alpha_2^{(3)} \bar{a}_2a_2a_3 + \\
&\quad + \alpha_0^{(3)}\alpha_3^{(3)} \bar{a}_0a_0\bar{a}_2a_2a_3 + \alpha_0^{(3)}\alpha_4^{(3)} \bar{a}_2a_2\bar{a}_0a_0a_3 + \\
&\quad + \alpha_0^{(3)}\alpha_5^{(3)} \bar{a}_2a_2\bar{a}_2a_2a_3 + \dots \\
\varphi(\bar{a}_3) &= \bar{\alpha}_0^{(3)} \bar{a}_3 + \bar{\alpha}_0^{(3)}\bar{\alpha}_1^{(3)} \bar{a}_3\bar{a}_0a_0 + \bar{\alpha}_0^{(3)}\bar{\alpha}_2^{(3)} \bar{a}_3\bar{a}_2a_2 + \\
&\quad + \bar{\alpha}_0^{(3)}\bar{\alpha}_3^{(3)} \bar{a}_3\bar{a}_0a_0\bar{a}_2a_2 + \bar{\alpha}_0^{(3)}\bar{\alpha}_4^{(3)} \bar{a}_3\bar{a}_2a_2\bar{a}_0a_0 + \\
&\quad + \bar{\alpha}_0^{(3)}\bar{\alpha}_5^{(3)} \bar{a}_3\bar{a}_2a_2\bar{a}_2a_2 + \dots \\
\varphi(a_4) &= \alpha_0^{(4)} a_4 + \alpha_0^{(4)}\alpha_1^{(4)} \bar{a}_3a_3a_4 + \alpha_0^{(4)}\alpha_2^{(4)} \bar{a}_3\bar{a}_0a_0a_3a_4 + \\
&\quad + \alpha_0^{(4)}\alpha_3^{(4)} \bar{a}_3\bar{a}_2a_2a_3a_4 + \dots \\
\varphi(\bar{a}_4) &= \bar{\alpha}_0^{(4)} \bar{a}_4 + \bar{\alpha}_0^{(4)}\bar{\alpha}_1^{(4)} \bar{a}_4\bar{a}_3a_3 + \bar{\alpha}_0^{(4)}\bar{\alpha}_2^{(4)} \bar{a}_4\bar{a}_3\bar{a}_0a_0a_3 + \\
&\quad + \bar{\alpha}_0^{(4)}\bar{\alpha}_3^{(4)} \bar{a}_4\bar{a}_3\bar{a}_2a_2a_3 + \dots,
\end{aligned}$$

for $n = 6, 7, 8$,

$$\begin{aligned}
\varphi(a_5) &= \alpha_0^{(5)} a_5 + \alpha_0^{(5)}\alpha_1^{(5)} \bar{a}_4a_4a_5 + \dots \\
\varphi(\bar{a}_5) &= \bar{\alpha}_0^{(5)} \bar{a}_5 + \bar{\alpha}_0^{(5)}\bar{\alpha}_1^{(5)} \bar{a}_5\bar{a}_4a_4 + \dots,
\end{aligned}$$

for $n = 7, 8$, and

$$\begin{aligned}
\varphi(a_6) &= \alpha_0^{(6)} a_6 + \dots \\
\varphi(\bar{a}_6) &= \bar{\alpha}_0^{(6)} \bar{a}_6 + \dots,
\end{aligned}$$

for $n = 8$, for some parameters $\alpha_i^{(l)}, \bar{\alpha}_i^{(l)} \in K$, with $\alpha_0^{(l)}, \bar{\alpha}_0^{(l)}$ non-zero, for all $l \in \{0, 1, \dots, n\}$, $\alpha_5^{(3)} = \bar{\alpha}_5^{(3)} = \alpha_1^{(4)} = \bar{\alpha}_1^{(4)} = \alpha_3^{(4)} = \bar{\alpha}_3^{(4)} = 0$ for $n = 6$, and $\alpha_3^{(4)} = \bar{\alpha}_3^{(4)} = \alpha_1^{(5)} = \bar{\alpha}_1^{(5)} = 0$ for $n = 7$. Denote $\alpha = \alpha_0^{(0)}\bar{\alpha}_0^{(0)}$. Note that $\alpha_i^{(i)}\bar{\alpha}_i^{(i)} = \alpha \neq 0$, for all $i \in \{0, 1, \dots, n\}$.

