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RESEARCH ARTICLE

Combinatorics of irreducible Gelfand-Tsetlin $sl(3)$-modules

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Abstract. In this paper we present an explicit description of all irreducible $\mathfrak{sl}(3)$-modules which admit a Gelfand-Tsetlin tableaux realization with respect to the standard Gelfand-Tsetlin subalgebra.

Introduction

In the present paper we will describe irreducible $\mathfrak{sl}(3)$-modules in a certain full subcategory of the category of Gelfand-Tsetlin modules (we will abbreviate Gelfand-Tsetlin by GT); namely the category GTT of GT-modules that admit a tableaux realization with respect to a GT-subalgebra [9]. This description provides a realization similar to the $\mathfrak{sl}(2)$ case (in the latter it is always possible to choose a basis of eigenvectors with respect to a Cartan subalgebra and write explicit formulas for the action of the generators of $\mathfrak{sl}(2)$).

Following [9]; we say that an $\mathfrak{sl}(n)$-module V admits a tableaux realization with respect to a GT-subalgebra Γ provided V decomposes as $V = \bigoplus_{\xi \in \Gamma^*} V_\xi$ where

$$V_\xi := \{v \in V : \exists k \in \mathbb{N} \text{ such that } (t - \xi(t))^k v = 0 \ \forall t \in \Gamma\},$$

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$dim(V_\xi) \leq 1$ for all $\xi \in \Gamma^*$ and the action of the generators of $\mathfrak{sl}(n)$ is given by the GT-formulas ([7], [11]). It was shown in [2] that in $\mathfrak{sl}(3)$, for any irreducible GT-module V, $dim(V_\xi) \leq 2$ for all $\xi \in \Gamma^*$. Moreover, there are explicit examples of GT-modules with $dim(V_\xi) = 2$ for some $\xi \in Supp(V)$. Hence GTT is a proper subcategory of GT.

In sections 1 and 2 we give the definitions and notations that we will use throughout the paper. The section 3 is devoted to the description of a basis for the irreducible $\mathfrak{sl}(3)$-modules in GTT which is the main result of the paper. As a direct consequence of this description it is possible to give simple conditions for a tableau such that the associated irreducible module has bounded weight multiplicities or 1-dimensional weight spaces. In section 4 we use the results of section 3 to answer when a highest weight $\mathfrak{sl}(3)$-module admits a tableaux realization (with respect to some GT-subalgebra). Finally, in the section 5 we give a characterization of the irreducible $\mathfrak{sl}(3)$-modules in GTT which are Harish-Chandra modules.

1. Gelfand-Tsetlin modules

Let $n \in \mathbb{N}$ fixed; for $k \in \{1, 2, \dots, n\}$ denotes by $\mathfrak{g}_k := \mathfrak{gl}(k)$; $U_k := U(\mathfrak{g}_k)$ the universal enveloping algebra of $\mathfrak{g}_k$ and $Z_k := Z(\mathfrak{g}_k)$ the center of $\mathfrak{g}_k$; let also $\mathfrak{g} := \mathfrak{g}_n$ and $U := U(\mathfrak{g})$.

If $\{E_{ij}\}$ denotes the canonical basis of $\mathfrak{g}$, we have a natural identification between $\mathfrak{g}_k$ and the subalgebra of $\mathfrak{g}$ generated by the matrices $\{E_{ij}\}_{i,j=1,\dots,k}$; i.e. consider $\mathfrak{g}_i$ as a subalgebra of $\mathfrak{g}_{i+1}$ with respect to the upper left corner embedding.

$$\left[\begin{array}{c|c|c|c|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \cline{1-1}
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cline{1-2}
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cline{1-3}
\ddots & \ddots & \ddots & \ddots & \vdots \\ \cline{1-4}
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{array}\right]$$

The chain of inclusions: $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ induces a chain of inclusions of the corresponding enveloping algebras.

Definition 1. Let Γ the subalgebra of U generated by $\{Z_k : k = 1, \dots, n\}$; this subalgebra is called standard **Gelfand-Tsetlin subalgebra** of U [3].

Remark 1. Z_m is a polynomial algebra in m variables $\{c_{mk} : k = 1, 2, \dots, m\}$,

$$c_{mk} = \sum_{(i_1, i_2, \dots, i_k) \in \{1, \dots, m\}^k} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_k i_1}$$

and the algebra Γ is a maximal commutative polynomial subalgebra of $U(\mathfrak{g})$ in $\frac{n(n+1)}{2}$ variables $\{c_{ij} : 1 \leq j \leq i \leq n\}$.

Definition 2. Let M be a $\mathfrak{g}$-module; $\chi : \Gamma \to \mathbb{C}$ a homomorphism and

$$M_\chi = \{v \in M : \exists k \in \mathbb{N} \text{ such that } (g - \chi(g))^k v = 0 \quad \forall g \in \Gamma\}.$$

The module M is called **Gelfand-Tsetlin module** (respect to Γ) if $M = \bigoplus_{\chi \in \Gamma^*} M_\chi$ and $dim(M_\chi) < \infty$ for all $\chi \in \Gamma^*$ [3].

Definition 3. An array of rows with complex entries $\{l_{ij} : 1 \leq j \leq i \leq n\}$ as follows:

$$\begin{array}{ccccccc}
\boxed{\lambda_{n1}} & \boxed{\lambda_{n2}} & & \cdots & & \boxed{\lambda_{n,n-1}} & \boxed{\lambda_{nn}} \\
& \boxed{\lambda_{n-1,1}} & & \cdots & & \boxed{\lambda_{n-1}^{n-1}} & \\
& & \cdots & \cdots & \cdots & & \\
& & \boxed{\lambda_{21}} & & \boxed{\lambda_{22}} & & \\
& & & \boxed{\lambda_{11}} & & &
\end{array}$$

is called **Gelfand-Tsetlin tableau**. A Gelfand-Tsetlin tableau is called **standard** if

$$\lambda_{ki} - \lambda_{k-1,i} \in \mathbb{Z}^{\geq 0} \quad \textit{and} \quad \lambda_{k-1,i} - \lambda_{k,i+1} \in \mathbb{Z}^{\geq 0}, \text{ for all} 1 \leq i \leq k \leq n-1.$$

In the finite dimensional case we have the following classical result [7]:

Theorem 1. If $L(\lambda)$ is a finite dimensional irreducible representation of $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$, there exist a bases $\{\xi_{[L]}\}$ of $L(\lambda)$ parameterized by all standard tableaux $[L]$ with top row $\lambda_{n1} = \lambda_1, \dots, \lambda_{nn} = \lambda_n$ and the $\mathfrak{gl}(n)$ generators acts by the formulas:

$$E_{k,k+1}(\xi_{[L]}) = -\sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k+1}(l_{ki} - l_{k+1,j})}{\prod_{j \neq i}^{k}(l_{ki} - l_{kj})}\right) \xi_{[L+\delta^{ki}]},$$

$$E_{k+1,k}(\xi_{[L]}) = \sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(l_{ki}-l_{k-1,j})}{\prod_{j\neq i}^{k}(l_{ki}-l_{kj})}\right)\xi_{[L-\delta^{ki}]},$$

$$E_{kk}(\xi_{[L]}) = \left(\sum_{i=1}^{k}\lambda_{ki} - \sum_{i=1}^{k-1}\lambda_{k-1,i}\right)\xi_{[L]},$$

Where $l_{ki} = \lambda_{ki} - i + 1$, $[L \pm \delta^{ki}]$ is the tableau obtained by $[L]$ adding ± 1 to the ki position of $[L]$; and if $[\tilde{L}]$ is not standard, the vector $\xi_{[\tilde{L}]}$ would be zero. Moreover, the action of the generators of Γ in the basis elements is given by:

$$c_{ij}(\xi_{[L]}) = \left(\sum_{k=1}^{i}(l_{ik}+i)^j \prod_{s\neq k}\left(1 - \frac{1}{l_{ik}-l_{is}}\right)\right)\xi_{[L]}$$

The formulas of the previous theorem are called **Gelfand-Tsetlin formulas** for $\mathfrak{gl}(n)$.

2. Gelfand-Tsetlin formulas for $\mathfrak{sl}(3)$

Remark 2. From now on we will prefer to use tableaux with entries l_{ij} instead of λ_{ij} because the formulas are symmetric with respect to the l_{ij}'s in the following sense. Let R_i denote the i-th row of the tableaux $[L]$ and S_i the i-th symmetric group. We have a natural action of the group $S_1 \times S_2 \times \cdots \times S_n$ on the set of GT-tableaux (with entries l_{ij}): $(\sigma_1, \ldots, \sigma_n)([L])$ is the tableau with the i-th row $\sigma_i(R_i)$, for $i = 1, 2, \ldots, n$.

