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RESEARCH ARTICLE

On derived π-length of a finite π-solvable group with supersolvable π-Hall subgroup

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To memory of L. A. Shemetkov

Abstract. It is proved that if π-Hall subgroup is a supersolvable group then the derived π-length of a π-solvable group G is at most $1 + \max_{r\in\pi} l_r^a(G)$, where $l_r^a(G)$ is the derived r-length of a π-solvable group G.

Introduction

All groups considered in this paper will be finite. All notation and definitions correspond to [1], [2].

Let $\mathbb{P}$ be the set of all prime numbers, and let π be the set of primes. In this paper, π' is the set of all primes not contained in π, i. e. $\pi = \mathbb{P} \setminus \pi'$. By π also denotes a function defined on the set of natural numbers $\mathbb{N}$ as follows: $\pi(a)$ is the set of primes dividing a positive integer a. For a group G and a subgroup H of G we believe that $\pi(G) = \pi(|G|)$ and $\pi(G : H) = \pi(|G : H|)$.

Fix a set of prime numbers π. If $\pi(m) \subseteq \pi$, then a positive integer m is called a π-number. The group G is called a π-group if $\pi(G) \subseteq \pi$, and a π'-group if $\pi(G) \subseteq \pi'$. If G is a π'-group, then $\pi(G) \cap \pi = \varnothing$. The chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_{n-1} \supseteq G_n = 1, \qquad (1)$$

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is called subnormal series of a group G, if subgroup G_{i+1} is normal subgroup of G_i for every i. The quotient groups G_i/G_{i+1} are called factors of the series (1).

The group is called a π-solvable group if it has a subnormal series (1) whose factors are solvable π-groups or π'-groups. The least number of π-factors of all such subnormal series of a group G is called the π-length of a π-solvable group G and is denoted by $l_\pi(G)$. Since π-factors of subnormal series (1) of a π-solvable group G are solvable, then every π-solvable group has subnormal series in which all π-factors are nilpotent. The least number of nilpotent π-factors of all such subnormal series of a group G is called the nilpotent π-length of a π-solvable group G and is denoted by $l^n_\pi(G)$. In case when π consists of a single prime $\{p\}$, i. e. $\pi = \{p\}$, we will obtain $l_\pi(G) = l^n_\pi(G) = l_p(G)$ for every π-solvable group. The equality $l_\pi(G) = l^n_\pi(G)$ is cleared to be a true for a π-solvable group with nilpotent π-Hall subgroup.

Recall that least positive integer m such that $G^{(m)} = 1$ is called the derived length of the group G and is denoted by $d(G)$. Here G' is the derived subgroup of G and $G^{(i)} = (G^{(i-1)})'$.

V. S. Monakhov suggested a new notation of the derived π-length of a π-solvable group. Let G be a π-solvable group. Then G has a subnormal series (1) whose factors are π'-groups or abelian π-groups. The least number of abelian π-factors of all such subnormal series of a group G is called the derived π-length of a π-solvable group G and is denoted by $l^a_\pi(G)$. Clearly, in the case $\pi = \pi(G)$ to $l^a_\pi(G)$ coincides with the derived length of G. The initial properties of the derived π-length and some of the results related to this notion, established in [4] – [6].

In 2001 V. S. Monakhov and O. A. Shpyrko [3] proved that $l^n_\pi(G) \le 1 + \max_{r\in\pi} l_r(G)$ if G is a π-solvable group in which the derived subgroup of a π-Hall subgroup is nilpotent. In this article, we received an analogue of this result for the derived π-length. Also, we obtain a new estimate of derived π-length of a π-solvable group whose all proper subgroups of a π-Hall subgroup are supersolvable.

1. Preliminary results

Lemma 1 ([4, Lemma 3]). *Let G be a π-solvable group. Then $d(G_\pi) \le l^a_\pi(G) \le l_\pi(G)d(G_\pi)$.*

Here and below, G_π is a π-Hall subgroup of a π-solvable group G.

Lemma 2 ([4, Lemma 4]). *Let G be a π-solvable group, and let t be a positive integer. Suppose that $l_{\pi}^{a}(G/N) \leq t$ for every non-trivial subgroup N of G, but $l_{\pi}^{a}(G) > t$. Then:*

(1) $O_{\pi'}(G) = 1$;

(2) *G has a unique minimal normal subgroup;*

(3) $F(G) = O_p(G) = F(O_{\pi}(G))$ *for some prime* $p \in \pi$;

(4) $O_{p'}(G) = 1$ *and* $C_G(F(G)) \subseteq F(G)$.