Invoking the relation at the vertex 0, we obtain

$$\begin{aligned}0 &= \alpha^{-2}\varphi\left(a_0\bar{a}_0\right)\\ &= a_0\bar{a}_0 + \left(\alpha_1^{(0)} + \bar{\alpha}_1^{(0)}\right) a_0\bar{a}_2a_2\bar{a}_0 +\\ &\quad + \left(\alpha_1^{(0)}\bar{\alpha}_1^{(0)} + \alpha_3^{(0)} + \bar{\alpha}_3^{(0)}\right) a_0\bar{a}_2a_2\bar{a}_2a_2\bar{a}_0 + \ldots,\end{aligned}$$

and thus $\alpha_1^{(0)} + \bar{\alpha}_1^{(0)} = 0 = \alpha_1^{(0)}\bar{\alpha}_1^{(0)} + \alpha_3^{(0)} + \bar{\alpha}_3^{(0)}$. Similarly, using the relation at the vertex 1, we obtain

$$0 = \alpha^{-2}\varphi\left(a_1\bar{a}_1\right) = a_1\bar{a}_1 + \left(\alpha_1^{(1)} + \bar{\alpha}_1^{(1)}\right) a_1a_2\bar{a}_0a_0\bar{a}_2\bar{a}_1 + \ldots,$$

and thus $\alpha_1^{(1)} + \bar{\alpha}_1^{(1)} = 0$. Further, invoking the relation at the vertex 2, we obtain

$$\begin{aligned}0 &= \alpha^{-2}\varphi\left(\bar{a}_1a_1 + a_2\bar{a}_2\right)\\ &= \bar{a}_1a_1 - \bar{\alpha}_1^{(1)}a_2\bar{a}_0a_0\bar{a}_2a_2\bar{a}_2 - \alpha_1^{(1)}a_2\bar{a}_2a_2\bar{a}_0a_0\bar{a}_2 +\\ &\quad + a_2\bar{a}_2 + \left(\alpha_1^{(2)} + \bar{\alpha}_1^{(2)}\right) a_2\bar{a}_0a_0\bar{a}_2 +\\ &\quad + \left(\alpha_3^{(2)} + \bar{\alpha}_3^{(2)} + \alpha_1^{(2)}\bar{\alpha}_2^{(2)}\right) a_2\bar{a}_0a_0\bar{a}_2a_2\bar{a}_2 +\\ &\quad + \left(\alpha_4^{(2)} + \bar{\alpha}_4^{(2)} + \alpha_2^{(2)}\bar{\alpha}_1^{(2)}\right) a_2\bar{a}_2a_2\bar{a}_0a_0\bar{a}_2 + \ldots,\end{aligned}$$

hence $\alpha_1^{(2)} + \bar{\alpha}_1^{(2)} = \alpha_3^{(2)} + \bar{\alpha}_3^{(2)} + \alpha_1^{(2)}\bar{\alpha}_2^{(2)} - \alpha_1^{(1)} = \alpha_4^{(2)} + \bar{\alpha}_4^{(2)} + \alpha_2^{(2)}\bar{\alpha}_1^{(2)} - \bar{\alpha}_1^{(1)} = 0$ (note that $a_2\bar{a}_2a_2\bar{a}_2 = \bar{a}_1a_1\bar{a}_1a_1 = 0$). Applying now $\alpha_1^{(1)} + \bar{\alpha}_1^{(1)} = 0$ we obtain the equality

$$\alpha_3^{(2)} + \bar{\alpha}_3^{(2)} + \alpha_1^{(2)}\bar{\alpha}_2^{(2)} + \alpha_4^{(2)} + \bar{\alpha}_4^{(2)} + \alpha_2^{(2)}\bar{\alpha}_1^{(2)} = 0.$$

