Algebra and Discrete Mathematics
Volume 13 **(2012)**. Number 1. pp. 111 – 127
© Journal "Algebra and Discrete Mathematics"
RESEARCH ARTICLE

Finite local nearrings on metacyclic Miller-Moreno p-groups

I. Yu. Raievska and Ya. P. Sysak

Communicated by G. Pilz

Abstract. In this paper the metacyclic Miller-Moreno p-groups which appear as the additive groups of finite local nearrings are classified.

Introduction

Nearrings are a generalization of associative rings in the sense that with respect to addition they need not be commutative and only one distributive law is assumed. In this paper the concept "nearring" means a left distributive nearring with a multiplicative identity. The reader is referred to the books by J. Meldrum [9] or G. Pilz [11] for terminology, definitions and basic facts concerning nearrings.

A nearring R is called local if the set of all non-invertible elements of R forms a subgroup of the additive group of R. A study of local nearrings was initiated by Maxson [5] who defined a number of their basic properties and proved in particular that the additive group of a finite zero-symmetric local nearring is a p-group. It follows from [3] that local nearrings with cyclic additive group are commutative local rings. In [6] Maxson described all non-isomorphic zero-symmetric local nearrings with non-cyclic additive group of order p^2 which are not nearfields. He also shown in [7] that every non-cyclic abelian p-group of order $p^n > 4$ is the additive group of a zero-symmetric local nearring which is not a ring. This result was

2000 Mathematics Subject Classification: *16Y30.*

Key words and phrases: *Nearring with identity, local nearring, additive group, Miller-Moreno group.*

Journal Algebra Discrete Math.

extended in [4] to infinite abelian p-groups of finite exponent. However in the case of finite non-abelian p-groups the situation is different. For instance, neither a generalized quaternion group nor a non-abelian group of order 8 can be the additive group of a local nearring, as it was noted in [8] (see also [2]). Some other examples of finite p-groups with this property can be found in [4]. On the other hand, it was proved by Maxson in [8] that each metacyclic Miller-Moreno group of order p^n and exponent p^{n-1} with $n \geq 3$ if $p > 2$ and $n \geq 4$ if $p = 2$ can be the additive group of a local nearring. The purpose of our paper is to give a full classification of the metacyclic Miller-Moreno p-groups which appear as the additive groups of finite local nearrings. Moreover, if G is such an additive group, then we describe all possible multiplications " $\cdot$ " on G for which the system $(G, +, \cdot)$ is a local nearring.

1. Preliminaries

Recall first some concepts concerned nearrings.

Definition 1. A set R with two binary operations $+$ and $\cdot$ is called a (left) nearring if the following statements hold:

1. $(R, +) = R^+$ is a (not necessarily abelian) group with neutral element 0;
2. $(R, \cdot)$ is a semigroup;
3. $x(y + z) = xy + xz$ for all $x, y, z \in R$.

If R is a nearring, then the group R^+ is called the *additive group* of R. As it follows from statement 3, for each subgroup M of R^+ and each element $x \in R$ the set $xM = \{x \cdot y | y \in M\}$ is a subgroup of R^+ and, in particular, $x \cdot 0 = 0$. If in addition $0 \cdot x = 0$, then the nearring R is called *zero-symmetric*, and if the semigroup $(R, \cdot)$ is a monoid, i.e. it has an identity element i, then R is a *nearring with identity* i. In the latter case the group R^* of all invertible elements of the monoid $(R, \cdot)$ is called the *multiplicative group* of R. A subgroup M of R^+ is called R^*-*invariant* if $rM \leq M$ for each $r \in R^*$, and M is an (R, R)-*subgroup*, if $xMy \subseteq M$ for arbitrary $x, y \in R$.

Definition 2. A nearring R with identity is said to be local if the set $L = R \setminus R^*$ of all non-invertible elements of R is a subgroup of R^+.

Some basic properties of local nearrings are described in the following lemma (see [1], Lemmas 3.2, 3.4 and 3.9).

Journal Algebra Discrete Math.

Journal Algebra Discrete Math.

Lemma 1. *Let R be a local nearring with identity i and L its subgroup of all non-invertible elements of R^+. Then the following statements hold:*

1) *L is an (R, R)-subgroup of R^+;*

2) *each proper R^*-invariant subgroup of R^+ is contained in L;*

3) *if R is finite, then R^+ is a p-group for some prime p, the subgroup L is normal in R^+ and the factor group R^+/L is elementary abelian.*

Definition 3. A finite non-abelian group whose proper subgroups are abelian is called a Miller-Moreno group.

Henceforth for each prime number p and positive integers m, n with $m \geq 2$ let $G(p^m, p^n)$ denote the semidirect product $<a> \rtimes <b>$ of two cyclic groups $<a>$ and $<b>$ of orders p^m and p^n, respectively, in which $b^{-1}ab = a^{1+p^{m-1}}$. It is well-known that each metacyclic Miller-Moreno p-group is isomorphic either to a quaternion group Q_8 or to a group $G(p^m, p^n)$ (see [12]).

Recall that the exponent of a finite p-group is the maximal order of its elements. The following assertion is easily verified.