Definition 4. The tableaux $[L]_1$ and $[L]_2$ are equivalent if there exist $(\sigma_1, \ldots, \sigma_n)$ such that $(\sigma_1, \ldots, \sigma_n)([L]_1) = [L]_2$ and we write in this case $[L]_1 \approx [L]_2$.

Remark 3. If we want to recover the tableaux with coefficients λ_{ij} we just need to remember the relations $l_{ki} = \lambda_{ki} - i + 1$.

In the particular case of $\mathfrak{gl}(3)$, let $a, b, c, x, y, z \in \mathbb{C}$ fixed complex numbers; from now on we will use $[L]$ to denote the fixed tableau;

$[L]$:=

a	b	c
x	y	
z		

Then, the GT-formulas for the generators of $\mathfrak{gl}(3)$ are:

$$E_{11}([L]) = z[L] \qquad E_{12}([L]) = -(x-z)(y-z)[L+\delta^{11}]$$
$$E_{22}([L]) = (x+y+1-z)[L] \qquad E_{21}([L]) = [L-\delta^{11}]$$
$$E_{33}([L]) = (a+b+c+2-x-y)[L]$$

$$E_{32}([L]) = \tfrac{(x-z)}{(x-y)}[L-\delta^{21}] - \tfrac{(y-z)}{(x-y)}[L-\delta^{22}]$$
$$E_{23}([L]) = \tfrac{(a-x)(b-x)(c-x)}{(x-y)}[L+\delta^{21}] - \tfrac{(a-y)(b-y)(c-y)}{(x-y)}[L+\delta^{22}].$$

As we want to restrict our attention to $\mathfrak{sl}(3)$, we have to consider just the tableaux such that $E_{11}([L]) + E_{22}([L]) + E_{33}([L]) = 0$ that implies $a+b+c+3=0$; then the GT-formulas for the generators of $\mathfrak{sl}(3)$ are given by:

$$\textbf{GT-formulas:} \quad \begin{cases} h_1([L]) = (2z-(x+y+1))[L] \\ h_2([L]) = (2(x+y+1)-z)[L] \\ E_{12}([L]) = -(x-z)(y-z)[L+\delta^{11}] \\ E_{21}([L]) = [L-\delta^{11}] \\ E_{32}([L]) = \tfrac{(x-z)}{(x-y)}[L-\delta^{21}] - \tfrac{(y-z)}{(x-y)}[L-\delta^{22}] \\ E_{23}([L]) = \dfrac{(a-x)(b-x)(c-x)}{(x-y)}[L+\delta^{21}] - \\ \qquad\qquad - \dfrac{(a-y)(b-y)(c-y)}{(x-y)}[L+\delta^{22}]. \end{cases}$$

Now we introduce some notation that will help us to simplify the desired description.

Notation 1. Let

$$[T] = \begin{array}{ccccc} l_{31} & & l_{32} & & l_{33} \\ & l_{21} & & l_{22} & \\ & & l_{11} & & \end{array}$$

be an arbitrary tableau, $B_1([T]) := \{l_{31}-l_{21}, l_{32}-l_{21}, l_{33}-l_{21}\}$ and $B_2([T]) := \{l_{31}-l_{22}, l_{32}-l_{22}, l_{33}-l_{22}\}$. We consider the following functions:

- $t_0([T]) := l_{21}-l_{22}$; $t_3([T]) := l_{21}-l_{11}$; $t_3^-([T]) := l_{22}-l_{11}$
- $t_i([T]) := \min\{\{B_i([T]) \cap \mathbb{Z}^{\geq 0}\} \cup \{+\infty\}\}$; $i = 1, 2$

- $t_i^-([T]) := \max\{\{B_i([T]) \cap \mathbb{Z}^{<0}\} \cup \{-\infty\}\}$; $i = 1, 2$.

In the cases that $t_i([T]) = +\infty$ or $t_i^-([T]) = -\infty$ for some $i = 1, 2$, we will write $t_i([T]) \notin \mathbb{Z}$ or $t_i^-([T]) \notin \mathbb{Z}$ respectively.

From now on in order to simplify the notation we will write t_0, t_1, t_2, t_1^-, t_2^-, t_3, t_3^- instead to $t_0([L])$, $t_1([L])$, $t_2([L])$, $t_1^-([L])$, $t_2^-([L])$, $t_3([L])$, $t_3^-([L])$, where $[L]$ is the fixed tableau as before.

3. Description of irreducible GTT $\mathfrak{sl}(3)$-modules

Given a tableau $[L]$, we can look at the set of all tableaux that can be obtained with non-zero coefficients from $[L]$ using the GT-formulas. It is natural to ask what are the possible tableaux $[L]$ that we can consider in order to obtain an $\mathfrak{sl}(3)$-module structure on the vector space generated by this set of tableaux.

The only problem (if we apply the GT-formulas) is a possibility of zero denominators. Thus we have to restrict our attention to the **lattice of tableaux** of $[L]$

$$Latt([L]) := \{[\tilde{L}] : [\tilde{L}] \text{ is obtained from } [L] \\ \text{using GT-formulas and } t_0([\tilde{L}]) \neq 0\}.$$

Here to obtain $[\tilde{L}]$ from $[L]$ using the GT-formulas means that there exist $X \in U(\mathfrak{sl}(3))$ such that $[\tilde{L}]$ appear with non-zero coefficient in $X([L])$.

The following result from [3] implies the existence of some GT-modules called *generic Gelfand-Tsetlin* modules.

Theorem 2. If $t_0, t_1, t_2, t_1^-, t_2^-, t_3, t_3^- \notin \mathbb{Z}$ then, the $\mathbb{C}$-vector space $V_{[L]}$ generated by the set of vectors $\{\xi_{[\tilde{L}]} : [\tilde{L}] \in Latt([L])\}$ defines an *irreducible* $sl(3)$-module with the action of $\mathfrak{sl}(3)$ given by the GT-formulas.

Corollary 3. If $t_0 \notin \mathbb{Z}$ then, the $\mathbb{C}$-vector space $V_{[L]}$ generated by the set of vectors $\{\xi_{[\tilde{L}]} : [\tilde{L}] \in Latt([L])\}$ has a structure of GT $sl(3)$-module; where the action of $\mathfrak{sl}(3)$ is given by the GT-formulas.

Definition 5. We say that an $\mathfrak{sl}(3)$-module V admits a tableaux realization with respect to a GT-subalgebra Γ provided that V is a GT-module (with respect to Γ), $dim(V_\xi) \leq 1$ for all $\xi \in \Gamma^*$ and the action of the generators of $\mathfrak{sl}(3)$ is given by the GT-formulas. Equivalently, a GT $\mathfrak{sl}(3)$-module is said to have a **tableaux realization** if it is isomorphic to $V_{[T]}$ for some tableau $[T]$. We say that a module is a **GTT-module** if it admits a tableaux realization.

In this section we will describe explicitly bases of all irreducible $\mathfrak{sl}(3)$-modules in GTT and then we will be able to calculate weight multiplicities (with respect to the standard Cartan subalgebra of $\mathfrak{sl}(3)$) of this modules in terms of the values of the constants $t_0, t_1, t_2, t_1^-, t_2^-, t_3, t_3^-$.

By the GT-formulas, we have that $Latt([L]) \subset \{[L]_{m,n,k} : m, n, k \in \mathbb{Z}\}$ where the tableaux $[L]_{m,n,k}$ is defined as:

$$[L]_{m,n,k} := \begin{array}{ccc} \boxed{a} & \boxed{b} & \boxed{c} \\ & \boxed{x+m} \;\; \boxed{y+n} & \\ & \boxed{z+k} & \end{array}$$

Then we can identify $Latt([L])$ with points of $\mathbb{R}^3$ with integer coordinates to describe a basis of the module $V_{[L]}$.