Assume that $n = 6$. Using the relation at the vertex 5, we obtain

$$0 = \alpha^{-2}\varphi\left(\bar{a}_4a_4\right) = \bar{a}_4a_4 + \left(\alpha_2^{(4)} + \bar{\alpha}_2^{(4)}\right) \bar{a}_4\bar{a}_3\bar{a}_0a_0a_3a_4 + \ldots,$$

and hence $\alpha_2^{(4)} + \bar{\alpha}_2^{(4)} = 0$. Similarly, using the relation at the vertex 4, we obtain

$$\begin{aligned}0 &= \alpha^{-2}\varphi\left(\bar{a}_3a_3 + a_4\bar{a}_4\right)\\ &= \bar{a}_3a_3 + \left(\alpha_1^{(3)} + \bar{\alpha}_1^{(3)} - \alpha_2^{(3)} - \bar{\alpha}_2^{(3)}\right) \bar{a}_3\bar{a}_0a_0a_3\\ &\quad + \left(\alpha_3^{(3)} + \bar{\alpha}_3^{(3)} + \alpha_2^{(3)}\bar{\alpha}_1^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)}\right) \bar{a}_3\bar{a}_0a_0\bar{a}_2a_2a_3\\ &\quad + \left(\alpha_4^{(3)} + \bar{\alpha}_4^{(3)} + \alpha_1^{(3)}\bar{\alpha}_2^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)}\right) \bar{a}_3\bar{a}_2a_2\bar{a}_0a_0a_3\\ &\quad + a_4\bar{a}_4 + \alpha_2^{(4)}\bar{a}_3\bar{a}_0a_0\bar{a}_2a_2a_3 + \bar{\alpha}_2^{(4)}\bar{a}_3\bar{a}_2a_2\bar{a}_0a_0a_3 + \ldots\end{aligned}$$

Note that we have $\bar{a}_3a_3\bar{a}_3a_3 = \bar{a}_4a_4 = 0$. Hence

$$\begin{aligned}\alpha_1^{(3)} + \bar{\alpha}_1^{(3)} - \alpha_2^{(3)} - \bar{\alpha}_2^{(3)} &= \alpha_3^{(3)} + \bar{\alpha}_3^{(3)} + \alpha_2^{(3)}\bar{\alpha}_1^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)} + \alpha_2^{(4)} \\ &= \alpha_4^{(3)} + \bar{\alpha}_4^{(3)} + \alpha_1^{(3)}\bar{\alpha}_2^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)} + \bar{\alpha}_2^{(4)} = 0.\end{aligned}$$

Applying $\alpha_2^{(4)} + \bar{\alpha}_2^{(4)} = 0$ to the above equations we obtain

$$\alpha_3^{(3)} + \bar{\alpha}_3^{(3)} + \alpha_2^{(3)}\bar{\alpha}_1^{(3)} + \alpha_4^{(3)} + \bar{\alpha}_4^{(3)} + \alpha_1^{(3)}\bar{\alpha}_2^{(3)} + \alpha_2^{(3)}\bar{\alpha}_2^{(3)} = 0$$

(note that the calculations are in a field $K$ of characteristic 3).