Lemma 2. *The exponent of $G(p^m, p^n)$ is equal to p^m for $m > n$ and to p^n for $m \leq n$. Moreover, if x is an element of maximal order in $G(p^m, p^n)$, then there exist generators a, b of this group such that either $a = x$ or $b = x$ and the relations $a^{p^m} = b^{p^n} = 1$ and $b^{-1}ab = a^{1+p^{m-1}}$ hold.*

Lemma 3. *Let the group $G(p^m, p^n)$ be additively written. Then for any natural numbers r, s, t the equalities $bs + ar = ar(1 - sp^{m-1}) + bs$ and $(ar + bs)t = ar(t - s\binom{t}{2}p^{m-1}) + bst$ hold.*

Proof. Let $q = 1 - p^{m-1}$. Since $-b + a + b = a(1 + p^{m-1})$ and $m \geq 2$, then $b + a = aq + b$, so $bs + ar = arq^s + bs$ for arbitrary integers $r \geq 0$ and $s \geq 0$. Taking into consideration, that

$$q^s = (1 - p^{m-1})^s \equiv 1 - sp^{m-1} \pmod{p^m}$$

by binomial's formula, giving $bs + ar = ar(1 - sp^{m-1}) + bs$. Next, $(ar + bs)t = ar(1 + q^s + \cdots + q^{s(t-1)}) + bst$ by induction on t. Therefore, $1 + q^s + \cdots + q^{s(t-1)} \equiv 1 + (1 - sp^{m-1}) + \cdots + (1 - s(t-1)p^{m-1}) = t - s\binom{t}{2}p^{m-1} \pmod{p^m}$, thus $(ar + bs)t = ar(t - s\binom{t}{2}p^{m-1}) + bst$. □

Lemma 4. *Put $G = G(p^m, p^n)$ where $m \leq n$. If $A = \operatorname{Aut} G$ is the group of all automorphisms of G and $<x>$ is a cyclic subgroup of G, then the following statements hold:*

1) *if* $m = n$ *and* $<x>$ *is a normal subgroup of order* p^m *in* G, *then* $|x^A| \leq p^{2m-2}(p-1)$;

2) *if* $p > 2$ *and* $< x >$ *is a non-normal subgroup of order* p^n, *then* $x^{-1} \notin x^A$.

Proof. If $< x >$ is a normal subgroup of order p^m in G, then either $< x >$ is a central subgroup of G or $a^{p^{m-1}} \in < x >$. Since the center $Z(G) =< a^p > \times < b^p >$ and the derived subgroup $G' =< a^{p^{m-1}} >$ are characteristic subgroups of G, it follows that in the first case $x^A \subseteq Z(G)$ and so $|x^A| \leq p^{2m-2}(p-1)$ and in the second case $G =<x> \rtimes <b>$ and $a^{p^{m-1}} \in <x^\alpha>$ for each $\alpha \in A$. The latter implies

$$| <x><x^\alpha> | = \frac{|x||x^\alpha|}{| <x> \cap <x>^\alpha |} \leq p^{2m-1},$$

so that the subgroup $<x><x^\alpha>$ is abelian and thus $x^\alpha \in C_G(x)$. Since $C_G(x) =<x> \times <b^p>$, for $m = n$ the number of all elements of order p^m in $C_G(x)$ is equal to $p^{2m-2}(p-1)$ and hence $|x^A| \leq p^{2m-2}(p-1)$, as desired.

Now let $< x >$ be a non-normal subgroup of order p^n in G. Then $<a> \cap <x>= 1$ and so $x = a^s b^t$ for some natural numbers s, t such that $(t, p) = 1$. In particular, $G =<a> \rtimes <x>$ and $[a, x] = [a, b]^t = a^{tp^{m-1}}$. Assume that $x^\alpha = x^{-1}$ for some automorphism $\alpha \in A$. Since $< a^{p^{m-1}} >^\alpha =< a^{p^{m-1}} >$, it follows that $a^\alpha = a^r x^u$ for some natural numbers r, u such that $(r, p) = 1$ and p is a divisor of u. Therefore $a^{rtp^{m-1}} = (a^{tp^{m-1}})^\alpha = [a^\alpha, x^\alpha] = [a^r x^u, x^{-1}] = [a, x^{-1}]^r = a^{-rtp^{m-1}}$ and hence $a^{2rtp^{m-1}} = 1$. But then p^m divides $2rtp^{m-1}$ what is impossible if $p > 2$. Therefore $x^{-1} \notin x^A$, as claimed. □

Lemma 5. *The exponent of the additive group of a finite nearring* R *with identity* i *is equal to the additive order of* i *which coincides with additive order of every invertible element of* R.

Proof. Indeed, if $i \cdot k = 0$ for some natural number k, then $x \cdot k = (x \cdot i) \cdot k = x \cdot (i \cdot k) = x \cdot 0 = 0$ for each $x \in R$. On the other hand, if $r \cdot l = 0$ for $r \in R^*$ and for a natural number l, then $i \cdot l = r^{-1}(r \cdot l) = 0$, so that the additive orders of r and i coincide. □

The following assertion is a direct consequence of Lemmas 2 and 5.

Corollary 1. *Let* R *be a nearring with identity* i *whose additive group* R^+ *is isomorphic to a group* $G(p^m, p^n)$. *Then there exist generators* a, b

Journal Algebra Discrete Math.

of R^+ satisfying relations $ap^m = bp^n = 0$ and $a + b = b + a(1 + p^{m-1})$ such that either a or b coincides with i.

Lemma 6. *Let R be a local nearring whose additive group R^+ is isomorphic to a group $G(p^m, p^n)$ and let L be the subgroup of all non-invertible elements of R. Then L is a subgroup in R^+ of index $|R^+ : L| = p$.*

Proof. Assume that $|R^+ : L| = p^k$ with $k \geq 2$. Since $R = R^* \cup L$ and $R^* \cap L = \varnothing$, we have $|R^*| = p^{m+n} - p^{m+n-k} = p^{m+n-k}(p^k - 1)$. In particular, the order of R^* is divisible by $p^k - 1$.