Let $m, n, k \in \mathbb{Z}^{\geq 0}$, applying the GT-formulas to $[L]$ we see that:

1) $[L + m\delta^{21}]$ appears in the decomposition of $E_{23}^m[L]$ with coefficient

$$\prod_{i=0}^{m-1} \frac{(a-x-i)(b-x-i)(c-x-i)}{(x-y+i)}$$

2) $[L - m\delta^{21}]$ appears in the decomposition of $E_{32}^m[L]$ with coefficient

$$\prod_{i=0}^{m-1} \frac{(x-z-i)}{(x-y-i)}$$

3) $[L + n\delta^{22}]$ appears in the decomposition of $E_{23}^n[L]$ with coefficient

$$\prod_{i=0}^{n-1} \frac{(a-y-i)(b-y-i)(c-x-i)}{(x-y-i)}$$

4) $[L - n\delta^{22}]$ appears in the decomposition of $E_{32}^n[L]$ with coefficient

$$\prod_{i=0}^{n-1} \frac{(y-z-i)}{(x-y+i)}$$

5) $E_{21}^k([L]) = [L - k\delta^{11}]$

6) $E_{12}^k([L]) = \prod_{i=0}^{k-1}(x - z - i)(y - z - i)[L + k\delta^{11}]$

As an immediate consequence of the above observation we have the following lemma:

Lemma 4. For $i = 1, 2, 3$ denote by A_i the conditions $t_i \notin \mathbb{Z}^{\geq 0}$ and by A_3^- the condition $t_3^- \notin \mathbb{Z}^{\geq 0}$. Then the following statements hold:

1) $[L + m\delta^{21}] \in Latt([L])$ for all $m \in \mathbb{Z}^+$ if A_1 and $t_0 \notin \mathbb{Z}^{<0}$.
2) $[L - m\delta^{21}] \in Latt([L])$ for all $m \in \mathbb{Z}^+$ if A_3 and $t_0 \notin \mathbb{Z}^{>0}$.
3) $[L - k\delta^{11}] \in Latt([L])$ for all $k \in \mathbb{Z}^+$.
4) $[L + k\delta^{11}] \in Latt([L])$ for all $k \in \mathbb{Z}^+$ if A_3 and A_3^-.
5) $[L + n\delta^{22}] \in Latt([L])$ for all $m \in \mathbb{Z}^+$ if A_2 and $t_0 \notin \mathbb{Z}^{>0}$.
6) $[L - n\delta^{22}] \in Latt([L])$ for all $m \in \mathbb{Z}^+$ if A_3^- and $t_0 \notin \mathbb{Z}^{<0}$.

Now we will answer the following question: what conditions on the entries of $[L]$ guarantee that $Latt([L]) = \mathbb{Z}^3$ (i.e. when $Latt([L])$ is the largest possible)?

Definition 6. Given $m, n, k \in \mathbb{Z}$, we say that the tableau $[L]_{m,n,k}$, can be obtained from $[L]$ by the path $r \to s \to t$, with $\{r, s, t\} = \{1, 2, 3\}$ if: From $[L]$ we can obtain $[L]_{(m,0,0)}$ if $r = 1$ (respectively $[L]_{(0,n,0)}$ if $r = 2$ and $[L]_{(0,0,k)}$ if $r = 3$); from $[L]_{(m,0,0)}$ we obtain $[L]_{(m,n,0)}$ if $s = 2$ or $[L]_{(m,0,k)}$ if $s = 3$ (respectively from $[L]_{(0,n,0)}$ we obtain $[L]_{(m,n,0)}$ if $s = 1$ or $[L]_{(0,n,k)}$ if $s = 3$ and from $[L]_{(0,0,k)}$ we obtain $[L]_{(m,0,k)}$ if $s = 1$ or $[L]_{(0,n,k)}$ if $s = 2$) and in the last step we obtain the tableau $[L]_{m,n,k}$.

Example 1. $[L]_{7,-1,4}$ is obtained from $[L]$ by the path $3 \to 1 \to 2$ means: from $[L]$ we obtain the tableau $[L]_{0,0,4}$; with this tableau we obtain $[L]_{7,0,4}$ and from this, we can obtain $[L]_{7,-1,4}$.

Proposition 5. $Latt([L]) = \mathbb{Z}^3$ if and only if

$$t_1, t_2, t_3, t_3^- \notin \mathbb{Z}^{\geq 0};\ t_0 \notin \mathbb{Z}.$$

Proof:. ($\Leftarrow$) Let $m, n, k \in \mathbb{Z}$. Using lemma 4 in each step of the path indicated below it is possible to obtain the tableau $[L]_{m,n,k}$. In each case the path will depend of the ordered triple of signs of m, n, k as follows:

$(+,+,+)$	$(-,-,-)$	$(+,-,+)$
$\begin{cases} 3 \to 2 \to 1; & \text{if } n \geq m \\ 3 \to 1 \to 2; & \text{if } m \geq n \end{cases}$	$\begin{cases} 1 \to 2 \to 3; & \text{if } m \geq n \\ 2 \to 1 \to 3; & \text{if } n \geq m \end{cases}$	$3 \to 1 \to 2$

$(+,+,-)$	$(-,-,+)$	$(+,-,-)$
$\begin{cases} 1\to 2\to 3; & \text{if } m\geq n \\ 2\to 1\to 3; & \text{if } n\geq m \end{cases}$	$\begin{cases} 1\to 2\to 3; & \text{if } m\geq n \\ 2\to 1\to 3; & \text{if } n\geq m \end{cases}$	$1\to 2\to 3$

$(-,+,+)$	$(-,+,-)$
$3\to 2\to 1$	$1\to 2\to 3$

($\Rightarrow$) Without loss of generality we can assume that some of these constants are zero. Then we conclude:

1) If $t_1=0$ then it is not possible to obtain $[L]_{1,0,0}$ from $[L]$.
2) If $t_2=0$ then $[L]_{0,1,0}\notin Latt([L])$.
3) If $t_3=0$ or $t_3^-=0$ then $[L]_{0,0,1}\notin Latt([L])$ or $[L]_{0,0,-1}\notin Latt([L])$ respectively. □

Now we have enough information about $Latt([L])$ in order to describe irreducible modules. For this we will use the following characterization.

The module $V_{[L]}$ is irreducible if and only if $Latt([\tilde{L}])=Latt([L])$ for all $[\tilde{L}]\in Latt([L])$.

Theorem 6. Let $[L]$ be such that $Latt([L])=\mathbb{Z}^3$. Then $V_{[L]}$ is irreducible if and only if

$$t_0,t_1,t_1^-,t_2,t_2^-,t_3,t_3^-\notin\mathbb{Z}.$$

Proof:. ($\Leftarrow$) Under the conditions it is possible to apply the GT-formulas to any tableau in $Latt([L])$ and we never obtain zero coefficients. Then for all $[\tilde{L}]\in Latt([L])$ we have $Latt([\tilde{L}])=\mathbb{Z}^3$ which implies $V_{[L]}$ irreducible.

($\Rightarrow$) If $t_1^-\in\mathbb{Z}^{<0}$ (respectively t_2^- or $t_3^-\in\mathbb{Z}^{<0}$) then $[L]\notin Latt([L]_{-1,0,0})$ (respectively $[L]\notin Latt([L]_{0,-1,0})$ or $[L]\notin Latt([L]_{0,0,-1})$). Hence $V_{[L]}$ can not be irreducible. □

Remark 4. Note that it is possible to obtain $Latt([L])=\mathbb{Z}^3$ in the case when some constants are negative integers, but not necessarily we obtain an irreducible module.

Remark 5. To know a basis of the module generated by $[L]$ it is enough to know the values of the constants $\{t_0,t_1,t_1^-,t_2,t_2^-,t_3,t_3^-\}$. For some subset A of $\{t_0,t_1,t_1^-,t_2,t_2^-,t_3,t_3^-\}$, the notation $A\subset\mathbb{Z}$ will means from now on that $A\subset\mathbb{Z}$ and the complement of A in $\{t_0,t_1,t_1^-,t_2,t_2^-,t_3,t_3^-\}$ has empty intersection with $\mathbb{Z}$. In particular $t_1,t_2\in\mathbb{Z}^{\geq 0}$ means that $\{t_0,t_1^-,t_2^-,t_3,t_3^-\}\bigcap\mathbb{Z}=\emptyset$.

Proposition 7. Let $[L]$ be a fixed tableau as before. Denote by $V_{[L]}$ the $\mathfrak{sl}(3)$-module generated by $Latt([L])$ using the GT-formulas. Then

1) If $t_1 \in \mathbb{Z}^{\geq 0}$, $V_{[L]}$ is an irreducible module with bases parameterized by $Latt([L]) = \{[L]_{m,n,k} : m \leq t_1\}$.
2) If $t_2 \in \mathbb{Z}^{\geq 0}$, $V_{[L]}$ is an irreducible module with bases parameterized by $Latt([L]) = \{[L]_{m,n,k} : n \leq t_2\}$.
3) If $t_3 \in \mathbb{Z}^{\geq 0}$, $V_{[L]}$ is an irreducible module with bases parameterized by $Latt([L]) = \{[L]_{m,n,k} : k - m \leq t_3\}$.
4) If $t_1^- \in \mathbb{Z}^{<0}$, the irreducible module that contains $[L]$ can be obtain as a quotient module of $V_{[L]}$; and the bases is parameterized by the set of tableaux $\{[L]_{m,n,k} : m > t_1^-\}$.
5) If $t_1^- \in \mathbb{Z}^{<0}$, the irreducible module that contains $[L]$ can be obtain as a quotient module of $V_{[L]}$; and the bases is parameterized by the set of tableaux $\{[L]_{m,n,k} : n > t_2^-\}$.
6) If $t_3 \in \mathbb{Z}^{<0}$, the irreducible module that contains $[L]$ can be obtain as a quotient module of $V_{[L]}$; and the bases is parameterized by the set of tableaux $\{[L]_{m,n,k} : k - m > t_3\}$.