Now assume that $n \in \{7, 6\}$. Using the relation at the vertex 5, we obtain

$$\begin{aligned}0 &= \alpha^{-2}\varphi\left(\bar{a}_4a_4 + a_5\bar{a}_5\right) \\ &= \bar{a}_4a_4 + \left(\alpha_1^{(4)} + \bar{\alpha}_1^{(4)}\right)\bar{a}_4\bar{a}_3a_3a_4 \\ &\quad + \left(\alpha_2^{(4)} + \bar{\alpha}_2^{(4)} - \alpha_3^{(4)} - \bar{\alpha}_3^{(4)}\right)\bar{a}_4\bar{a}_3\bar{a}_0a_0a_3a_4 \\ &\quad + a_5\bar{a}_5 + \left(\alpha_1^{(5)} + \bar{\alpha}_1^{(5)}\right)\bar{a}_4\bar{a}_3a_3a_4 + \dots \\ &= \bar{a}_4a_4 + a_5\bar{a}_5 + \left(\alpha_1^{(4)} + \bar{\alpha}_1^{(4)} + \alpha_1^{(5)} + \bar{\alpha}_1^{(5)}\right)\bar{a}_4\bar{a}_3a_3a_4 \\ &\quad + \left(\alpha_2^{(4)} + \bar{\alpha}_2^{(4)} - \alpha_3^{(4)} - \bar{\alpha}_3^{(4)}\right)\bar{a}_4\bar{a}_3\bar{a}_0a_0a_3a_4 + \dots,\end{aligned}$$

and hence

$$\alpha_1^{(4)} + \bar{\alpha}_1^{(4)} + \alpha_1^{(5)} + \bar{\alpha}_1^{(5)} = 0 = \alpha_2^{(4)} + \bar{\alpha}_2^{(4)} - \alpha_3^{(4)} - \bar{\alpha}_3^{(4)}$$

(for $n = 7$, we have $\alpha_1^{(4)} + \bar{\alpha}_1^{(4)} = \alpha_2^{(4)} + \bar{\alpha}_2^{(4)} = 0$).

Assume that $n = 7$. Using the relation at the vertex 4, we obtain

$$\begin{aligned}0 &= \alpha^{-2}\varphi\left(\bar{a}_3a_3 + a_4\bar{a}_4\right) \\ &= \bar{a}_3a_3 + \left(\alpha_1^{(3)} + \bar{\alpha}_1^{(3)}\right)\bar{a}_3\bar{a}_0a_0a_3 + \left(\alpha_2^{(3)} + \bar{\alpha}_2^{(3)}\right)\bar{a}_3\bar{a}_2a_2a_3 \\ &\quad + \left(\alpha_3^{(3)} + \bar{\alpha}_3^{(3)} + \alpha_2^{(3)}\bar{\alpha}_1^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)} - \alpha_5^{(3)} - \bar{\alpha}_5^{(3)}\right)\bar{a}_3\bar{a}_0a_0\bar{a}_2a_2a_3 \\ &\quad + \left(\alpha_4^{(3)} + \bar{\alpha}_4^{(3)} + \alpha_1^{(3)}\bar{\alpha}_2^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)} - \alpha_5^{(3)} - \bar{\alpha}_5^{(3)}\right)\bar{a}_3\bar{a}_2a_2\bar{a}_0a_0a_3 \\ &\quad + a_4\bar{a}_4 + \left(\alpha_1^{(4)} + \bar{\alpha}_1^{(4)}\right)\bar{a}_3\bar{a}_0a_0a_3 + \left(\alpha_1^{(4)} + \bar{\alpha}_1^{(4)}\right)\bar{a}_3\bar{a}_2a_2a_3 \\ &\quad + \alpha_2^{(4)}\bar{a}_3\bar{a}_0a_0\bar{a}_2a_2a_3 + \bar{\alpha}_2^{(4)}\bar{a}_3\bar{a}_2a_2\bar{a}_0a_0a_3 + \dots\end{aligned}$$