Here $F(X)$ is the Fitting subgroup of a group X, i. e. $F(X)$ is the largest normal nilpotent subgroup of X.

Lemma 3 ([4, Theorem 1]). *If G is a π-solvable group in which a Sylow p-subgroup is abelian for every $p \in \pi$, then $l_{\pi}^{a}(G) = d(G_{\pi}) \leq |\pi(G_{\pi})|$.*

Lemma 4 ([4, Theorem 2]). *Let G be a π-solvable group. If G_{π} is abelian, then $l_{\pi}^{a}(G) \leq 1$.*

Lemma 5 ([5, Lemma 2.6]). *If G is a π-solvable group and $\pi = \pi_1 \cup \pi_2$, then $l_{\pi}^{a}(G) \leq l_{\pi_1}^{a}(G) + l_{\pi_2}^{a}(G)$.*

Lemma 6 ([5, Theorem 3.1]). *Let G be a p-solvable group. If a Sylow p-subgroup of G is bicyclic, then $l_{p}^{a}(G) \leq 2$ for every $p > 2$ and $l_{p}^{a}(G) \leq 3$ for $p = 2$.*

The group is called a bicyclic group if it is the product of two cyclic subgroups.

Lemma 7 ([7, Theorem 2]). *Let G be a group of odd order. If every Sylow subgroup of G is bicyclic, then the derived subgroup of G is nilpotent.*

A group is called a Schmidt group if it is a non-nilpotent groups all of whose proper subgroups are nilpotent. O. Yu. Schmidt pioneered the study of such groups [8]. A whole paragraph from Huppert's monography is dedicated to Schmidt groups, (see [1, III.5]). A survey of results on the existence of Schmidt subgroups in finite groups and some of their applications in the theory of group classes given in [9].

Lemma 8 ([10, Theorem 2]). *Let G be a p-solvable group. If a Sylow p-subgroup of G is isomorphic to a Sylow Subgroup of a Schmidt group, then $l_{p}^{a}(G) \leq 1$.*

The group is called a Miller-Moreno group if it is a non-abelian group and all of its proper subgroups are abelian. Non-nilpotent Miller-Moreno groups are a special case of Schmidt groups and the structure of these groups is easily derived from the properties of Schmidt groups. Nilpotent Miller-Moreno groups are the groups of prime-power order.

We denote by $\mathfrak{U}$ a class of all supersolvable groups. Then $G^{\mathfrak{U}}$ is $\mathfrak{U}$-residual of G, i. e. $G^{\mathfrak{U}}$ is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{U}$.

Lemma 9 ([11, Theorem 22], [12]). *Let G be a minimal non-supersolvable group, i. e. G is a non-supersolvable group and all proper subgroups of G are supersolvable. Then:*

(1) *G is solvable and $|\pi(G)| \le 3$;*

(2) *$G^{\mathfrak{U}} = P$ is a Sylow p-subgroup of G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(G)$;*

(3) *$P' \subseteq \Phi(P) \subseteq Z(P)$;*

(4) *if Q is a complement to P in G, then $Q/Q \cap \Phi(G)$ is either a cyclic group of prime-power order or a Miller-Moreno group.*

Lemma 10 ([13, Theorem 26.3], [14, Theorem 1]). *The minimal non-supersolvable groups are one of the following types:*

(1) *$G = [P]Q$ is a Schmidt group;*

(2) *$G = [P]Q$, where P is a Sylow p-subgroup of Schmidt type (see the definition in [14]); Q is a cyclic Sylow q-subgroup; $[P]\Phi(Q)$ and $[\Phi(P)]Q$ are supersolvable, $[P, \Phi(Q)] = P$;*

(3) *$G = [P]Q$, where P is a Sylow p-subgroup of Schmidt type; Q is a Sylow q-subgroup; $\Phi(Q) > C_Q(P) \lhd G$; $Q/C_Q(P)$ is either a non-abelian group of order q^3 and exponent q or a Miller-Moreno group of prime-power order containing a cyclic maximal subgroup, $p \equiv 1 \pmod q$; $[P, Q'] = P$, $[\Phi(P)]Q$, $[P]Q_1$ are supersolvable, where Q_1 is any subgroup of Q;*

