Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 14 **(2012)**. Number 1. pp. 29 – 36

The representation type of elementary abelian p-groups with respect to the modules of constant Jordan type

Vitalij M. Bondarenko, Iryna V. Lytvynchuk

Communicated by V. V. Kirichenko

Abstract. We describe the representation type of elementary abelian p-groups with respect to the modules of constant Jordan type and offer two conjectures (for such modules) in the general case, one of which suggests that any non-wild group is of finite representation type in each dimension.

Introduction

Modules of constant Jordan type for a finite group G (or, more generally, for an arbitrary finite group scheme) were introduced by Carlson, Friedlander and Pevtsova in 2008 [1], and have been studied in many papers (see, e.g., [2]–[6]). In this paper we study the case when G is an elementary abelian p-group by considering the question of its representation type with respect to the modules of such type.

We use the matrix language instead of the module one, and usually say "representation" instead of "matrix representation". Through the paper, k is an algebraically closed field of characteristic $p > 0$. All representations are over k unless otherwise stated (when one considers representations over the free algebra $k\langle x, y\rangle$).

2010 MSC: 16G60, 20C20.

Key words and phrases: elementary abelian groups, matrix representations, representation type, constant Jordan type.

Let $G = \langle g_1, \dots g_r \rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian p-group. A matrix representation λ of G is said to be of *constant Jordan type* if the Jordan canonical form of the nilpotent matrix $a_1\lambda(g_1-1)+\dots+a_r\lambda(g_r-1)$ (that corresponds to a radical element $a_1(g_1-1)+\dots+a_r(g_r-1)$ of kG) is independent of $a_1, \dots, a_r \in k$, not all of which are equal to zero. If this Jordan canonical form consists of Jordan blocks of size $t_1, \dots, t_s$, then one says that the representation λ has *Jordan type* $JT(\lambda) = [t_1]\dots[t_s]$.

We call G of *cJ-finite representation type over k* if there are, up to equivalence, only finitely many indecomposable representations of constant Jordan type, and of *cJ-infinite representation type* if otherwise. In the last case G is called of *cJ-semiinfinite representation type* if there are only finitely many indecomposable representations in each dimension.

Modifying Drozd's definition [7], introduce the notion of a wild group with respect to the representations of constant Jordan type.

Let G be an elementary abelian p-group. We say that a matrix representation γ of G over the free associative k-algebra $\Sigma = k\langle x, y\rangle$ is *cJ-perfect* if, for any matrix representations φ and φ' of Σ over k, the representations $\gamma\otimes\varphi$ and $\gamma\otimes\varphi'$ of G over k satisfy the next conditions[1]:

1) $\gamma\otimes\varphi$ is of constant Jordan type;

2) $\gamma\otimes\varphi$ and $\gamma\otimes\varphi'$ are equivalent implies φ and φ' are equivalent;

3) $\gamma\otimes\varphi$ is indecomposable if φ is indecomposable[2].

A cJ-perfect representation γ is said to be *proportional* if (under the above notation) $JT(\gamma\otimes\varphi') = [JT(\gamma\otimes\varphi)]^q$ whenever $\dim\varphi' = q\dim\varphi$ (it is sufficient to require this property only for $\dim\varphi = 1$).

We call the group G of *cJ-wild representation type* or *cJ-wild* (*over k*) if it has a proportional cJ-perfect representation over Σ [3].

In this paper we prove the following theorem.

Theorem 1. *An elementary abelian p-group $G = (\mathbb{Z}/p)^r$ is of*

cJ-finite representation type if $r = 1$ (for any p),

cJ-semiinfinite representation type if $r = p = 2$,

cJ-wild representation type if otherwise.

[1] For a fix φ of dimension s, the matrix $(\gamma\otimes\varphi)(g)$ is obtained from a matrix $\gamma(g)$ by change x and y, respectively, on the matrices $\varphi(x)$ and $\varphi(y)$ (and $a \in k$ on the scalar matrix aE_s, where E_s is the identity matrix of size s).

[2] The opposite directions to 2) and 3) are evident.

[3] Note that there are formally a weaker definition, when the condition of proportionality is ignored, but in fact both the definitions are equivalent (from the point of view of determination of the cJ-representation type of G).