On the other hand, for each element $r \in R^*$ the mapping $x \mapsto rx$ with $x \in R$ is an automorphism of R^+ because $r(x + y) = rx + ry$ for every $x, y \in R$. Therefore R^* can be viewed as a subgroup of the group of automorphisms $\operatorname{Aut} R^+$. As it follows from [10], Corollary of Theorem 2, the order of the group $\operatorname{Aut} G(p^m, p^n)$ and so $\operatorname{Aut} R^+$ is a number of the form $p^l(p - 1)$ for some $l \geq m + n$, and therefore it is divisible by $p^k - 1$ only if $k = 1$. This contradiction shows that $|R^+ : L| = p$. □

2. Nearrings with identity on metacyclic Miller-Moreno p-groups

As it was mentioned in the Introduction, the quaternion group Q_8 cannot be isomorphic to the additive group of a nearring R with identity. Therefore in what follows the additive group of R is isomorphic to a group $G(p^m, p^n)$. Thus it follows from Corollary 1 that $R^+ = <a> + <b>$ with elements a, b one of which coincides with identity element of R and the relations $ap^m = bp^n = 0$ and $a + b = b + a(1 + p^{m-1})$ are valid. Moreover, each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 \leq x_1 < p^m$ and $0 \leq x_2 < p^n$.

Consider first the case when a coincides with identity element of R, so that $xa = ax = x$ for each $x \in R$. Then R^+ is of exponent p^m by Lemma 5 and so $m \geq n$. Furthermore, for each $x \in R$ there exist integers $\alpha(x)$ and $\beta(x)$ such that $xb = a\alpha(x) + b\beta(x)$. It is clear that modulo p^m and p^n, respectively, these integers are uniquely determined by x and so some mappings $\alpha : R \to \mathbb{Z}_{p^m}$ and $\beta : R \to \mathbb{Z}_{p^n}$ are determined.

Lemma 7. *Let $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ be elements of R. If a coincides with identity element of R, then*

$$\begin{aligned} xy = a[(x_1y_1 + \alpha(x)y_2) - (x_1x_2\tbinom{y_1}{2} + \alpha(x)x_2y_1y_2 \\ + \alpha(x)\beta(x)\tbinom{y_2}{2}))p^{m-1}] + b(x_2y_1 + \beta(x)y_2). \end{aligned}$$

Journal Algebra Discrete Math.

Moreover, for the mappings $\alpha : R \to \mathbb{Z}_{p^m}$ *and* $\beta : R \to \mathbb{Z}_{p^n}$ *the following statements hold :*

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring* R *is zero-symmetric;*

(1) $\alpha(a) = 0$ *and* $\beta(a) = 1$*;*

(2) $\alpha(x) \equiv 0 \ (\bmod p^{m-n})$*, except the case* $p = 2 = m$ *and* $n = 1$*;*

(3) $x_1(\beta(x) - 1) \equiv x_2\alpha(x) \ (\bmod p)$*;*

(4) $$\begin{aligned}\alpha(xy) &= x_1\alpha(y) + \alpha(x)\beta(y)\\ &\quad - [x_1x_2\tbinom{\alpha(y)}{2} + \alpha(x)x_2\alpha(y)\beta(y) + \alpha(x)\beta(x)\tbinom{\beta(y)}{2}]p^{m-1};\end{aligned}$$

(5) $\beta(xy) = x_2\alpha(y) + \beta(x)\beta(y)$.

Proof. By the left distributive law, we have

$$xy = (xa)y_1 + (xb)y_2 = (ax_1 + bx_2)y_1 + (a\alpha(x) + b\beta(x))y_2.$$

Furthermore, Lemma 3 implies that

$$(ax_1 + bx_2)y_1 = ax_1(y_1 - x_2\binom{y_1}{2}p^{m-1}) + bx_2y_1,$$

$$(a\alpha(x) + b\beta(x))y_2 = a\alpha(x)(y_2 - \beta(x)\binom{y_2}{2}p^{m-1}) + b\beta(x)y_2$$

and

$$\begin{aligned}&bx_2y_1 + a\alpha(x)(y_2 - \beta(x)\tbinom{y_2}{2}p^{m-1})\\ &\quad = a\alpha(x)(y_2 - \beta(x)\tbinom{y_2}{2}p^{m-1})(1 - x_2y_1p^{m-1}) + bx_2y_1.\end{aligned}$$

Thus

$$(*) \qquad \begin{aligned}xy = a[&(x_1y_1 + \alpha(x)y_2) - (x_1x_2\tbinom{y_1}{2} + \alpha(x)x_2y_1y_2\\ &+ \alpha(x)\beta(x)\tbinom{y_2}{2})p^{m-1}] + b(x_2y_1 + \beta(x)y_2),\end{aligned}$$

as desired.

As $0 \cdot a = a \cdot 0 = 0$, the nearring R is zero-symmetric if and only if $0 = 0 \cdot b = a\alpha(0) + b\beta(0)$ whence $\alpha(0) = \beta(0) = 0$. Similarly, from the equality $b = ab = a\alpha(a) + b\beta(a)$ it follows that $\alpha(a) = 0$ and $\beta(a) = 1$. Since $(xb)p^n = x(bp^n) = 0$ and $xb = a\alpha(x) + b\beta(x)$, we have $0 = (a\alpha(x) + b\beta(x))p^n = a\alpha(x)(p^n - \beta(x)\binom{p^n}{2}p^{m-1})$ by Lemma 3 and hence

$\alpha(x) \equiv 0 \pmod{p^{m-n}}$, except the case $p = 2 = m$ and $n = 1$ in which the group R^+ is dihedral of order 8. Next, if $y = a(1 - p^{m-1}) + b$, then

$$xy = a[\alpha(x) + x_1 - (x_1 + \alpha(x)x_2)p^{m-1}] + b(x_2 + \beta(x))$$

by formula (*). On the other hand , $y = b + a$ and so

$$xy = xb + x = a(\alpha(x) + x_1 - x_1\beta(x)p^{m-1}) + b(x_2 + \beta(x))$$

by Lemma 3. Comparing both results for xy, we obtain

$$(x_1 + \alpha(x)x_2)p^{m-1} \equiv x_1\beta(x)p^{m-1} (\mod p^m).$$

Thus $x_1(\beta(x) - 1) \equiv x_2\alpha(x) \pmod{p}$ and so statement (3) holds.