(4) *$G = [P]([Q]R)$, where P is a Sylow p-subgroup of Schmidt type; Q and R are the cyclic Sylow q- and r-subgroups, $q > r$; $[P]Q$, $[P]R$ and $[Q]R$ are non-nilpotent; $[P, Q] = P$; $[Q, R] = Q$; $\Phi(P) < \Phi(P){\cdot}[P, R] \le P$; $\Phi(P) \times \Phi(Q) \le Z([P]Q)$; $\Phi(R) = Z([Q]R)$, $p \equiv 1 \pmod{qr}$ and $q \equiv 1 \pmod r$.*

Lemma 11. *Let G be a π-solvable group, and let G_π be a minimal non-supersolvable group. Then $l_p(G) \le 1$ and $l_p^a(G) \le 2$ for $p \in \pi((G_\pi)^{\mathfrak{U}})$.*

Proof. By hypothesis, $(G_\pi)^{\mathfrak{U}} = G_p$. First of all, we prove that $l_p(G) \le 1$. The group $G_\pi O_{p'}(G)/O_{p'}(G)$ is a π-Hall subgroup of $G/O_{p'}(G)$ and

$$(G_\pi O_{p'}(G)/O_{p'}(G))^{\mathfrak{U}} = (G_\pi)^{\mathfrak{U}} O_{p'}(G)/O_{p'}(G) \simeq$$

$$\simeq G_p O_{p'}(G)/O_{p'}(G) \simeq G_p \simeq (G_\pi)^{\mathfrak{U}}$$

by properties residuals. The group $G_\pi O_{p'}(G)/O_{p'}(G)$ is a minimal non-supersolvable group and, by induction, $l_p(G/O_{p'}(G)) \leq 1$, so $l_p(G) \leq 1$. Hence we can assume that $O_{p'}(G) = 1$. Therefore, $F(G) = O_p(G)$ and $C_G(O_p(G)) \subseteq O_p(G)$.

Assume that $O_p(G)$ is a proper subgroup of G_p. Clearly, the group $O_p(G)\Phi(G_p)/\Phi(G_p)$ is a normal subgroup of $G_\pi/\Phi(G_p)$. Since $G_p/\Phi(G_p)$ is a minimal normal subgroup of $G_\pi/\Phi(G_p)$ by Lemma 9 (2) and

$$O_p(G)\Phi(G_p)/\Phi(G_p) \subseteq G_p/\Phi(G_p),$$

then

$$O_p(G)\Phi(G_p)/\Phi(G_p) = 1 \text{ or } O_p(G)\Phi(G_p) = G_p.$$

If $O_p(G)\Phi(G_p)/\Phi(G_p) = 1$, then $O_p(G) \subseteq \Phi(G_p)$. Since, by Lemma 9 (3), $\Phi(G_p) \subseteq Z(G_p)$, we have

$$O_p(G) \subseteq Z(G_p),\ G_p \subseteq C_G(O_p(G)) \subseteq O_p(G),$$

we have a contradiction. If $O_p(G)\Phi(G_p) = G_p$, then $O_p(G) = G_p$. Therefore, $O_p(G) = G_p$. Hence $l_p(G) \leq 1$.

By Lemma 9 (3), $d(G_p) \leq 2$, and $l_p^a(G) \leq 2$ by Lemma 1. □

2. Main results

Theorem 1. *Let G be a π-solvable group. If the derived subgroup of G_π is nilpotent, then $l_\pi^a(G) \leq 1 + \max_{r\in\pi} l_r^a(G)$.*

Proof. Let G be a π-solvable group, and let the derived subgroup of G_π be a nilpotent. We use induction on $|G|$. Let N is a normal subgroup of G. Since $G_\pi N/N \simeq G_\pi/(G_\pi \cap N)$, then their derived subgroups are isomorphic.

$$(G_\pi/(G_\pi \cap N))' = (G_\pi)'(G_\pi \cap N)/(G_\pi \cap N) \simeq$$

$$\simeq (G_\pi)'/((G_\pi)' \cap N) \simeq (G_\pi N/N)'.$$

Therefore, the conditions of the lemma are inherited by all quotient groups. By Lemma 2, $O_{\pi'}(G) = 1$, G has a unique minimal normal subgroup

$$C_G(F(G)) \subseteq F(G) = O_p(G) = F(O_\pi(G))$$

for some prime $p \in \pi$. Clearly, $F(G) \subseteq G_\pi$.