1. Propositions on *cJ*-wildness

Proposition 1. *The group $G = \mathbb{Z}/p \times \mathbb{Z}/p$ is cJ-wild for any $p > 2$.*

Proof. Denote the natural generators of G by g_1 and g_2, and consider the next matrix representation γ of G over $\Sigma = k\langle x, y\rangle$:[4]

$$\gamma(g_1 - 1) = \begin{pmatrix} \gamma_{11(1)} & \gamma_{12(1)} \\ 0 & 0 \end{pmatrix}, \quad \gamma(g_2 - 1) = \begin{pmatrix} \gamma_{11(2)} & \gamma_{12(2)} \\ 0 & 0 \end{pmatrix}$$

with all the matrices $\gamma_{ij(r)}$ and 0 of size 5×5, where

$$\gamma_{11(1)} = \begin{pmatrix} 0 & 1 & x & y & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{12(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\gamma_{11(2)} = \begin{pmatrix} 0 & 0 & 1 & x & y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{12(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

We will prove that γ is a proportional cJ-perfect representation.

It is easy to see that the reprensentation $\gamma \otimes \varphi$, where φ is a representation of Σ of dimension s (which is uniquely determined by the $(\gamma\otimes\varphi)(g_1-1)$ and $(\gamma\otimes\varphi)(g_2-1)$), is of constant Jordan type $[3]^s[2]^{3s}[1]^s$.

Let φ, φ' be representations of Σ having the same dimension s and let $A_q = (\gamma\otimes\varphi)(g_q-1)$, $A'_q = (\gamma\otimes\varphi')(g_q-1))$, where $q = 1, 2$. Consider the matrix equalities (in the variable X)

$$A_1 X = X A'_1\ , \qquad A'_2 X = X A'_2\ , \tag{$*$}$$

viewing all their matrices as block with blocks of size $s \times s$. The equality $(A_q X)_{ij} = (X A'_q)_{ij}$ of the corresponding $s \times s$ blocks of the matrices $A_q X$ and $X A'_q$ is denoted by $(q.ij)$; $q \in \{1, 2\}, i, j \in \{1, 2, \ldots, 10\}$. For simplicity we write $[ij]$ instead of X_{ij} (X_{ij} is a $s \times s$ block of X).

[4]In fact, we define γ on $g_1 - 1, g_2 - 1 \in kG$ (instead of g_1, g_2), but we identify this representation of kG with the corresponding that of G. As before, we use here the multiplicative notation for the group G. A similar remark applies to Proposition 2.

We write down those equations of the form $(q.ij)$ that are needed for our proof.

$(1.12):\ [22]+\varphi(x)[32]+\varphi(y)[42]=[11],$
$(1.13):\ [23]+\varphi(x)[33]+\varphi(y)[43]=[11]\varphi'(x),$
$(1.14):\ [24]+\varphi(x)[34]+\varphi(y)[44]=[11]\varphi'(y),$
$(1.22):\ [62]=[21],$ $(1.26):\ [66]=[22],$ $(1.32):\ [72]=[31],$
$(1.36):\ [76]=[32],$ $(1.37):\ [77]=[33],$ $(1.38):\ [78]=[34],$
$(1.39):\ [79]=[35],$ $(1.42):\ [82]=[41],$ $(1.46):\ [86]=[42],$
$(1.47):\ [87]=[43],$ $(1.48):\ [88]=[44],$ $(1.49):\ [89]=[45],$
$(1.4,10):\ [8,10]=0,$ $(1.52):\ [92]=[51],$ $(1.56):\ [96]=[52],$
$(1.57):\ [97]=[53],$ $(1.58):\ [98]=[54],$ $(1.59):\ [99]=[55],$
$(1.5,10):\ [9,10]=0,$ $(1.62):\ 0=[61],$ $(1.66):\ 0=[62],$
$(1.67):\ 0=[63],$ $(1.68):\ 0=[64],$ $(1.69):\ 0=[65],$
$(1.72):\ 0=[71],$ $(1.76):\ 0=[72],$ $(1.77):\ 0=[73],$
$(1.78):\ 0=[74],$ $(1.79):\ 0=[75],$ $(1.82):\ 0=[81],$
$(1.86):\ 0=[82],$ $(1.87):\ 0=[83],$ $(1.88):\ 0=[84],$
$(1.89):\ 0=[85],$ $(1.92):\ 0=[91],$ $(1.96):\ 0=[92],$
$(1.97):\ 0=[93],$ $(1.98):\ 0=[94],$ $(1.99):\ 0=[95],$
$(1.10,2):\ 0=[10,1],$ $(1.10,6):\ 0=[10,2],$ $(1.10,7):\ 0=[10,3],$
$(1.10,8):\ 0=[10,4],$ $(1.10,9):\ 0=[10,5].$
$(2.26):\ [76]=0,$ $(2.27):\ [77]=[22],$ $(2.28):\ [78]=[23],$
$(2.29):\ [79]=[24],$ $(2.36):\ [86]=0,$ $(2.37):\ [87]=[32],$
$(2.38):\ [88]=[33],$ $(2.39):\ [89]=[34],$ $(2.3,10):[8,10]=[35],$
$(2.46):\ [96]=0,$ $(2.47):\ [97]=[42],$ $(2.48):\ [98]=[43],$
$(2.49):\ [99]=[44],$ $(2.4,10):[9,10]=[45],$ $(2.56):\ [10,6]=0,$
$(2.57):\ [10,7]=[52],$ $(2.58):\ [10,8]=[53],$ $(2.59):\ [10,9]=[54],$
$(2.5,10):[10,10]=[55].$