Finally, the associativity of multiplication in R implies that

$$x(yb) = (xy)b = a\alpha(xy) + b\beta(xy).$$

Furthermore, substituting $yb = a\alpha(y) + b\beta(y)$ instead of y in formula (*), we also have

$$\begin{aligned} x(yb) = a[(x_1\alpha(y) + \alpha(x)\beta(y)) - (x_1x_2\tbinom{\alpha(y)}{2} + \alpha(x)x_2\alpha(y)\beta(y) \\ + \alpha(x)\beta(x)\tbinom{\beta(y)}{2}))p^{m-1}] + b(x_2\alpha(y) + \beta(x)\beta(y)). \end{aligned}$$

Comparing the coefficients under a and b in two expressions obtained for $x(yb)$, we derive statements (4) and (5) of the lemma. □

Consider now the case when b coincides with identity element of R and so $xb = bx = x$ for each $x \in R$. Then R^+ is of exponent p^n by Lemma 5 and thus $m \leq n$. As above, there exist integers $\alpha(x)$ and $\beta(x)$ which are uniquely determined by x modulo p^m and p^n, respectively, such that $xa = a\alpha(x) + b\beta(x)$.

Lemma 8. *If b coincides with identity element in R, then*

$$xy = a[\alpha(x)y_1 + x_1y_2 - x_1x_2\binom{y_2}{2}p^{m-1}] + b[\beta(x)y_1 + x_2y_2].$$

Also for the mappings $\alpha : R \to \mathbb{Z}_{p^m}$ and $\beta : R \to \mathbb{Z}_{p^n}$ the following statements hold:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring R is zero-symmetric;*

(1) $\alpha(b) = 1$ *and* $\beta(b) = 0$*;*

Journal Algebra Discrete Math.

(2) $\beta(x) \equiv 0 \pmod{p^{n-m+1}}$;

(3) $\alpha(x)(1-x_2) \equiv 0 \pmod{p}$;

(4) $\alpha(xy) = \alpha(x)\alpha(y) + x_1\beta(y) - x_1x_2\binom{\beta(y)}{2}p^{m-1}$;

(5) $\beta(xy) = \beta(x)\alpha(y) + x_2\beta(y)$.

Proof. Clearly R is zero-symmetric if and only if $0 = 0 \cdot a = a\alpha(0) + b\beta(0)$, whence $\alpha(0) = \beta(0) = 0$. From the equality $a = ba = a\alpha(b) + b\beta(b)$ it follows that $\alpha(b) = 1$ and $\beta(b) = 0$. Using the left distributive law, we have also

$$xy = (xa)y_1 + (xb)y_2 = (a\alpha(x) + b\beta(x))y_1 + (ax_1 + bx_2)y_2.$$

Next, Lemma 3 implies that

$$(a\alpha(x) + b\beta(x))y_1 = a\alpha(x)(y_1 - \beta(x)\binom{y_1}{2}p^{m-1}) + b\beta(x)y_1,$$

$$(ax_1 + bx_2)y_2 = ax_1(y_2 - x_2\binom{y_2}{2}p^{m-1}) + bx_2y_2$$

and

$$\begin{aligned}&b\beta(x)y_1 + ax_1(y_2 - x_2\tbinom{y_2}{2}p^{m-1})\\ &\quad = ax_1(y_2 - x_2\tbinom{y_2}{2}p^{m-1})(1 - \beta(x)y_1p^{m-1}) + b\beta(x)y_1.\end{aligned}$$

Therefore

$$(**)\qquad \begin{aligned}xy = a[(\alpha(x)y_1 + x_1y_2) - (\alpha(x)\beta(x)\tbinom{y_1}{2} + x_1x_2\tbinom{y_2}{2}+\\ + x_1\beta(x)y_1y_2)p^{m-1}] + b(\beta(x)y_1 + x_2y_2).\end{aligned}$$

Substituting $y = a(1 - p^{m-1}) + b$ in formula (**), we derive

$$\begin{aligned}xy = a[\alpha(x)(1 - p^{m-1}) + x_1 - (\alpha(x)\beta(x)\tbinom{1-p^{m-1}}{2} + x_1x_2\tbinom{1}{2}\\ + x_1\beta(x)(1 - p^{m-1}))p^{m-1}] + b(\beta(x)(1 - p^{m-1}) + x_2).\end{aligned}$$

In the case $p > 2$ or $p = 2$ and $m \geq 3$ this implies

(*i*) $xy = a[x_1 + \alpha(x) - (\alpha(x) + x_1\beta(x))p^{m-1}] + b(\beta(x)(1 - p^{m-1}) + x_2)$.

If $p = 2$ and $m = 2$, then

(*ii*) $\quad xy = a[x_1 - \alpha(x) + 2(\alpha(x)\beta(x) + x_1\beta(x))] + b(x_2 - \beta(x))$.

Journal Algebra Discrete Math.

On the other hand, $y = b + a$ and thus

$$(iii) \qquad xy = x + xa = a[x_1 + \alpha(x) - x_2\alpha(x)p^{m-1}] + b(x_2 + \beta(x)).$$

Comparing formulas (i) and (ii) with formula (iii), we obtain in the first case the congruences

$$(\alpha(x) + x_1\beta(x))p^{m-1} \equiv x_2\alpha(x)p^{m-1}(\mod p^m)$$

and

$$\beta(x)p^{m-1} \equiv 0(\mod p^n)$$

and in the second case the congruences

$$-\alpha(x) + 2\alpha(x)\beta(x) + 2x_1\beta(x) \equiv \alpha(x) - 2x_2\alpha(x)(\mod 2^2)$$

and

$$\beta(x) \equiv -\beta(x)(\mod 2^n).$$

In both cases this gives the congruences

$$\alpha(x)(1 - x_2) \equiv 0 \ (\mod p \ ) \quad \text{and} \quad \beta(x) \equiv 0 \ (\mod p^{n-m+1}),$$

i.e. statements (3) and (2), respectively. But then

$$(\alpha(x)\beta(x)\binom{y_1}{2} + x_1\beta(x)y_1y_2)p^{m-1} \equiv 0 \ (\mod p^m)$$

and so formula (**) becomes the equality

$$xy = a[(\alpha(x)y_1 + x_1y_2) - x_1x_2\binom{y_2}{2}p^{m-1}] + b[\beta(x)y_1 + x_2y_2],$$

as desired. Finally, replacing in this equality the element y by $ya = a\alpha(y) + b\beta(y)$ and using that $x(ya) = (xy)a$, we have got the expressions for $\alpha(xy)$ and $\beta(xy)$ contained in statements (4) and (5) of the lemma. □