Proof. The cases $1, 2, 3$ are obvious from the GT-formulas and the irreducibility is guaranteed by the Theorem 6. In each of the cases $4, 5, 6$ we can apply the GT-formulas and obtain in $Latt([L])$ a tableaux $[\widetilde{L}]$ that satisfies $t_1([\tilde{L}]) \in \mathbb{Z}^{\geq 0}$ (respectively $t_2([\tilde{L}]) \in \mathbb{Z}^{\geq 0}$ or $t_3([\tilde{L}]) \in \mathbb{Z}^{\geq 0}$). Then the irreducible module that contains $[L]$ is isomorphic to the quotient module $V_{[L]}/V_{[\widetilde{L}]}$. □

Corollary 8. Using Proposition 7 we can characterize the set of tableaux that parameterizes a basis of the irreducible module that contains $[L]$ as follows:

1) For $t_1 \in \mathbb{Z}^{\geq 0}$; $\{[L]_{m,n,k} : t_1([L]_{m,n,k}) \in \mathbb{Z}^{\geq 0}\}$.
2) For $t_2 \in \mathbb{Z}^{\geq 0}$; $\{[L]_{m,n,k} : t_2([L]_{m,n,k}) \in \mathbb{Z}^{\geq 0}\}$.
3) For $t_3 \in \mathbb{Z}^{\geq 0}$; $\{[L]_{m,n,k} : t_3([L]_{m,n,k}) \in \mathbb{Z}^{\geq 0}\}$.
4) For $t_1^- \in \mathbb{Z}^{<0}$; $\{[L]_{m,n,k} : t_1^-([L]_{m,n,k}) \in \mathbb{Z}^{<0}\}$.
5) For $t_2^- \in \mathbb{Z}^{<0}$; $\{[L]_{m,n,k} : t_2^-([L]_{m,n,k}) \in \mathbb{Z}^{<0}\}$.
6) For $t_3^- \in \mathbb{Z}^{<0}$; $\{[L]_{m,n,k} : t_3^-([L]_{m,n,k}) \in \mathbb{Z}^{<0}\}$.

Corollary 9. If A denotes the set $\{t_1, t_2, t_1^-, t_2^-, t_3, t_3^-\}$, $[L]$ satisfies the conditions $A_1 := A \bigcap \mathbb{Z}^{\geq 0}$ and $A_2 := A \bigcap \mathbb{Z}^{<0}$ and those conditions implies $t_0 \neq 0$; then, a base for the irreducible module that contains $[L]$ can be parameterized by:

$$\{[L]_{m,n,k} : A_1([L]_{m,n,k}) \in \mathbb{Z}^{\geq 0} \text{ and } A_2([L]_{m,n,k}) \in \mathbb{Z}^{<0}\}$$

Definition 7. For each tableau $[T]$ satisfying the conditions of corollary 9 we will denote by $I_{[T]}$ the irreducible $\mathfrak{sl}(3)$-module generated by $[T]$ with the basis parameterized by the set of tableaux described as before. This basis we will be denote by $\mathcal{B}_{[T]}$.

We can take advantage of knowing these bases to calculate the weights dimensions of modules with tableaux realization.

If we want to know the action of h_1 and h_2 in the module $I_{[L]}$ it is enough to describe the action of h_1 and h_2 in tableaux of type $[L]_{m,n,k}$.

- $h_1([\mathbf{L}]_{m,n,k}) = (2(z+k) - (x+y+1+n+m))[\mathbf{L}]_{m,n,k}$
- $h_2([\mathbf{L}]_{m,n,k}) = (2(x+y+1+n+m) - (z+k))[\mathbf{L}]_{m,n,k}$.

Set $\lambda^{(1)}_{m,n,k} := 2(z+k) - (x+y+1+n+m)$ and $\lambda^{(2)}_{m,n,k} := 2(x+y+1+n+m) - (z+k)$. Since x, y, z are fixed, we have a natural identification between weights of the module $I_{[L]}$ and points in $\mathbb{Z} \times \mathbb{Z}$ as follows:

$$(\lambda^{(1)}_{m,n,k}, \lambda^{(2)}_{m,n,k}) \leftrightsquigarrow (2k, 2(m+n)) \leftrightsquigarrow (k, n+m) \leftrightsquigarrow (\alpha, \beta).$$

Theorem 10. For each $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$ the dimension of the weight space $(I_{[L]})_{(2(z+\alpha)-(x+y+1+\beta), 2(x+y+1+\beta)-(z+\alpha))}$ is equal to the cardinality of the set

$$T_{(\alpha,\beta)} := \{[L]_{t,\beta-t,\alpha} : t \in \mathbb{Z}\} \bigcap \mathcal{B}_{[L]}$$

Proof. It is sufficient to note that the vector associated with a tableaux $[L]_{m,n,k}$ has weight $(2(z+\alpha) - (x+y+1+\beta), 2(x+y+1+\beta) - (z+\alpha))$ if and only if $m + n = \beta$ and $k = \alpha$. □

Now we will describe explicitly bases and weight multiplicities of all irreducible $\mathfrak{sl}(3)$-modules that admit a tableaux realization. To do that we have to consider all possible combinations of conditions defining non-isomorphic modules (some of these conditions define isomorphic modules in the sense of the Definition 4; for instance, a module defined by a tableau $[L]$ satisfying the conditions $t_1 \in \mathbb{Z}^{\geq 0}$ is naturally isomorphic to the module defined by the tableau $\sigma([L])$ where $\sigma \in S_1 \times S_2 \times S_3$; in particular to a module defined by a tableau satisfying the conditions $t_2 \in \mathbb{Z}^{\geq 0}$).

First we consider the conditions that give infinite dimensional weight spaces.

Conditions	$\mathcal{B}_{[L]}$
	$\{L_{m,n,k} : m,n,k \in \mathbb{Z}\}$
$t_2 \in \mathbb{Z}^{\geq 0}$	$\{L_{m,n,k} : n \leq t_2\}$
$t_3 \in \mathbb{Z}^{\geq 0}$	$\{L_{m,n,k} : k \leq m + t_3\}$
$t_1^- \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m > t_1^-\}$
$t_3 \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m < k - t_3\}$
$t_2, t_3 \in \mathbb{Z}^{\geq 0}$	$\{L_{m,n,k} : k - t_3 \leq m; n \leq t_2\}$
$t_1 \in \mathbb{Z}^{\geq 0}, t_3 \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m < k - t_3; m \leq t_1\}$
$t_2^-, t_3 \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m < k - t_3; n > t_2^-\}$
$t_1 \in \mathbb{Z}^{\geq 0}, t_2^- \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m \leq t_1; n > t_2^-\}$
$t_3 \in \mathbb{Z}^{\geq 0}, t_1^- \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m \geq k - t_3; m > t_1^-\}$
$t_2, t_3 \in \mathbb{Z}^{\geq 0}; t_1^- \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m \geq k - t_3; n \leq t_2; m > t_1^-\}$
$t_1 \in \mathbb{Z}^{\geq 0}; t_2^-, t_3 \in \mathbb{Z}^{<0}$	$\{L_{m,n,k} : m < k - t_3; n > t_2^-; m \leq t_1\}$
$t_3 \in \mathbb{Z}^{\geq 0}, t_3^- \in \mathbb{Z}^{<0}$ *	$\{L_{m,n,k} : n + t_3^- < k \leq m + t_3\}$

In all other cases we have $dim(V_{(\lambda^{(1)},\lambda^{(2)})}) < \infty$ for all weight space.