We have the following corollaries:

1) $(1.h2)$, $(1.h6)-(1.h9) \Rightarrow [h1]=[h2]=[h3]=[h4]=[h5]=0$ for $h=6,7,8,9,10$;
2) $(1.h-4,2)$ and $[h2]=0 \Rightarrow [h-4,1]=0$ for $h=6,7,8,9$;
3) $(2.29),(1.39),(2.3,10),(1.4,10) \Rightarrow [24]=[79]=[35]=[8,10]=0$;
4) (2.28), (1.38), (2.39), (1.49), $(2.4,10)$, $(1.5,10) \Rightarrow$ $[23]=[78]=[34]=[89]=[45]=[9,10]=0$;
5) (2.57), (1.56), $(2.46) \Rightarrow [10,7]=[52]=[96]=0$;
6) (2.59), (1.58), (2.48), (1.47), (2.37), (1.36), $(2.26) \Rightarrow$ $[10,9]=[54]=[98]=[43]=[87]=[32]=[76]=0$;
7) (2.58), (1.57), (2.47), (1.46), $(2.36) \Rightarrow$ $[10,8]=[53]=[97]=[42]=[86]=0$;

8) $(2.5,10)$, (1.59), (2.49), (1.48), (2.38), (1.37), (2.27), $(1.26) \Rightarrow$ $[10,10]=[55]=[99]=[44]=[88]=[33]=[77]=[22]=[66]$;
9) (1.12) and $32=42=0$ (see 6), 7)) $\Rightarrow [22]=[11]$;
10) (1.13) and $[23]=[43]=0$ (see 4),6)), $[33]=[22]=[11]$ (see 8),9)) $\Rightarrow \varphi(x)[11]=[11]\varphi'(x)$;
11) (1.14) and $[24]=[34]=0$ (see 3),4)), $[44]=[22]=[11]$ (see 8),9)) $\Rightarrow \varphi(y)[11]=[11]\varphi'(y)$.

From 1)– 9) (see after the symbol $\Rightarrow$) and $[10,6]=0$ (see (2.56)) it follows that the matrix X is a block upper triangular matrix with equal diagonal blocks. Then X is invertible if and only if so is $[11]$, and hence 10) and 11) imply that φ and φ' are equivalent if so are $\gamma\otimes\varphi$ and $\gamma\otimes\varphi'$.

Further, in the case $\varphi=\varphi'$ the equalities $(*)$ (resp. $\varphi(x)[11]=[11]\varphi(x)$ and $\varphi(y)[11]=[11]\varphi(y)$) define the algebra endomorphism of $\gamma\otimes\varphi$ (resp. φ). Since any matrix X, which satisfies $(*)$, is a block upper triangular matrix with equal diagonal blocks, then the endomorphism algebra of $\gamma\otimes\varphi$ is local if so is the endomorphism algebra of φ. Therefore $\gamma\otimes\varphi$ is indecomposable if φ is indecomposable. □

Proposition 2. *The group $G=\mathbb{Z}/2\times\cdots\times\mathbb{Z}/2$ ($n>2$ times) is cJ-wild.*