3. Local nearrings on metacyclic Miller-Moreno p-groups

Let R be a local nearring whose additive group R^+ is isomorphic to a group $G(p^m, p^n)$ and let L denote the subgroup in R^+ of all non-invertible elements from R. As it was mentioned above, $R = R^* \cup L$ and $R^+ = < a > + < b >$ with elements a, b satisfying the relations $ap^m = bp^n = 0$ and $a + b = b + a(1 + p^{m-1})$. Furthermore, $< a >$ is a normal subgroup of R^+ and each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 < x_1 < p^m$ and $0 < x_2 < p^n$.

Journal Algebra Discrete Math.

Lemma 9. 1) *If $a \in R^*$, then $p^{m+n} > 8$ and $m > n$.*

2) *If $b \in R^*$, then $p = 2$ and $m \leq n$.*

Proof. If $a \in R^*$, then R^+ is a group of exponent p^m by Lemma 5 and so $m \geq n$. It was shown in [8] that the dihedral group of order 8 is not the additive group of a local nearring, so that $p^{m+n} > 8$. Assume that $m = n$. As it was shown in the proof of Lemma 6, the multiplicative group R^* can be viewed as a subgroup of $\mathrm{Aut}\, R^+$ such that $R^*x \subseteq x^{\mathrm{Aut}\, R^+}$ for each $x \in R$ and, in particular, $R^* = R^*a \subseteq a^{\mathrm{Aut}\, R^+}$. Since $|a^{\mathrm{Aut}\, R^+}| \leq p^{2m-2}(p-1)$ by Lemma 4, it follows that $|R^*| \leq p^{2m-2}(p-1)$. On the other hand, the subgroup L is of index p in R^+, so that $|R^*| = |R| - |L| = |R| - \frac{1}{p}|R| = p^{2m} - p^{2m-1} = p^{2m-1}(p-1)$, contrary to the latter inequality. Thus $m > n$, proving statement 1).

Now let $b \in R^*$. Then R^+ is a group of exponent p^n and so $m \leq n$. Since the subgroup $<b>$ is non-normal in R^+, it follows that $-b \notin b^{\mathrm{Aut}\, R^+}$ by Lemma 4. As $R^* = R^*b \subseteq b^{\mathrm{Aut}\, R^+}$, this means that $-b \notin R^*$. But then $-b \in L$ and if $p > 2$, then $b \in L$, contrary to the assumption. Thus $p = 2$ and $m \leq n$, proving statement 2). □

Theorem 1. *Let R be a local nearring whose additive group R^+ is isomorphic to a group $G(p^m, p^n)$. Then $R^+ = <a> + <b>$, one of the elements a, b coincides with identity element of R and the following statements hold:*

1) *$ap^m = bp^n = 0$ and $a + b = b + a(1 + p^{m-1})$;*

2) *if a coincides with identity element of R, then $p^{m+n} > 8$, $m > n$, $L = <ap> + <b>$ and $R^* = \{ax_1 + bx_2 \mid x_1 \not\equiv 0 \ (mod\ p)\}$;*

3) *if b coincides with identity element of R, then $p = 2$, $m \leq n$, $L = <a> + <b2>$ and $R^* = \{ax_1 + bx_2 \mid x_2 \equiv 1 \ (\bmod 2)\}$.*

Proof. By Corollary 1, there exists the decomposition $R^+ = <a> + <b>$ in which one of the elements a or b coincides with identity element of R and statement 1) holds.

If a coincides with identity element, then $m > n$ and $b \in L$ by Lemma 9, so that $L = <ap> + <b>$. Since $xL \subseteq L$ for each $x \in R$ by Lemma 1, we have $xb = a\alpha(x) + b\beta(x) \in L$ whence $\alpha(x) \equiv 0 \ (\bmod p)$ for each $x \in R$. Furthermore, $R = R^* \cup L$ and so an element $x = ax_1 + bx_2$ belongs to R^* if and only if $x_1 \not\equiv 0 \ (\bmod p)$.

Similarly, if b coincides with identity element of R, then $p = 2$, $m \leq n$ and $a \in L$ by Lemma 9. Therefore $L = <a> + <b2>$ and $\beta(x) \equiv$

Journal Algebra Discrete Math.

$0 \pmod 2$ for each $x \in R$. Thus an element $x = ax_1 + bx_2$ belongs to R^* if and only if $x_2 \equiv 1 \pmod 2$. □

Applying statements 2) and 3) of Theorem 1 to Lemmas 7 and 8, respectively, we obtain the following formulas for multiplying elements $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ in the local nearring R whose the additive group is isomorphic to a group $G(p^m, p^n)$.