Conditions	$\mathcal{B}_{[L]}$	Dimension of $V_{(\alpha,\beta)}$
$t_1, t_3 \in \mathbb{Z}^{\geq 0}$	$k - t_3 \leq m \leq t_1$	$\begin{cases} 0 & \text{if } \alpha > t_1 + t_3 \\ t_1 + t_3 - \alpha + 1, & \text{if } \alpha \leq t_1 + t_3 \end{cases}$
$t_1, t_2 \in \mathbb{Z}^{\geq 0}$	$\begin{cases} n \leq t_2; \\ m \leq t_1 \end{cases}$	$\begin{cases} 0 & \text{if } \beta > t_1 + t_2 \\ t_1 + t_2 - \beta + 1, & \text{if } \beta \leq t_1 + t_2 \end{cases}$
$t_1^-, t_3 \in \mathbb{Z}^{<0}$	$t_1^- < m < k - t_3$	$\begin{cases} 0 & \text{if } \alpha \leq t_1^- + t_3 \\ \alpha - t_3 - t_1^- - 1, & \text{if } \alpha > t_1^- + t_3 \end{cases}$
$t_1^-, t_2^- \in \mathbb{Z}^{<0}$	$\begin{cases} m > t_1^-; \\ n > t_2^- \end{cases}$	$\begin{cases} 0 & \text{if } \beta \leq t_1^- + t_2^- \\ \beta - t_2^- - t_1^- - 1, & \text{if } \beta > t_1^- + t_2^- \end{cases}$
$t_1, t_2, t_3 \in \mathbb{Z}^{\geq 0}$	$\begin{cases} n \leq t_2; \\ k - t_3 \leq m \leq t_1 \end{cases}$	$\begin{cases} 0 & \text{if } \beta > t_1 + t_2 \\ 0 & \text{if } \alpha > t_1 + t_3 \\ t_1 + t_2 - \beta + 1, & \text{if } \beta - \alpha \geq t_2 - t_3 \\ t_1 + t_3 - \alpha + 1, & \text{if } \beta - \alpha \leq t_2 - t_3 \end{cases}$
$\begin{cases} t_1 \in \mathbb{Z}^{\geq 0}; \\ t_1^- \in \mathbb{Z}^{<0} \end{cases}$	$t_1^- < m \leq t_1$	$t := t_1 - t_1^-$
$\begin{cases} t_3 \in \mathbb{Z}^{\geq 0}, \\ t_2^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m \geq k - t_3; \\ n > t_2^- \end{cases}$	$\begin{cases} 0 & \text{if } \beta - \alpha > t_2^- - t_3 \\ \beta - \alpha - t_2^- + t_3, & \text{if } \beta - \alpha \leq t_2^- - t_3 \end{cases}$
$\begin{cases} t_2 \in \mathbb{Z}^{\geq 0}, \\ t_3 \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m < k - t_3; \\ n \leq t_2 \end{cases}$	$\begin{cases} 0 & \text{if } \beta - \alpha \geq t_2 - t_3 \\ \alpha - \beta + t_2 - t_3, & \text{if } \beta - \alpha < t_2 - t_3 \end{cases}$
$\begin{cases} t_3 \in \mathbb{Z}^{\geq 0}; \\ t_1^-, t_2^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m \geq k - t_3; \\ n > t_2^-; \\ m > t_1^- \end{cases}$	$\begin{cases} 0 & \text{if } \beta \leq t_1^- + t_2^- + 1 \\ 0 & \text{if } \beta - \alpha \leq t_2^- - t_3 \\ \beta - \alpha - t_2^- + t_3, & \text{if } \alpha \geq t_1^- + t_3 + 1 \\ \beta - t_1^- - t_2^- - 1, & \text{if } \alpha \leq t_1^- + t_3 + 1 \end{cases}$

Conditions	$\mathcal{B}_{[L]}$	Dimension of $V_{(\alpha,\beta)}$
$\begin{cases} t_2 \in \mathbb{Z}^{\geq 0}; \\ t_1^-, t_3 \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} t_1^- < m < k - t_3; \\ n \leq t_2 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha \leq t_3 + t_1^- + 1 \\ 0 & \text{if } \beta - \alpha \geq t_2 - t_3 \\ \alpha - t_3 - t_1^- - 1, & \text{if } \beta \leq t_1^- + t_2 + 1 \\ \alpha - \beta - t_3 + t_2, & \text{if } \beta \geq t_1^- + t_2 + 1 \end{cases}$
$t_1^-, t_2^-, t_3 \in \mathbb{Z}^{<0}$	$\begin{cases} t_1^- < m < k - t_3; \\ n > t_2^- \end{cases}$	$\begin{cases} 0 & \text{if } \alpha \leq t_3 + t_1^- \\ 0 & \text{if } \beta \leq t_1^- + t_2^- \\ \beta - t_2^- - t_1^- - 1, & \text{if } \beta - \alpha \leq t_2^- - t_3 \\ \alpha - t_3 - t_1^- - 1, & \text{if } \beta - \alpha \geq t_2^- - t_3 \end{cases}$
$\begin{cases} t_2, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_2^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m \geq k - t_3; \\ t_2^- < n \leq t_2 \end{cases}$	$\begin{cases} 0 & \text{if } \beta - \alpha \leq t_2^- - t_3 \\ t := t_2 - t_2^-, & \text{if } \beta - \alpha \geq t_2 - t_3 \\ \beta - \alpha + t_3 - t_2^-, & \text{if } \beta - \alpha \leq t_2 - t_3 \end{cases}$
$\begin{cases} t_1, t_2, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_1^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} n \leq t_2; \\ t_1^- < m \leq t_1; \\ k - t_3 \leq m \end{cases}$	$\begin{cases} 0 & \text{if } \beta > t_1 + t_2 \\ 0 & \text{if } \alpha > t_1 + t_3 \\ t_1 + t_2 - \beta + 1, & \text{if } \beta - \alpha \geq t_2 - t_3 \wedge \\ & \beta \geq t_2 + t_1^- + 1 \\ t_1 + t_3 - \alpha + 1, & \text{if } \beta - \alpha \leq t_2 - t_3 \wedge \\ & \alpha \geq t_3 + t_1^- + 1 \\ t := t_1 - t_1^-, & \text{if } \alpha \leq t_3 + t_1^- + 1 \wedge \\ & \beta \leq t_2 + t_1^- + 1 \end{cases}$
$\begin{cases} t_2 \in \mathbb{Z}^{\geq 0}; \\ t_2^-, t_3 \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m < k - t_3; \\ t_2^- < n \leq t_2 \end{cases}$	$\begin{cases} 0 & \text{if } \beta - \alpha \geq t_2 - t_3 \\ t := t_2 - t_2^-, & \text{if } \beta - \alpha \leq t_2^- - t_3 \\ \alpha - \beta - t_3 + t_2, & \text{if } \beta - \alpha \geq t_2^- - t_3 \end{cases}$
$\begin{cases} t_2 \in \mathbb{Z}^{\geq 0}; \\ t_2^-, t_1^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m > t_1^-; \\ t_2^- < n \leq t_2 \end{cases}$	$\begin{cases} 0 & \text{if } \beta \leq t_1^- + t_2^- + 1 \\ t := t_2 - t_2^-, & \text{if } \beta \geq t_2 + t_1^- + 1 \\ \beta - t_1^- - t_2^- - 1, & \text{if } \beta \leq t_2 + t_1^- + 1 \end{cases}$
$\begin{cases} t_1, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_1^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m \geq k - t_3; \\ t_1^- < m \leq t_1 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha > t_1 + t_3 \\ t := t_1 - t_1^-, & \text{if } \alpha \leq t_1^- + t_3 + 1 \\ t_1 + t_3 - \alpha, & \text{if } \alpha \geq t_1^- + t_3 + 1 \end{cases}$
$\begin{cases} t_1, t_2 \in \mathbb{Z}^{\geq 0}; \\ t_1^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} n \leq t_2; \\ t_1^- < m \leq t_1 \end{cases}$	$\begin{cases} 0 & \text{if } \beta > t_1 + t_2 \\ t := t_1 - t_1^-, & \text{if } \beta \leq t_1^- + t_2 + 1 \\ t_1 + t_2 - \beta + 1, & \text{if } \beta \geq t_1^- + t_2 + 1 \end{cases}$
$\begin{cases} t_2 \in \mathbb{Z}^{\geq 0}; \\ t_2^-, t_3^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} n < k - t_3^-; \\ t_2^- < n \leq t_2 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha \leq t_2^- + t_3^- + 1 \\ t := t_2 - t_2^-, & \text{if } \alpha \geq t_2 + t_3^- + 1 \\ \alpha - t_3^- - t_2^- - 1, & \text{if } \alpha \leq t_2 + t_3^- + 1 \end{cases}$
$\begin{cases} t_1, t_2 \in \mathbb{Z}^{\geq 0}; \\ t_3 \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m < k - t_3; \\ n \leq t_2; \\ m \leq t_1 \end{cases}$	$\begin{cases} 0 & \text{if } \beta - \alpha \geq t_2 - t_3 \\ 0 & \text{if } \beta > t_2 + t_1 \\ t_1 + t_2 - \beta + 1, & \text{if } \alpha \geq t_1 + t_3 + 1 \\ \alpha - \beta - t_3 + t_2, & \text{if } \alpha \leq t_1 + t_3 + 1 \end{cases}$
$\begin{cases} t_1, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_2^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} k - t_3 \leq m \leq t_1; \\ n > t_2^- \end{cases}$	$\begin{cases} 0 & \text{if } \beta - \alpha \leq t_2^- - t_3 \\ 0 & \text{if } \alpha > t_3 + t_1 \\ t_1 + t_3 - \alpha + 1, & \text{if } \beta \geq t_1 + t_2^- + 1 \\ \beta - \alpha - t_2^- + t_3, & \text{if } \beta \leq t_1 + t_2^- + 1 \end{cases}$