Note that we have $\bar{a}_3a_3\bar{a}_3a_3\bar{a}_3a_3 = \bar{a}_4a_4\bar{a}_4a_4 = 0$. Hence

$$\begin{aligned}\alpha_1^{(3)} + \bar{\alpha}_1^{(3)} + \alpha_1^{(4)} + \bar{\alpha}_1^{(4)} &= 0, \\ \alpha_2^{(3)} + \bar{\alpha}_2^{(3)} + \alpha_1^{(4)} + \bar{\alpha}_1^{(4)} &= 0, \\ \alpha_3^{(3)} + \bar{\alpha}_3^{(3)} + \alpha_2^{(3)}\bar{\alpha}_1^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)} - \alpha_5^{(3)} - \bar{\alpha}_5^{(3)} + \alpha_2^{(4)} &= 0, \\ \alpha_4^{(3)} + \bar{\alpha}_4^{(3)} + \alpha_1^{(3)}\bar{\alpha}_2^{(3)} - \alpha_2^{(3)}\bar{\alpha}_2^{(3)} - \alpha_5^{(3)} - \bar{\alpha}_5^{(3)} + \bar{\alpha}_2^{(4)} &= 0.\end{aligned}$$

Summing up the last two equalities and applying the equality $\alpha_2^{(4)}+\bar\alpha_2^{(4)}=0$ we obtain

$$\alpha_3^{(3)}+\bar\alpha_3^{(3)}+\alpha_2^{(3)}\bar\alpha_1^{(3)}+\alpha_4^{(3)}+\bar\alpha_4^{(3)}+\alpha_1^{(3)}\bar\alpha_2^{(3)}+\alpha_2^{(3)}\bar\alpha_2^{(3)}+\alpha_5^{(3)}+\bar\alpha_5^{(3)}=0$$

(again note that the calculations are in a field $K$ of characteristic 3).

Assume that $n=8$. Applying the relation at the vertex 4, we obtain

$$\begin{aligned}
0 &= \alpha^{-2}\varphi\left(\bar a_3a_3+a_4\bar a_4\right)\\
&= \bar a_3a_3+\left(\alpha_1^{(3)}+\bar\alpha_1^{(3)}\right)\bar a_3\bar a_0a_0a_3+\left(\alpha_2^{(3)}+\bar\alpha_2^{(3)}\right)\bar a_3\bar a_2a_2a_3\\
&\quad+\left(\alpha_3^{(3)}+\bar\alpha_3^{(3)}+\alpha_2^{(3)}\bar\alpha_1^{(3)}\right)\bar a_3\bar a_0a_0\bar a_2a_2a_3\\
&\quad+\left(\alpha_4^{(3)}+\bar\alpha_4^{(3)}+\alpha_1^{(3)}\bar\alpha_2^{(3)}\right)\bar a_3\bar a_2a_2\bar a_0a_0a_3\\
&\quad+\left(\alpha_2^{(3)}\bar\alpha_2^{(3)}+\alpha_5^{(3)}+\bar\alpha_5^{(3)}\right)\bar a_3\bar a_2a_2\bar a_2a_2a_3\\
&\quad+a_4\bar a_4+\left(\alpha_1^{(4)}+\bar\alpha_1^{(4)}\right)\bar a_3\bar a_0a_0a_3+\left(\alpha_1^{(4)}+\bar\alpha_1^{(4)}\right)\bar a_3\bar a_2a_2a_3\\
&\quad+\left(\alpha_2^{(4)}+\alpha_3^{(4)}-\alpha_1^{(4)}\bar\alpha_1^{(4)}\right)\bar a_3\bar a_0a_0\bar a_2a_2a_3\\
&\quad+\left(\bar\alpha_2^{(4)}+\bar\alpha_3^{(4)}-\alpha_1^{(4)}\bar\alpha_1^{(4)}\right)\bar a_3\bar a_2a_2\bar a_0a_0a_3\\
&\quad+\left(\alpha_3^{(4)}+\bar\alpha_3^{(4)}-\alpha_1^{(4)}\bar\alpha_1^{(4)}\right)\bar a_3\bar a_2a_2\bar a_2a_2a_3+\dots
\end{aligned}$$