Let K be the derived subgroup of G_π. By hypothesis of the theorem subgroup K is nilpotent. Since p'-Hall subgroup $K_{p'}$ of K is a normal subgroup of G_π, it follows

$$K_{p'} \subseteq C_G(F(G)) \subseteq F(G),\ K_{p'} = 1.$$

Thus, K is a p-group, $G_{\pi\backslash\{p\}}$ is abelian and a Sylow q-subgroup of G is abelian for every $q \in \pi\backslash\{p\}$. So $l^a_q(G) = 1$ for every $q \in \pi\backslash\{p\}$ by Lemma 4. Therefore, $\max_{r\in\pi} l^a_r(G) = l^a_p(G)$.

Let $\pi_1 = \pi \setminus \{p\}$. By Lemma 5, $l^a_\pi(G) \leq l^a_{\pi_1}(G) + l^a_p(G)$. Since G_{π_1} is abelian, we have $l^a_{\pi_1}(G) \leq 1$ by Lemma 4. Now $l^a_\pi(G) \leq 1 + l^a_p(G) \leq 1 + \max_{r\in\pi} l^a_r(G)$. □

Corollary 1. *Let G be a π-solvable group. If a Sylow p-subgroup of G is cyclic for every $p \in \pi$, then $l^a_\pi(G) \leq 2$.*

Proof. By Lemma 4, $l^a_p(G) \leq 1$ for all $p \in \pi$, so $\max_{r\in\pi} l^a_r(G) \leq 1$ and, by [1, Theorem IV.2.11], G_π is a supersolvable. By [1, Theorem VI.9.1], the derived subgroup of G_π is nilpotent. By Theorem 1, $l^a_\pi(G) \leq 2$. □

Corollary 2. *Let G be a π-solvable group, and let a Sylow p-subgroup of G be bicyclic for every $p \in \pi$. Then $l^a_\pi(G) \leq 6$. If $2 \notin \pi$, then $l^a_\pi(G) \leq 3$.*

Proof. Let $\pi = \{2\} \cup \tau$. By Lemma 5, $l^a_\pi(G) \leq l^a_2(G) + l^a_\tau(G)$. By Lemma 6, $l^a_2(G) \leq 3$ and $l^a_t(G) \leq 2$ for all $t \in \tau$, so $\max_{t\in\tau} l^a_t(G) \leq 2$. By Lemma 7, the derived subgroup of a τ-Hall subgroup is nilpotent. By Theorem 1, we have that

$$l^a_\tau(G) \leq 1 + \max_{t\in\tau} l^a_t(G) \leq 3.$$

Now $l^a_\pi(G) \leq 6$. If $2 \notin \pi$, then $\pi = \tau$ and $l^a_\pi(G) = l^a_\tau(G) \leq 3$. □

Let H be a subgroup of a group G. A subgroup K of G is called a complement of H in G if $G = HK$ and $H \cap K = 1$. Yu. M. Gorchakov [15] showed that complementability of all subgroups is equivalents to complementability subgroups of prime order. The group G is called completely factorable if all of its subgroups are complemented. In 1937 Ph. Hall [16] found that *finite groups in which all subgroups are complemented exhausted by supersolvable groups with elementary abelian Sylow subgroups.*

Corollary 3. *Let G be a π-solvable group. If G_π is completely factorable, then $l^a_\pi(G) \leq 2$.*

Proof. By [16], G_π of G is supersolvable and a Sylow p-subgroup of G is an elementary abelian for all $p \in \pi$. By [1, Theorem VI.9.1], the derived subgroup of G_π is nilpotent. By Lemma 4 and Theorem 1, $l_\pi^a(G) \le 2$. □

Corollary 4. *Let G be a π-solvable group. If G_π is supersolvable, then $l_\pi^a(G) \le 1 + \max_{r\in\pi} l_r^a(G)$.*