Conditions	$\mathcal{B}_{[L]}$	Dimension of $V_{(\alpha,\beta)}$
$\begin{cases} t_2 \in \mathbb{Z}^{\geq 0}; \\ t_2^-, t_1^-, t_3 \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} t_1^- < m < k - t_3; \\ t_2^- < n \leq t_2 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha \leq t_1^- + t_3 + 1 \\ 0 & \text{if } \beta - \alpha \geq t_2 - t_3 \\ 0 & \text{if } \beta \leq t_1^- + t_2^- + 1 \\ t := t_2 - t_2^-, & \text{if } \beta - \alpha \leq t_2^- + t_3 \wedge \\ & \beta \geq t_2 + t_1^- + 1 \\ \alpha - \beta - t_3 + t_2, & \text{if } \beta - \alpha \leq t_2^- + 1 \wedge \\ & \beta \geq t_2 + t_1^- + 1 \\ \alpha - t_3 - t_1^- - 1, & \text{if } \beta - \alpha \leq t_2^- + 1 \wedge \\ & \beta \leq t_2 + t_1^- + 1 \\ \beta - t_2^- - t_1^- - 1, & \text{if } \beta - \alpha \geq t_2^- + t_3 \wedge \\ & \beta \leq t_2 + t_1^- + 1 \end{cases}$
$\begin{cases} t_1, t_2 \in \mathbb{Z}^{\geq 0}; \\ t_1^-, t_3^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m \leq t_1; \\ t_2^- < n \leq t_2; \\ k \leq m + t_3 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha - \beta < t_3^- - t_1 + 1 \\ 0 & \text{if } \beta > t_2 + t_1 \\ \alpha - \beta + t_1 - t_3^-, & \text{if } \alpha - \beta \leq t_3^- - t_1^- \wedge \\ & \alpha \leq t_3^- + t_2 + 1 \\ t_1 + t_2 - \beta + 1, & \text{if } \beta \geq t_1^- + t_2 + 1 \wedge \\ & \alpha \geq t_2 + t_3^- + 1 \\ t := t_1 - t_1^-, & \text{if } \beta \leq t_1^- + t_2 + 1 \wedge \\ & \beta - \alpha \leq t_1^- - t_3^- \end{cases}$
$\begin{cases} t_2 \in \mathbb{Z}^{\geq 0}; \\ t_2^-, t_1^-, t_3^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} t_1^- < m; \\ t_2^- < n \leq t_2; \\ n < k - t_3^- \end{cases}$	$\begin{cases} 0 & \text{if } \alpha \leq t_2^- + t_3^- + 1 \\ 0 & \text{if } \beta \leq t_1^- + t_2^- + 1 \\ t := t_2 - t_2^-, & \text{if } \alpha \geq t_2 + t_3^- + 1 \wedge \\ & \beta \geq t_2 + t_1^- + 1 \\ \alpha - t_3^- - t_2^- - 1, & \text{if } \beta - \alpha \geq t_1^- - t_3^- \wedge \\ & \alpha \leq t_2 + t_3^- + 1 \\ \beta - t_2^- - t_1^- - 1, & \text{if } \beta - \alpha \leq t_1^- - t_3^- \wedge \\ & \beta \leq t_2 + t_1^- + 1 \end{cases}$
$\begin{cases} t_1, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_1^-, t_2^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} t_1^- < m \leq t_1; \\ t_2^- < n; \\ k - t_3 \leq m \end{cases}$	$\begin{cases} 0 & \text{if } \alpha > t_1 + t_3 \\ 0 & \text{if } \beta - \alpha < t_2^- - t_3 + 1 \\ 0 & \text{if } \beta < t_1^- + t_2^- \\ t := t_1 - t_1^-, & \text{if } \beta \geq t_1 + t_2^- - 1 \wedge \\ & \alpha \leq t_3 + t_1^- + 1 \\ \beta - \alpha - t_2^- + t_3, & \text{if } \beta \leq t_2^- + t_1 + 1 \wedge \\ & \alpha > t_3 + t_1^- \\ \beta - t_2^- - t_1^- - 1, & \text{if } \beta \leq t_2^- + t_1 - 1 \wedge \\ & \alpha \leq t_3 + t_1^- + 1 \\ t_1 + t_3 - \alpha + 1, & \text{if } \beta \geq t_2^- + t_1 - 1 \wedge \\ & \alpha > t_3 + t_1^- \end{cases}$
$\begin{cases} t_2, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_1^-, t_2^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} t_1^- < m; \\ t_2^- < n \leq t_2; \\ k - t_3 \leq m \end{cases}$	$\begin{cases} 0 & \text{if } \beta \leq t_1^- + t_2^- + 1 \\ 0 & \text{if } \beta - \alpha \leq t_2^- - t_3 \\ t := t_2 - t_2^-, & \text{if } \beta \geq t_2 + t_1^- + 1 \wedge \\ & \beta - \alpha \geq t_2 - t_3 \\ \beta - t_2^- - t_1^- - 1, & \text{if } \beta \leq t_2 + t_1^- + 1 \wedge \\ & \alpha \leq t_3 + t_1^- + 1 \\ \beta - \alpha - t_3 - t_2^-, & \text{if } \beta - \alpha \leq t_2 - t_3 \wedge \\ & \alpha > t_3 + t_1^- \end{cases}$