Corollary 2. *If a coincides with identity element of R and $xb = a\alpha(x) + b\beta(x)$, then*

$$xy = a(x_1y_1 + \alpha(x)y_2 - x_1x_2\binom{y_1}{2}p^{m-1}) + b(x_2y_1 + \beta(x)y_2),$$

where the mappings $\alpha : R \to \mathbb{Z}_{p^m}$ and $\beta : R \to \mathbb{Z}_{p^n}$ satisfy the conditions:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring R is zero-symmetric;*

(1) $\alpha(a) = 0$ *and* $\beta(a) = 1$*;*

(2) $\alpha(x) \equiv 0 \pmod{p^{m-n}}$*;*

(3) $x_1(\beta(x) - 1) \equiv 0 \pmod p$*;*

(4) $\alpha(xy) = x_1\alpha(y) + \alpha(x)\beta(y) - x_1x_2\binom{\alpha(y)}{2}p^{m-1}$*;*

(5) $\beta(xy) = x_2\alpha(y) + \beta(x)\beta(y)$*.*

Corollary 3. *If b coincides with identity element of R and $xa = a\alpha(x) + b\beta(x)$, then*

$$xy = a(\alpha(x)y_1 + x_1y_2 - x_1x_2\binom{y_2}{2}2^{m-1}) + b(\beta(x)y_1 + x_2y_2),$$

where mappings $\alpha : R \to \mathbb{Z}_{2^m}$ and $\beta : R \to \mathbb{Z}_{2^n}$ satisfy the conditions:

(0) $\alpha(0) = \beta(0) = 0$ *if and only if the nearring R is zero-symmetric;*

(1) $\alpha(b) = 1$ *and* $\beta(b) = 0$*;*

(2) $\beta(x) \equiv 0 \pmod{2^{n-m+1}}$*;*

(3) $\alpha(x) \equiv 0 \pmod 2$ *if and only if* $x_2 \equiv 0 \pmod 2$*;*

(4) $\alpha(xy) = \alpha(x)\alpha(y) + x_1\beta(y) - x_1x_2\binom{\beta(y)}{2}2^{m-1}$*;*

(5) $\beta(xy) = \beta(x)\alpha(y) + x_2\beta(y)$*.*

Journal Algebra Discrete Math.

4. Groups $G(p^m, p^n)$ as the additive groups of local nearrings

The following theorem gives sufficient conditions for existing a finite local nearring whose additive group is isomorphic to a group $G(p^m, p^n)$. Together with Theorem 1 and remarks mentioned in the Introduction this completes our classification of all metacyclic Miller-Moreno p-groups which are the additive groups of local nearrings.

Theorem 2. *For an arbitrary prime p and natural numbers m, n such that either $p^{m+n} > 8$ and $m > n$ or $p = 2$ and $1 < m \leq n$ there exists a local nearring R whose additive group R^+ is isomorphic to a group $G(p^m, p^n)$.*

Proof. Let R be an additively written group $G(p^m, p^n)$ with generators a, b satisfying the relations $ap^m = bp^n = 0$ and $a + b = b + a(1 + p^{m-1})$. Then $R = <a> + <b>$ and each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2$ with coefficients $0 < x_1 < p^m$ and $0 < x_2 < p^n$. In order to define a multiplication " $\cdot$ " on R in such a manner that $(R, +, \cdot)$ is a local nearring, we separately consider two cases.

1. The case $p^{m+n} > 8$ and $m > n$.

For arbitrary elements $x = ax_1 + bx_2$ and $y = ay_1 + by_2$ of R we define the multiplication " $\cdot$ " by the rule

$$x \cdot y = a(x_1 y_1 + \alpha(x) y_2 - x_1 x_2 \binom{y_1}{2} p^{m-1}) + b(x_2 y_1 + \beta(x) y_2),$$

where $\alpha : R \to \mathbb{Z}_{p^m}$ and $\beta : R \to \mathbb{Z}_{p^n}$ are mappings satisfying conditions (1) - (5) of Corollary 2. As an example, we can for instance take

$$\alpha(x) = 0 \quad \text{for all} \quad x \in R \quad \text{and} \quad \beta(x) = \begin{cases} 1 & \text{if} \quad x_1 \not\equiv 0 \ (\mod p\ ), \\ 0 & \text{if} \quad x_1 \equiv 0 \ (\mod p\ ). \end{cases}$$

It is easy to see that the element a is a multiplicative identity for $(R, \cdot)$ and $x \cdot b = a\alpha(x) + b\beta(x)$ for each $x \in R$. We show that with respect to the operations " $+$ " and " $\cdot$ " the system $(R, +, \cdot)$ is a nearring with identity element a. Clearly it suffices to check that if $z = az_1 + bz_2$ is an arbitrary element of R, then $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x \cdot y) \cdot b = x \cdot (y \cdot b)$.

Indeed, we have

$$x \cdot z = a(x_1 z_1 + \alpha(x) z_2 - x_1 x_2 \binom{z_1}{2} p^{m-1}) + b(x_2 z_1 + \beta(x) z_2)$$

Journal Algebra Discrete Math.

and

$$\begin{aligned}&b[x_2y_1+\beta(x)y_2]+a[x_1z_1+\alpha(x)z_2-x_1x_2\tbinom{z_1}{2}p^{m-1}]=\\&\quad=a[(1-(x_2y_1+\beta(x)y_2)p^{m-1})(x_1z_1+\alpha(x)z_2-\\&\quad-x_1x_2\tbinom{z_1}{2}p^{m-1})]+b[x_2y_1+\beta(x)y_2]\end{aligned}$$

by Lemma 3. Since $\alpha(x)\equiv 0 \pmod p$ by condition (2) of Corollary 2 and so $a(\alpha(x)p^{m-1})=0$, this implies

$$\begin{aligned}x\cdot y+x\cdot z&=a[x_1(y_1+z_1)+\alpha(x)(y_2+z_2)-(x_1x_2\tbinom{y_1}{2}+x_1x_2\tbinom{z_1}{2}+\\&\quad+x_1x_2y_1z_1+x_1\beta(x)y_2z_1)p^{m-1}]+b[x_2(y_1+z_1)+\beta(x)(y_2+z_2)].\end{aligned}$$