Note that we have $\bar a_3a_3\bar a_3a_3\bar a_3a_3=\bar a_4a_4\bar a_4a_4=0$. Hence

$$\begin{aligned}
\alpha_1^{(3)}+\bar\alpha_1^{(3)}+\alpha_1^{(4)}+\bar\alpha_1^{(4)}&=0,\\
\alpha_2^{(3)}+\bar\alpha_2^{(3)}+\alpha_1^{(4)}+\bar\alpha_1^{(4)}&=0,\\
\alpha_3^{(3)}+\bar\alpha_3^{(3)}+\alpha_2^{(3)}\bar\alpha_1^{(3)}+\alpha_2^{(4)}+\bar\alpha_3^{(4)}-\alpha_1^{(4)}\bar\alpha_1^{(4)}&=0,\\
\alpha_4^{(3)}+\bar\alpha_4^{(3)}+\alpha_1^{(3)}\bar\alpha_2^{(3)}+\bar\alpha_2^{(4)}+\alpha_3^{(4)}-\alpha_1^{(4)}\bar\alpha_1^{(4)}&=0,\\
\alpha_2^{(3)}\bar\alpha_2^{(3)}+\alpha_5^{(3)}+\bar\alpha_5^{(3)}+\alpha_3^{(4)}+\bar\alpha_3^{(4)}-\alpha_1^{(4)}\bar\alpha_1^{(4)}&=0.
\end{aligned}$$

Summing up the last three equalities and applying the equality $\alpha_2^{(4)}+\bar\alpha_2^{(4)}-\alpha_3^{(4)}-\bar\alpha_3^{(4)}=0$, we obtain

$$\alpha_3^{(3)}+\bar\alpha_3^{(3)}+\alpha_2^{(3)}\bar\alpha_1^{(3)}+\alpha_4^{(3)}+\bar\alpha_4^{(3)}+\alpha_1^{(3)}\bar\alpha_2^{(3)}+\alpha_2^{(3)}\bar\alpha_2^{(3)}+\alpha_5^{(3)}+\bar\alpha_5^{(3)}=0.$$

Assume that $n=\{6,7,8\}$. Applying the relation at the vertex 3, we obtain

$$\begin{aligned}
0 &= \alpha^{-2}\varphi\left(\bar a_0a_0+\bar a_2a_2+a_3\bar a_3+\bar a_2a_2\bar a_2a_2\bar a_0a_0\right)\\
&= \bar a_0a_0+\alpha_1^{(0)}\bar a_0a_0\bar a_2a_2+\bar\alpha_1^{(0)}\bar a_2a_2\bar a_0a_0+\alpha_1^{(0)}\bar\alpha_1^{(0)}\bar a_2a_2\bar a_0a_0\bar a_2a_2\\
&\quad+\left(\alpha_2^{(0)}+\bar\alpha_2^{(0)}\right)\bar a_0a_0\bar a_2a_2\bar a_0a_0\\
&\quad+\bar\alpha_3^{(0)}\bar a_2a_2\bar a_2a_2\bar a_0a_0+\alpha_3^{(0)}\bar a_0a_0\bar a_2a_2\bar a_2a_2\\
&\quad+\bar a_2a_2+\bar\alpha_1^{(2)}\bar a_0a_0\bar a_2a_2+\alpha_1^{(2)}\bar a_2a_2\bar a_0a_0+\left(\alpha_2^{(2)}+\bar\alpha_2^{(2)}\right)\bar a_2a_2\bar a_2a_2\\
&\quad+\left(\alpha_1^{(2)}\bar\alpha_2^{(2)}+\alpha_4^{(2)}\right)\bar a_2a_2\bar a_2a_2\bar a_0a_0\\
&\quad+\left(\alpha_2^{(2)}\bar\alpha_1^{(2)}+\bar\alpha_3^{(2)}\right)\bar a_0a_0\bar a_2a_2\bar a_2a_2
\end{aligned}$$