Conditions	$\mathcal{B}_{[L]}$	Dimension of $V_{(\alpha,\beta)}$
$\begin{cases} t_3 \in \mathbb{Z}^{\geq 0}; \\ t_1^-, t_2^-, t_3^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} t_1^- < m; \\ t_2^- < n; \\ n + t_3^- < k \leq m + t_3 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha \leq t_3^- + t_2^- + 1 \\ 0 & \text{if } \beta - \alpha \leq t_2^- - t_3 \\ 0 & \text{if } \beta \leq t_1^- + t_2^- + 1 \\ \beta - \alpha - t_2^- + t_3, & \text{if } 2\alpha - \beta \geq t_3^- + t_3 + 1 \wedge \\ & \ \alpha \geq t_1^- + t_3 + 1 \\ \beta - t_2^- - t_1^- - 1, & \text{if } \alpha \leq t_1^- + t_3 + 1 \wedge \\ & \ \beta - \alpha \leq t_1^- - t_3^- \\ \alpha - t_3^- - t_2^- - 1, & \text{if } 2\alpha - \beta > t_3^- + t_3 \wedge \\ & \ \alpha > t_1^- + t_3 \end{cases}$
$\begin{cases} t_1, t_2, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_2^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m \leq t_1; \\ t_2^- < n \leq t_2; \\ k \leq m + t_3 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha > t_1 + t_3 \\ 0 & \text{if } \alpha - \beta > t_3 - t_2^- - 1 \\ 0 & \text{if } \beta > t_2 + t_1 \\ \beta - \alpha + t_3 - t_2^-, & \text{if } \beta - \alpha \leq t_2 - t_3 \wedge \\ & \ \beta \leq t_2^- + t_1 + 1 \\ t_1 + t_2 - \beta + 1, & \text{if } \beta \geq t_2^- + t_1 + 1 \wedge \\ & \ \beta - \alpha \geq t_2 - t_3 \\ t_1 + t_3 - \alpha + 1, & \text{if } \beta \geq t_2^- + t_1 + 1 \wedge \\ & \ \beta - \alpha \leq t_2 - t_3 \\ t := t_2 - t_2^-, & \text{if } \beta \leq t_2^- + t_1 + 1 \wedge \\ & \ \beta - \alpha \geq t_2 - t_3 \end{cases}$
$\begin{cases} t_1, t_2 \in \mathbb{Z}^{\geq 0}; \\ t_2^-, t_3^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} m \leq t_1; \\ t_2^- < n \leq t_2; \\ k \leq m + t_3 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha - \beta < t_3^- - t_1 + 1 \\ 0 & \text{if } \beta > t_2 + t_1 \\ 0 & \text{if } \alpha < t_2^- + t_3^- + 2 \\ t_1 + t_2 - \beta + 1, & \text{if } \beta \geq t_2^- + t_1 + 1 \wedge \\ & \ \alpha \geq t_2 + t_3^- + 1 \\ \alpha - t_2^- - t_3^- - 1, & \text{if } \alpha \leq t_2 + t_3^- + 1 \wedge \\ & \ \beta \leq t_1 + t_2^- + 1 \\ \alpha - \beta + t_1 - t_3^-, & \text{if } \alpha \leq t_2 + t_3^- + 1 \wedge \\ & \ \beta \geq t_1 + t_2^- + 1 \\ t := t_1 - t_1^-, & \text{if } \beta \leq t_2^- + t_1 + 1 \wedge \\ & \ \alpha \geq t_2 + t_3^- + 1 \end{cases}$
$\begin{cases} t_1, t_2, t_3 \in \mathbb{Z}^{\geq 0}; \\ t_1^-, t_3^- \in \mathbb{Z}^{<0} \end{cases}$	$\begin{cases} n \leq t_2; \\ n + t_3^- < k \leq m + t_3; \\ t_1^- < m \leq t_1 \end{cases}$	$\begin{cases} 0 & \text{if } \alpha > t_1 + t_3 \\ 0 & \text{if } \beta - \alpha \geq t_1 - t_3^- \\ 0 & \text{if } \beta > t_1 + t_2 \\ t := t_1 - t_1^- & \text{if } \alpha \leq t_3 + t_1^- + 1 \wedge \\ & \ \beta - \alpha \leq t_1^- - t_3^- \wedge \\ & \beta \leq t_1^- + t_2 + 1 \\ \alpha - \beta - t_3^- + t_1 & \text{if } 2\alpha - \beta \leq t_3^- + t_3 + 1 \wedge \\ & \ \beta - \alpha \leq t_1^- - t_3^- \wedge \\ & \ \alpha \geq t_3^- + t_2 + 1 \\ t_1 + t_3 - \alpha - 1 & \text{if } 2\alpha - \beta \geq t_3^- + t_3 + 1 \wedge \\ & \ \alpha \geq t_3 + t_1^- + 1 \wedge \\ & \ \beta - \alpha \leq t_2 - t_3 \\ t_1 + t_2 - \beta - 1 & \text{if } \alpha - \beta \leq t_3 - t_2 \wedge \\ & \ \alpha \geq t_2 + t_3^- + 1 \wedge \\ & \ \beta \geq t_1^- + t_2 + 1 \end{cases}$

And finally we have a description of the set of tableaux that define finite dimensional $\mathfrak{sl}(3)$-modules. They have to satisfies the following conditions:

- **Conditions**: $t_1, t_2, t_3 \in \mathbb{Z}^{\geq 0}; t_1^-, t_2^-, t_3^- \in \mathbb{Z}^{<0}$
- $\mathcal{B}_{[L]}$: $\{[L]_{(m,n,k)} : t_2^- < n \leq t_2; n + t_3^- < k \leq m + t_3; t_1^- < m \leq t_1\}$
- **Weight Multiplicities**:

$$\begin{cases} 0 & \text{if } \alpha < t_3^- + t_2^- \wedge \ \beta - \alpha \geq t_1 - t_3^- \\ 0 & \text{if } \beta - \alpha < t_2^- - t_3 - 1 \wedge \ \alpha > t_1 + t_3 \\ 0 & \text{if } \beta \leq t_1^- + t_2^- + 1 \\ t_1 - t_1^- & \text{if } \beta - \alpha \leq t_1^- - t_3^- \wedge \alpha \leq t_1^- + t_3 + 1 \\ t_2 - t_2^- & \text{if } \beta - \alpha \geq t_2 - t_3 \wedge \alpha \geq t_2 + t_3^- + 1 \end{cases}$$

$$\begin{cases} \alpha - \beta + t_1 - t_3^- & \text{if } 2\alpha - \beta \leq t_3^- + t_3 + 1 \wedge \beta - \alpha \geq t_1^- - t_3^- \\ & \quad \wedge \alpha \leq t_2 + t_3^- + 1 \wedge \beta \geq t_1 + t_2^- + 1 \\ t_1 + t_3 - \alpha + 1 & \text{if } 2\alpha - \beta \geq t_3^- + t_3 + 1 \wedge \alpha \geq t_3 + t_1^- + 1 \\ & \quad \wedge \alpha - \beta \geq t_3 - t_2 \wedge \beta \leq t_1^- + t_2 + 1 \\ t_1 + t_2 - \beta + 1 & \text{if } \alpha \geq t_3^- + t_2 + 1 \wedge \beta - \alpha \geq t_2 - t_3 \\ & \quad \wedge \beta \geq t_2 + t_1^- + 1 \wedge \beta \geq t_1 + t_2^- + 1 \\ \alpha - t_3^- - t_2^- - 1 & \text{if } 2\alpha - \beta \leq t_3^- + t_3 + 1 \wedge \alpha \leq t_2 + t_3^- + 1 \\ & \quad \wedge \beta - \alpha \geq t_1^- - t_3^- \wedge \beta \leq t_1 + t_2^- + 1 \\ \beta - \alpha + t_3 - t_2^- & \text{if } 2\alpha - \beta \geq t_3 + t_3^- + 1 \wedge \alpha \geq t_1^- + t_3 + 1 \\ & \quad \wedge \alpha - \beta \geq t_3 - t_2 \wedge \beta \leq t_1 + t_2^- + 1 \\ \beta - t_1^- - t_2^- - 1 & \text{if } \beta - \alpha \leq t_1^- - t_3^- \wedge \alpha \leq t_1^- + t_3 + 1 \\ & \quad \wedge \beta \geq t_2 + t_1^- + 1 \wedge \beta \leq t_1 + t_2^- + 1 \end{cases}$$

As an immediate consequence of the above description we can characterize irreducible modules in GTT with 1-dimensional weight spaces and those with bounded multiplicities.

Definition 8. A weight $\mathfrak{g}$-module V is called **pointed** if $dim(V_\lambda) = 1$ for all weight λ such that $dim(V_\lambda) \neq 0$.

Corollary 11. The irreducible $\mathfrak{sl}(3)$-module generated by $[L]$ is a pointed module if and only if $[L]$ satisfies the following conditions:

$$t_1 = 0, \quad t_1^- = -1 \qquad \text{or } t_2 = 0, \ t_2^- = -1.$$

Definition 9. A weight module V is **bounded** if there exist $N \in \mathbb{N}$ such that $dim(V_\lambda) \leq N$ for all weight λ.

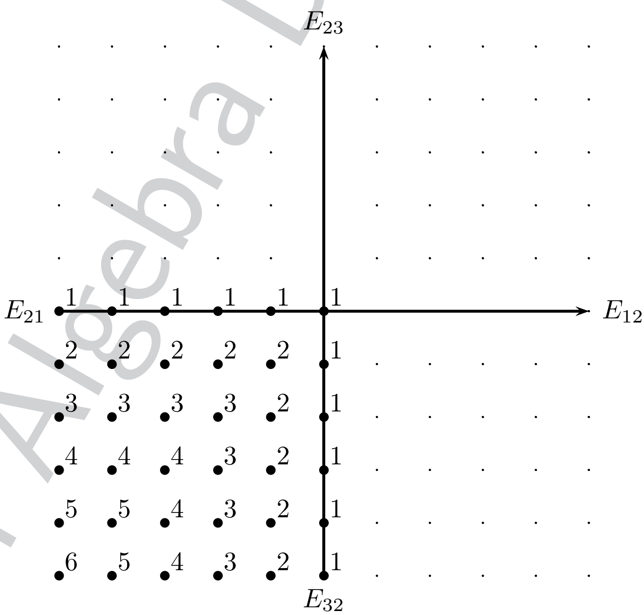
Corollary 12. The irreducible $\mathfrak{sl}(3)$-module generated by $[L]$ is bounded if and only if $[L]$ satisfies the following conditions:

$$t_1 \in \mathbb{Z}^{\geq 0}; \quad t_1^- \in \mathbb{Z}^{<0} \qquad \text{or } t_2 \in \mathbb{Z}^{\geq 0},\ t_2^- \in \mathbb{Z}^{<0}$$

Example 2. Let be $c = -3 - \pi - \sqrt{2}$, the following tableau satisfies $t_1 = 0, t_2 = 0, t_3 = 0$; $t_1^-, t_2^-, t_3^- \notin \mathbb{Z}$. Hence we are in the case $t_1, t_2, t_3 \in \mathbb{Z}^{\geq 0}$.