Proof. Denote the natural generators of G by $g_1,g_2,\dots,g_n$, and consider the next matrix representation γ of G over $\widehat{\Sigma}=k\langle x,y,x^{-1},y^{-1}\rangle$:[5]

$\gamma(g_i-1)=\begin{pmatrix}0&\gamma_i\\0&0\end{pmatrix}$, $1\le i\le n$, with the zero diagonal blocks of sizes 2×2 and $(n+1)\times(n+1)$, and $\gamma_i=\begin{pmatrix}0_i & E & 0'_i\end{pmatrix}$ for $i\neq n$, $\gamma_n=\begin{pmatrix}0_n & 0'_n & S\end{pmatrix}$, where $S=\begin{pmatrix}x&1\\0&y\end{pmatrix}$, E is the identity matrix of size 2×2 and 0_i, 0_n (resp. $0'_i$, $0'_n$) are the zero matrices of size $2\times(i-1)$ (resp. $2\times(n-i)$).

We prove that γ is a proportional cJ-perfect representation (in the same way as in the proof of Proposition 1).

It is easy to see that the reprensentation $\gamma\otimes\varphi$, where φ is a representation of $\widehat{\Sigma}$ of dimension s, is of constant Jordan type $[2]^{2s}[1]^{(n-1)s}$ (taking into account invertibility of the matrices $\varphi(x)$ and $\varphi(y)$)[6].

Let φ, φ' be representations of $\widehat{\Sigma}$ of the same dimension s and let $A_q=(\gamma\otimes\varphi)(g_q-1)$, $A'_q=(\gamma\otimes\varphi')(g_q-1))$, where $q=1,2,\dots,n$.

---

[5] The replacing Σ by $\widehat{\Sigma}$ is possible sinse the algebra $\widehat{\Sigma}$ is wild [8]: the representation $\gamma: x\to\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$, $y\to\begin{pmatrix}1&x&0\\0&1&y\\0&0&1\end{pmatrix}$ of $\widehat{\Sigma}$ over Σ is perfect (in the usual sense [9]).

[6] A representation φ of $\widehat{\Sigma}$ is that of Σ for which $\varphi(x)$ and $\varphi(y)$ are invertible.

Consider the matrix equalities $A_qX = XA'_q$, $q = 1, 2, \dots n$, viewing all their matrices as block with blocks of size $s \times s$. The equality $(A_qX)_{ij} = (XA'_q)_{ij}$ of the corresponding blocks of the matrices A_qX and XA'_q is denoted by $(q.ij)$; $q \in \{1, 2, \dots, n\}, i, j \in \{1, 2, \dots, n+3\}$.

We again write $[ij]$ instead of X_{ij}. It follows from the equalities

$(1.23) : [43] = [21], \quad (2.13) : [43] = 0,$

$(1.j3) : 0 = [j1]$ and $(1.j4) : 0 = [j2]$ for $j = 3, \dots, n+3,$

$(i.1j) : [i+2, j] = 0$ for $i = 3, \dots, n-1, j = 3, \dots, i+1$ (if $n > 3$),

$(n-1.2j) : [n+2, j] = 0$ for $j = 3, \dots, n,$

$(n-1.2, n+1) : [n+2, n+1] = [21],$

$(n.2, n+2) : \varphi(y)[n+3, n+2] = 0$ ($\varphi(y)$ is invertible),

$(n.1j) : [n+3, j] + [n+2, j] = 0$ for $j = 3, \dots, n+1$

that X is a block upper triangular matrix.

Further, from the equalities

$(1.13) : [33] = [11], \quad (1.24) : [44] = [22], \quad (2.14) : [44] = [11],$

$(2.25) : [55] = [22], \quad (n.1, n+3) : [n+3, n+3] = [11],$

$(i.2, i+3) : [i+3, i+3] = [22]$ for $i = 3, \dots, n-1$ (if $n > 3$)

it follows that all the diagonal blocks of X are equal.