$$\begin{aligned}
&+\left(\alpha_3^{(2)}+\bar\alpha_4^{(2)}\right)\bar a_2a_2\bar a_0a_0\bar a_2a_2\\
&+\alpha_1^{(2)}\bar\alpha_1^{(2)}\bar a_0a_0\bar a_2a_2\bar a_0a_0\\
&+a_3\bar a_3-\left(\alpha_1^{(3)}+\bar\alpha_2^{(3)}\right)\bar a_0a_0\bar a_2a_2-\left(\bar\alpha_1^{(3)}+\alpha_2^{(3)}\right)\bar a_2a_2\bar a_0a_0\\
&-\left(\alpha_2^{(3)}+\bar\alpha_2^{(3)}\right)\bar a_2a_2\bar a_2a_2\\
&+\left(-\bar\alpha_4^{(3)}-\alpha_5^{(3)}-\alpha_2^{(3)}\bar\alpha_1^{(3)}\right)\bar a_2a_2\bar a_2a_2\bar a_0a_0\\
&+\left(-\alpha_3^{(3)}-\bar\alpha_5^{(3)}-\alpha_1^{(3)}\bar\alpha_2^{(3)}\right)\bar a_0a_0\bar a_2a_2\bar a_2a_2\\
&+\left(-\alpha_4^{(3)}-\bar\alpha_3^{(3)}-\alpha_2^{(3)}\bar\alpha_2^{(3)}\right)\bar a_2a_2\bar a_0a_0\bar a_2a_2\\
&+\left(-\alpha_1^{(3)}\bar\alpha_1^{(3)}-\alpha_3^{(3)}-\bar\alpha_4^{(3)}\right)\bar a_0a_0\bar a_2a_2\bar a_0a_0\\
&+\alpha^4\bar a_2a_2\bar a_2a_2\bar a_0a_0+\dots\\
=\,&\bar a_0a_0+\bar a_2a_2+a_3\bar a_3\\
&+\left(\alpha_1^{(0)}+\bar\alpha_1^{(2)}-\alpha_1^{(3)}-\bar\alpha_2^{(3)}\right)\bar a_0a_0\bar a_2a_2\\
&+\left(\bar\alpha_1^{(0)}+\alpha_1^{(2)}-\bar\alpha_1^{(3)}-\alpha_2^{(3)}\right)\bar a_0a_0\bar a_2a_2\\
&+\left(\alpha_2^{(2)}+\bar\alpha_2^{(2)}-\alpha_2^{(3)}-\bar\alpha_2^{(3)}\right)\bar a_2a_2\bar a_2a_2\\
&+\left(\bar\alpha_3^{(0)}+\alpha_1^{(2)}\bar\alpha_2^{(2)}+\alpha_4^{(2)}-\bar\alpha_4^{(3)}-\alpha_5^{(3)}-\alpha_2^{(3)}\bar\alpha_1^{(3)}+\alpha^4\right)\bar a_2a_2\bar a_2a_2\bar a_0a_0\\
&+\left(\alpha_3^{(0)}+\alpha_2^{(2)}\bar\alpha_1^{(2)}+\bar\alpha_3^{(2)}-\alpha_3^{(3)}-\bar\alpha_5^{(3)}-\alpha_1^{(3)}\bar\alpha_2^{(3)}\right)\bar a_0a_0\bar a_2a_2\bar a_2a_2\\
&+\left(\alpha_1^{(0)}\bar\alpha_1^{(0)}+\alpha_3^{(2)}+\bar\alpha_4^{(2)}-\alpha_4^{(3)}-\bar\alpha_3^{(3)}-\alpha_2^{(3)}\bar\alpha_2^{(3)}\right)\bar a_2a_2\bar a_0a_0\bar a_2a_2\\
&+\left(\alpha_2^{(0)}+\bar\alpha_2^{(0)}+\alpha_1^{(2)}\bar\alpha_1^{(2)}-\alpha_1^{(3)}\bar\alpha_1^{(3)}-\alpha_3^{(3)}-\bar\alpha_4^{(3)}\right)\bar a_0a_0\bar a_2a_2\bar a_0a_0\\
&+\dots
\end{aligned}$$