On the other hand, $y+z=(ay_1+by_2)+(az_1+bz_2)=a(y_1+z_1-y_2z_1p^{m-1})+b(y_2+z_2)$ because $by_2+az_1=a(1-y_2p^{m-1})z_1+by_2$ by Lemma 3 and $b(y_2z_1p^{m-1})=0$ because $m>n$. Therefore

$$\begin{aligned}x\cdot(y+z)&=a[x_1(y_1+z_1)+\alpha(x)(y_2+z_2)-(x_1y_2z_1+\\&\quad+x_1x_2\tbinom{y_1+z_1-y_2z_1p^{m-1}}{2}))p^{m-1}]+b[x_2(y_1+z_1)+\beta(x)(y_2+z_2)]\end{aligned}$$

and hence

$$\begin{aligned}x\cdot y+x\cdot z-x\cdot(y+z)&=a[-(x_1x_2\tbinom{y_1}{2}+x_1x_2\tbinom{z_1}{2}+x_1x_2y_1z_1+\\&\quad+x_1\beta(x)y_2z_1)p^{m-1}+(x_1y_2z_1+x_1x_2\tbinom{y_1+z_1-y_2z_1p^{m-1}}{2}))p^{m-1}]=\\&\quad=ax_1(1-\beta(x))y_2z_1p^{m-1}.\end{aligned}$$

Since $x_1(\beta(x)-1)\equiv 0 \pmod p$ by condition (3) of Corollary 2, it follows that $ax_1(1-\beta(x))y_2z_1p^{m-1}=0$ and thus $x\cdot(y+z)=x\cdot y+x\cdot z$, as desired.

Next, $y\cdot b=a\alpha(y)+b\beta(y)$ and so $x\cdot(y\cdot b)=x\alpha(y)+(x\cdot b)\beta(y)$. Applying Lemma 3, we also have

$$x\alpha(y)=(ax_1+bx_2)\alpha(y)=ax_1(\alpha(y)-x_2\binom{\alpha(y)}{2}p^{m-1})+bx_2\alpha(y),$$

$$\begin{aligned}(x\cdot b)\beta(y)&=(a\alpha(x)+b\beta(x))\beta(y)=a\alpha(x)(\beta(y)-\beta(x)\tbinom{\beta(y)}{2}p^{m-1})+\\&\quad+b\beta(x)\beta(y)\end{aligned}$$

and

$$\begin{aligned}&bx_2\alpha(y)+a\alpha(x)(\beta(y)-\beta(x)\tbinom{\beta(y)}{2}p^{m-1})=\\&\quad=a\alpha(x)(\beta(y)-\beta(x)\tbinom{\beta(y)}{2}p^{m-1})(1-x_2\alpha(y)p^{m-1})+bx_2\alpha(y).\end{aligned}$$

As $\alpha(x)\equiv 0 \pmod p$, it follows that

$$\begin{aligned}&\alpha(x)\beta(x)\tbinom{\beta(y)}{2}p^{m-1})=x_2\alpha(x)\alpha(y)\beta(y)p^{m-1}=\\&\quad=x_2\beta(x)\alpha(y)\tbinom{\beta(y)}{2}p^{2m-2}=0,\end{aligned}$$

Journal Algebra Discrete Math.

whence

$$x\cdot(y\cdot b) = a[x_1\alpha(y)+\alpha(x)\beta(y)-x_1x_2\binom{\alpha(y)}{2}p^{m-1}]+b[x_2\alpha(y)+\beta(x)\beta(y)].$$

However $x_1\alpha(y) + \alpha(x)\beta(y) - x_1x_2\binom{\alpha(y)}{2}p^{m-1} = \alpha(x \cdot y)$ and $x_2\alpha(y) + \beta(x)\beta(y) = \beta(x \cdot y)$ by conditions (4) and (5) of Corollary 2, so that $(x \cdot y) \cdot b = a\alpha(x \cdot y) + b\beta(x \cdot y) = x \cdot (y \cdot b)$. Thus, the system $(R, +, \cdot)$ is a nearring with identity element a.

We show finally that $R = (R, +, \cdot)$ is a local nearring in which the subgroup $L =<ap> + <b>$ consists of all non-invertible elements of R. Obviously, it is enough to prove that the element $x = ax_1 + bx_2$ is right invertible if and only if $x_1 \not\equiv 0 \pmod{p}$ because in this case the set $R \setminus L$ coincides with the multiplicative group R^* by Lemma 1.

Indeed, if x is right invertible, then there exists an element $y = ay_1+by_2$ such as $x\cdot y = a$. From our definition of $x\cdot y$ it follows that $x_1y_1+\alpha(x)y_2 - x_1x_2\binom{y_1}{2}p^{m-1} \equiv 1 \pmod{p^m}$. But then $x_1y_1 \equiv 1 \pmod{p}$ because $\alpha(x) \equiv 0 \pmod{p}$ by condition (2) of Corollary 2, so that $x_1 \not\equiv 0 \pmod{p}$.

Conversely, if the latter holds, then $\beta(x) \equiv 1 \pmod{p}$ by condition (3) of Corollary 2 and so there exist natural numbers y_1, y_2 such that $x_1y_1 \equiv 1 \pmod{p^m}$ and $x_2y_1 + \beta(x)y_2 \equiv 0 \pmod{p^n}$. Taking $y = ay_1 + by_2$ and $z_1 = 1 + \alpha(x)y_2 - x_1x_2\binom{y_1}{2}p^{m-1}$, we have $x \cdot y = az_1$. Since $\alpha(x) \equiv 0 \pmod{p}$, there exists such a natural number s that $z_1s \equiv 1 \pmod{p^m}$, whence $x \cdot (ys) = (x \cdot y)s = (az_1)s = a$. Thus the element x is right invertible and so the nearring R is local.