Then $(n.1, n+2) : \varphi(x)[n+2, n+2] + [n+3, n+2] = [11]\,\varphi'(x)$ and $(n.2, n+3) : \varphi(y)[n+3, n+3] = [21] + [22]\varphi'(y)$ become of the forms $\varphi(x)[11] = [11]\varphi'(x)$ and $\varphi(y)[11] = [11]\varphi'(y)$, and the proof is finished in the same way as the proof of Proposition 1. □

2. Proof of Theorem 1

The group $\mathbb{Z}/p$ (of finite representation type) is of course of cJ-finite representation type. The group $G = \langle g_1, g_2 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is of cJ-semiinfinite representation type since by Theorem 5 [8] its representations

a) $\quad g_1 \to (1), \qquad g_2 \to (1),$

b) $$g_1 \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g_2 \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

c) $$g_1 \to \left(\begin{array}{c|cc} E_s & E_s & \bar{0} \\ \hline 0 & \multicolumn{2}{c}{E_{s+1}} \end{array}\right), \qquad g_2 \to \left(\begin{array}{c|cc} E_s & \bar{0} & E_s \\ \hline 0 & \multicolumn{2}{c}{E_{s+1}} \end{array}\right),$$

$$d) \qquad g_1 \to \left(\begin{array}{c|c} E_{s+1} & \begin{matrix} E_s \\ \widetilde{0} \end{matrix} \\ \hline 0 & E_s \end{array}\right), \quad g_2 \to \left(\begin{array}{c|c} E_{s+1} & \begin{matrix} \widetilde{0} \\ E_s \end{matrix} \\ \hline 0 & E_s \end{array}\right),$$

where s runs through all the natural numbers, $\overline{0}$ and $\widetilde{0}$ are the zero column and row matrices, form a complete set of pairwise nonequivalent indecomposable representations of constant Jordan type.

In all other cases G is cJ-wild by Propositions 1 and 2.

3. Conjectures

We state the following conjectures in the general case.

Conjecture 1. Let G be a finite group and k be an algebraically closed field of characteristic $p > 0$. If the group G is not wild with respect to the modules of constant Jordan type, then it is of finite representation type in each dimension (with respect to such modules).

Conjecture 2. Let G and k be as in Conjecture 1. If G is wild, then it is wild with respect to the modules of constant Jordan type.

For elementary abelian p-groups our conjectures follow from Theorem 1 (if one takes into account Theorem 1 [10] in the case of the second conjecture).

References

[1] J. F. Carlson, E. M. Friedlander, J. Pevtsova, *Modules of constant Jordan type*, J. Reine Angew. Math. **614** (2008), pp. 191-234.

[2] J. F. Carlson, E. M. Friedlander, *Exact category of modules of constant Jordan type*, Progr. in Math. **269** (2009), pp. 267-290.

[3] D. J. Benson, *Modules of constant Jordan type with one non-projective block*, Algebras and Representation Theory **13** (2010), pp. 315-318.

[4] D. J. Benson, *Modules of Constant Jordan Type with Small Non-Projective Part*, Algebras and Representation Theory (2011), doi: 10.1007/s10468-011-9291-5.

[5] S. Baland *Modules of constant Jordan type with two non-projective blocks*, J. Algebra **346** (2011), pp. 343-350.

[6] J. F. Carlson, E. M. Friedlander, A. A. Suslin, *Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$*, Comment. Math. Helv. **86** (2011), pp. 609 –657.

[7] Y. A. Drozd, *Tame and wild matrix problems*, Lecture Notes in Mathematics **832**, Springer-Verlag, Berlin/New York (1980), pp. 242-258.

[8] V. M Bondarenko, I. V. Lytvynchuk, *On some tame and wild matrix problems of constant rank*, Nauk. Visn. Uzhgorod. Univ., Ser. Mat. Inform. **23** (2012), N1, pp. 19-27 (in Russian).

[9] V. M. Bondarenko, *Linear operators on S-graded vector spaces*, Linear Algebra Appl. **365** (2003), pp. 45-90.

[10] V. M. Bondarenko, Ju. A. Drozd, *Representation type of finite groups*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **71** (1977), pp. 24–42 (in Russian); English trans. in J. Soviet Math. **20** (1982), N6, pp. 2515-2528.

Contact information

V. M. Bondarenko Institute of Mathematics, Tereshchenkivska 3, 01601 Kyiv, Ukraine
E-Mail: `vit-bond@imath.kiev.ua`

I. V. Lytvynchuk Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko Univ., Volodymyrska str., 64, 01033 Kyiv, Ukraine
E-Mail: `iryna.l@ukr.net`

Received by the editors: 15.12.2011
and in final form 15.08.2012.