Proof. By [1, Theorem VI.9.1], the derived subgroup of G_π is nilpotent. By Theorem 1, $l_\pi^a(G) \le 1 + \max_{r\in\pi} l_r^a(G)$. □

Corollary 5. *Let G be a π-solvable group. If G_π is a Schmidt group, then $l_\pi^a(G) \le 3$.*

Proof. Let G be a π-solvable group, and let $G_\pi = [P]Q$ be a Schmidt group, when P is a normal Sylow p-subgroup, and Q is a non-normal Sylow q-subgroup. Since Q is cyclic, we have $l_q^a(G) \le 1$ by Lemma 4. By Lemma 8, $l_p(G) \le 1$. Since either P is abelian or $P' = Z(P)$ [8]–[9], we have $d(P) \le 2$. By Lemma 1, $l_p^a(G) \le 2$. By Lemma 5, $l_\pi^a(G) \le l_p^a(G) + l_q^a(G) \le 3$. □

Corollary 6. *Let G be a π-solvable group. If G_π is a Miller-Moreno group, then $l_\pi^a(G) \le 2$.*

Proof. Assume that G_π is not a group of prime-power order. Then G_π is a Schmidt group in which every Sylow subgroup is abelian. So the derived subgroup of G_π is abelian and $\max_{r\in\pi} l_r^a(G) \le 1$ by Lemma 3. By Theorem 1, $l_\pi^a(G) \le 2$.

Let $G_\pi = G_p$ be a group of prime-power order. We use induction on $|G|$. If N is a non-trivial normal subgroup of G, then G_pN/N is an abelian or a Miller-Moreno group. So $l_p^a(G/N) \le 2$ either by Lemma 4 or by induction. By Lemma 2, G has a unique minimal normal subgroup,

$$O_{p'}(G) = 1,\ F(G) = O_p(G),\ C_G(F(G)) \subseteq F(G).$$

If $F(G) = G_p$, then $l_p^a(G) = d(G_p) = 2$. Let $F(G)$ be a proper subgroup of G_p. Then $F(G) \subseteq M$, when M is some maximal subgroup of G_p. By condition, M is abelian. So $M \subseteq C_G(F(G))$ and $F(G) = M$. Now $G_p/F(G)$ has prime order and $l_p^a(G/F(G)) \le 1$ by Lemma 4. Since $F(G)$ is abelian, we have $l_p^a(G) \le 2$. □

Theorem 2. *Let G be a π-solvable group. If every proper subgroup of G_π is supersolvable, then $l_\pi^a(G) \le 2 + \max_{r\in\pi} l_r^a(G)$.*

Proof. If G_π is supersolvable, then $l^a_\pi(G) \le 1 + \max_{r\in\pi} l^a_r(G)$ by Corollary 4. Let G_π be a non-supersolvable group. Then G_π is one of the four types listed in Lemma 10. Notation for G_π corresponds to Lemma 10. By Lemma 11, $l^a_p(G) \le 2$.

If G_π is a group of type (1)–(2), then Q is cyclic and $l^a_q(G) \le 1$ by Lemma 4 and, by Lemma 5,

$$l^a_\pi(G) \le l^a_p(G) + l^a_q(G) \le 2 + 1 \le 2 + \max_{r\in\pi} l^a_r(G).$$

If G_π is a group of type (3), then, by Lemma 5,

$$l^a_\pi(G) \le l^a_p(G) + l^a_q(G) \le 2 + l^a_q(G) \le 2 + \max_{r\in\pi} l^a_r(G).$$

Let G_π be a group of type (4). Then $G_\pi = [P]([Q]R)$, where Q and R are cyclic Sylow q- and r-subgroups. By Lemma 5, $l^a_\pi(G) \le l^a_{\{p,q\}}(G) + l^a_r(G)$. Since $\{p,q\}$-Hall subgroup of group G is supersolvable, we have $l^a_{\{p,q\}}(G) \le 1 + \max_{t\in\{p,q\}} l^a_t(G)$ by Corollary 4. By Lemma 4, $l^a_r(G) \le 1$, and by Lemma 5,

$$l^a_\pi(G) \le l^a_{\{p,q\}}(G) + l^a_r(G) \le 1 + \max_{t\in\{p,q\}} l^a_t(G) + 1 \le 2 + \max_{t\in\pi} l^a_t(G). \quad \square$$

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