Note that $\bar a_2a_2\bar a_2a_2\bar a_0a_0$, $\bar a_0a_0\bar a_2a_2\bar a_2a_2$, $\bar a_2a_2\bar a_0a_0\bar a_2a_2$, $\bar a_0a_0\bar a_2a_2\bar a_0a_0$ are linearly independent, for $n=7,8$, and

$$\bar a_2a_2\bar a_2a_2\bar a_0a_0+\bar a_0a_0\bar a_2a_2\bar a_2a_2+\bar a_2a_2\bar a_0a_0\bar a_2a_2+\bar a_0a_0\bar a_2a_2\bar a_0a_0=0,$$

for $n=6$. Denote $\beta=\alpha_2^{(0)}+\bar\alpha_2^{(0)}+\alpha_1^{(2)}\bar\alpha_1^{(2)}-\alpha_1^{(3)}\bar\alpha_1^{(3)}-\alpha_3^{(3)}-\bar\alpha_4^{(3)}$, for $n=6$, and $\beta=0$, for $n=7,8$. Then we have

$$\left(\bar\alpha_3^{(0)}+\alpha_1^{(2)}\bar\alpha_2^{(2)}+\alpha_4^{(2)}-\bar\alpha_4^{(3)}-\alpha_5^{(3)}-\alpha_2^{(3)}\bar\alpha_1^{(3)}+\alpha^4\right)-\beta=0,$$

$$\left(\alpha_3^{(0)}+\alpha_2^{(2)}\bar\alpha_1^{(2)}+\bar\alpha_3^{(2)}-\alpha_3^{(3)}-\bar\alpha_5^{(3)}-\alpha_1^{(3)}\bar\alpha_2^{(3)}\right)-\beta=0,$$

$$\left(\alpha_1^{(0)}\bar\alpha_1^{(0)}+\alpha_3^{(2)}+\bar\alpha_4^{(2)}-\alpha_4^{(3)}-\bar\alpha_3^{(3)}-\alpha_2^{(3)}\bar\alpha_2^{(3)}\right)-\beta=0.$$

Summing up the above three equations we obtain

$$\begin{aligned} 0 &= \alpha^4 + 3\beta + \left(\alpha_1^{(0)}\bar{\alpha}_1^{(0)} + \alpha_3^{(0)} + \bar{\alpha}_3^{(0)}\right) \\ &\quad + \left(\alpha_3^{(2)} + \bar{\alpha}_3^{(2)} + \alpha_1^{(2)}\bar{\alpha}_2^{(2)} + \alpha_4^{(2)} + \bar{\alpha}_4^{(2)} + \alpha_2^{(2)}\bar{\alpha}_1^{(2)}\right) \\ &\quad - \Big(\alpha_3^{(3)} + \bar{\alpha}_3^{(3)} + \alpha_2^{(3)}\bar{\alpha}_1^{(3)} + \alpha_4^{(3)} + \bar{\alpha}_4^{(3)} + \alpha_1^{(3)}\bar{\alpha}_2^{(3)} + \alpha_2^{(3)}\bar{\alpha}_2^{(3)} \\ &\qquad + \alpha_5^{(3)} + \bar{\alpha}_5^{(3)}\Big) \\ &= \alpha^4. \end{aligned}$$

This is a contradiction, because $\alpha \neq 0$. This shows that $P^f(\Delta)$ is not isomorphic to $P(\Delta)$.

(ii) It follows from (i) and [1, Corollary 9.3.2] that the categories $\operatorname{proj} P^f(\Delta)$ and $\operatorname{proj} P(\Delta)$ are not equivalent 1-Calabi-Yau categories with the same Auslander-Reiten quiver $Q_\Delta$. In particular, $\operatorname{proj} P(\Delta)$ is standard and $\operatorname{proj} P^f(\Delta)$ is nonstandard. □

*Remark.* We would like to mention that is not clear if there exist non-standard additively finite $K$-linear triangulated categories of Calabi-Yau dimension one over algebraically closed fields $K$ of characteristic different from 2 and 3.

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