$$[L] := \begin{array}{ccc} \pi & \sqrt{2} & c \\ \pi & \sqrt{2} & \\ \pi & & \end{array}$$

1) **Basis**: $\{L_{(m,n,k)} : k \leq m \leq 0; n \leq 0\}$
2) **Weights Multiplicities**:



4. On tableaux realizations of highest weight $\mathfrak{sl}(3)$-modules

In this section we will discuss the tableaux realizations of highest weight $\mathfrak{sl}(3)$-modules with respect to different choices of GT-subalgebras [6].

i) Let $\Gamma_1 := \Gamma$ the standard GT-subalgebra obtained by the inclusions with respect to the left upper corner. The formulas in this case are given by:

$$\begin{cases} h_1([L]) = (2z - (x+y+1))[L] \\ h_2([L]) = (2(x+y+1) - z)[L] \\ E_{12}([L]) = -(x-z)(y-z)[L+\delta^{11}] \\ E_{23}([L]) = \frac{(a-x)(b-x)(c-x)}{(x-y)}[L+\delta^{21}] - \frac{(a-y)(b-y)(c-y)}{(x-y)}[L+\delta^{22}] \end{cases}$$

Then, looking at the formulas, the only possible tableau that can represent a highest weight vector is:

$$[T] := \begin{array}{ccccc} x & & y & & c \\ & x & & y & \\ & & x & & \end{array}$$

where $c = -3 - x - y$ and the highest weight is $\lambda = (x - y - 1, x + 2y + 2)$. But in this case we can not represent highest weights with tableau where $t_0([T]) = 0$ (i.e. $\lambda = (-1, 3x + 2)$, $x \in \mathbb{C}$). then we obtain highest weight tableau for $\lambda \neq (-1, h_2)$ with $h_2 \in \mathbb{C}$.

ii) Let Γ_2 the GT-subalgebra induced by the inclusions with respect to the lower right corner. The GT formulas in this case are given by:

$$\begin{cases} h_2([L]) = (2z - (x+y+1))[L] \\ h_1([L]) = (2(x+y+1) - z)[L] \\ E_{23}([L]) = -(x-z)(y-z)[L+\delta^{11}] \\ E_{12}([L]) = \frac{(a-x)(b-x)(c-x)}{(x-y)}[L+\delta^{21}] - \frac{(a-y)(b-y)(c-y)}{(x-y)}[L+\delta^{22}] \end{cases}$$

Then, if $c := -3 - x - y$; the possible highest weights vectors are represented by the following tableau:

$$[T] := \begin{array}{ccccc} x & & y & & c \\ & x & & y & \\ & & x & & \end{array}$$

with highest weight $\lambda = (x+2y+2, x-y-1)$; (as in the case of Γ_1 we have the restriction $x \neq y$; i.e. $\lambda \neq (3x+2, -1)$; $x \in \mathbb{C}$); then we obtain highest weight tableaux realization for $\lambda \neq (h_1, -1)$ with $h_1 \in \mathbb{C}$.

iii) Let Γ_3 the GT-subalgebra induced by the subalgebras inclusions:

$$\langle E_{31} \rangle \subset \langle E_{11}, E_{13}, E_{31}, E_{33} \rangle \subset \mathfrak{gl}(3)$$

The GT-formulas in this case are given by:

$$\begin{cases} h_1([L]) = [(2(x+y+1)-z) + (2z-(x+y+1))][L] \\ h_2([L]) = -(2z-(x+y+1))[L] \\ E_{12}([L]) = [L-\delta^{11}] \\ E_{23}([L]) = \dfrac{(a-x)(b-x)(c-x)(y-z)}{(x-y)}[L+\delta^{21}+\delta^{11}] - \\ \qquad - \dfrac{(a-y)(b-y)(c-y)(x-z)}{(x-y)}[L+\delta^{22}+\delta^{11}] \end{cases}$$

Then, the possible highest weights vectors are represented by the following tableau:

$$[T_1] := \begin{array}{ccc} x & z-1 & \tilde{c} \\ & x \quad z-1 & \\ & z & \end{array}$$

where $\tilde{c} = -2-x-z$ and the highest weight is $\lambda = (x+2z, x-z)$. Then we obtain highest weight tableaux for $x \neq z-1$ that means $\lambda \neq (3z-1, -1)$ with $z \in \mathbb{C}$. Then, with Γ_3 we obtain tableaux realizations of highest weight modules such that the highest weight satisfies $\lambda \neq (h_1, -1)$ with $h_1 \in \mathbb{C}$.

Proposition 13. If $\lambda \neq (-1, -1)$; the irreducible highest weight $\mathfrak{sl}(3)$-module with highest weight λ admits a tableaux realization with respect to some GT-subalgebra.

5. Harish Chandra $\mathfrak{sl}(3)$-modules in *GTT*

Let $\mathcal{B}$ a Chevalley basis for $\mathfrak{sl}(3)$ given by:

$$X_\alpha := E_{12} \qquad Y_\alpha := E_{21} \qquad H_\alpha := E_{11} - H_{22} \qquad X_{\alpha+\beta} := E_{13}$$
$$X_\beta := E_{23} \qquad Y_\beta := E_{32} \qquad H_\beta := E_{22} - E_{33} \qquad Y_{\alpha+\beta} := E_{31}$$

and set $\tilde{\mathfrak{g}}$ the Lie subalgebra $\langle X_\alpha, Y_\alpha, H_\alpha\rangle \cong \mathfrak{sl}(2)$.

Definition 10. An $\mathfrak{sl}(3)$-module V is called **left (respectively right) Harish-Chandra module** if can be expressed as a sum of lowest weight (respectively highest weight) $\mathfrak{sl}(2)$-modules.

Definition 11. An $\mathfrak{sl}(3)$-module V is called **Harish-Chandra module** if can be expressed as a sum of finite dimensional $\mathfrak{sl}(2)$-modules. Equivalently; if the module is a left and right Harish-Chandra module.

Lemma 14. Let V be an irreducible $\mathfrak{sl}(3)$-module and $0 \neq v \in V$. If there exists $n \in \mathbb{Z}^{\geq 0}$ (respectively $n \in \mathbb{Z}^{<0}$) such that $X_\alpha^n v = 0$ then, for all $u \in V$ there exist $r = r(u) \in \mathbb{Z}^{\geq 0}$ (respectively $r \in \mathbb{Z}^{<0}$) such that $X_\alpha^r u = 0$.

Proof:. As V is irreducible, each $u \in V$ can be expressed as $u = \sum_k a_k v$ where a_k are elements of $U(\mathfrak{sl}(3))$. Then the statement of lemma is a consequence of the fact that for all $N \in \mathbb{Z}$ we have:

$$X_\alpha^N X_\beta = N X_{\alpha+\beta} X_\alpha^{N-1} + X_\beta X_\alpha^N, \qquad X_\alpha^N H_\alpha = H_\alpha X_\alpha^N - 2N X_\alpha^N$$
$$X_\alpha^N Y_{\alpha+\beta} = Y_{\alpha+\beta} X_\alpha^N - 2Y_\beta X_\alpha^{N-1}, \qquad X_\alpha^N H_\beta = H_\beta X_\alpha^N + N X_\alpha^N$$
$$X_\alpha Y_\alpha = Y_\alpha X_\alpha^N + N H_\alpha X_\alpha^{N-1} - 2N X_\alpha^{N-1}. \qquad \square$$

Corollary 15. An irreducible $\mathfrak{sl}(3)$-module V is a Harish-Chandra module (with respect to $\tilde{\mathfrak{g}}$) if and only if there exist $0 \neq v \in V$ and $n \in \mathbb{N}$ such that $X_\alpha^{\pm n} v = 0$.

As a consequence of the description of bases for irreducible $\mathfrak{sl}(3)$-modules in GTT we have the following corollaries:

Corollary 16. The irreducible $\mathfrak{sl}(3)$-module generated by $[L]$ is a left (respectively right) Harish-Chandra module (with respect to $\tilde{\mathfrak{g}}$) if and only if

$$t_3 \in \mathbb{Z}^{\geq 0} \qquad \text{(respectively } t_3^- \in \mathbb{Z}^{<0}\text{)}$$

Corollary 17. The irreducible $\mathfrak{sl}(3)$-module generated by $[L]$ is a Harish-Chandra module (with respect to $\tilde{\mathfrak{g}}$) if and only if at least the conditions holds:

$$t_3 \in \mathbb{Z}^{\geq 0}, \qquad t_3^- \in \mathbb{Z}^{<0}$$

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