2. The case $p = 2$ and $m \le n$.

Define now the multiplication "$\cdot$" by the rule

$$x \cdot y = a(\alpha(x)y_1 + x_1y_2 - x_1x_2\binom{y_2}{2}2^{m-1}) + b(\beta(x)y_1 + x_2y_2),$$

where the mappings $\alpha : R \to \mathbb{Z}_{2^m}$ and $\beta : R \to \mathbb{Z}_{2^n}$ satisfy conditions (1) - (5) of Corollary 3. In particular, we can put

$$\alpha(x) = \begin{cases} 1 & \text{if} \quad x_2 \not\equiv 0 \pmod{2}, \\ 0 & \text{if} \quad x_2 \equiv 0 \pmod{2} \end{cases} \quad \text{and} \quad \beta(x) = 0 \quad \text{for all} \quad x \in R.$$

Clearly the element b is a multiplicative identity of R and $x \cdot a = a\alpha(x) + b\beta(x)$ for each $x \in R$. As in the above case, in order to show that the

Journal Algebra Discrete Math.

system $(R,+,\cdot)$ is a nearring with identity element b, it is enough to verify that $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(x\cdot y)\cdot a=x\cdot(y\cdot a)$.

Indeed,

$$x\cdot z=a(\alpha(x)z_1+x_1z_2-x_1x_2\binom{z_2}{2}2^{m-1})+b(\beta(x)z_1+x_2z_2)$$

by the definition and

$$\begin{aligned}&b[\beta(x)y_1+x_2y_2]+a[\alpha(x)z_1+x_1z_2-x_1x_2\tbinom{z_2}{2}2^{m-1}]=\\&\quad=a[(1-(\beta(x)y_1+x_2y_2)2^{m-1})(\alpha(x)z_1+x_1z_2-x_1x_2\tbinom{z_2}{2}2^{m-1})]+\\&\qquad+b[\beta(x)y_1+x_2y_2]\end{aligned}$$

by Lemma 3. Since $\beta(x)\equiv 0\ (\bmod 2^{n-m+1})$ by condition (2) of Corollary 3, this implies that

$$\begin{aligned}x\cdot y+x\cdot z&=a[\alpha(x)(y_1+z_1)+x_1(y_2+z_2)-(x_1x_2\tbinom{y_2}{2}+x_1x_2\tbinom{z_2}{2}+\\&\quad+x_2y_2\alpha(x)z_1+x_1x_2y_2z_2)2^{m-1}]+b[\beta(x)(y_1+z_1)+x_2(y_2+z_2)].\end{aligned}$$

On the other hand, $y+z=a(y_1+z_1-y_2z_12^{m-1})+b(y_2+z_2)$ and so

$$\begin{aligned}x\cdot(y+z)&=a[\alpha(x)(y_1+z_1-y_2z_12^{m-1})+x_1(y_2+z_2)-\\&\quad-x_1x_2\tbinom{y_2+z_2}{2}2^{m-1}]+b[\beta(x)(y_1+z_1-y_2z_12^{m-1})+x_2(y_2+z_2)].\end{aligned}$$

Thus

$$\begin{aligned}-x\cdot(y+z)+x\cdot y+x\cdot z&=-b[\beta(x)(y_1+z_1-y_2z_12^{m-1})+\\&\quad+x_2(y_2+z_2)]+a[(\alpha(x)y_2z_1+x_1x_2\tbinom{y_2+z_2}{2}-x_1x_2\tbinom{y_2}{2}-x_1x_2\tbinom{z_2}{2}-\\&\quad-\alpha(x)x_2y_2z_1-x_1x_2y_2z_2)2^{m-1}]+b[\beta(x)(y_1+z_1)+x_2(y_2+z_2)].\end{aligned}$$

However

$$\begin{aligned}&a[(\alpha(x)y_2z_1+x_1x_2\tbinom{y_2+z_2}{2}-x_1x_2\tbinom{y_2}{2}-x_1x_2\tbinom{z_2}{2}-\\&\quad-\alpha(x)x_2y_2z_1-x_1x_2y_2z_2)2^{m-1}]=a\alpha(x)(1-x_2)y_2z_22^{m-1}=0\end{aligned}$$

by condition (3) of Corollary 3 and therefore

$$\begin{aligned}-x\cdot(y+z)+x\cdot y+x\cdot z&=-b[\beta(x)(y_1+z_1-y_2z_12^{m-1})+\\&\quad+x_2(y_2+z_2)]+b[\beta(x)(y_1+z_1)+x_2(y_2+z_2)]=\\&\quad=b\beta(x)y_2z_12^{m-1}=0,\end{aligned}$$

as required.

Journal Algebra Discrete Math.

Next, $y \cdot a = a\alpha(y) + b\beta(y)$ and so $x \cdot (y \cdot a) = (x \cdot a)\alpha(y) + x\beta(y) = (a\alpha(x) + b\beta(x))\alpha(y) + (ax_1 + bx_2)\beta(y)$. Applying Lemma 3, we have

$$(a\alpha(x) + b\beta(x))\alpha(y) = a\alpha(x)(\alpha(y) - \beta(x)\binom{\alpha(y)}{2}2^{m-1}) + b\beta(x)\alpha(y),$$

$$(ax_1 + bx_2)\beta(y) = ax_1(\beta(y) - x_2\binom{\beta(y)}{2}2^{m-1}) + bx_2\beta(y)$$

and

$$\begin{aligned} b\beta(x)\alpha(y) + ax_1(\beta(y) - x_2\tbinom{\beta(y)}{2}2^{m-1}) &= \\ = ax_1(\beta(y) - x_2\tbinom{\beta(y)}{2}2^{m-1})(1 - \beta(x)\alpha(y)2^{m-1}) &+ b\beta(x)\alpha(y). \end{aligned}$$

From this and the congruence $\beta(x) \equiv 0 \pmod{2^{n-m+1}}$ it follows that

$$x\cdot(y\cdot a) = a[\alpha(x)\alpha(y)+x_1\beta(y)-x_1x_2\binom{\beta(y)}{2}2^{m-1}]+b[\beta(x)\alpha(y)+x_2\beta(y)].$$

In view of conditions (4) and (5) of Corollary 3, the coefficients in brackets under a and b in the right part of the last equality coincide with $\alpha(x\cdot y)$ and $\beta(x \cdot y)$, respectively. Therefore $x \cdot (y \cdot a) = a\alpha(x \cdot y) + b\beta(x \cdot y) = (x \cdot y) \cdot a$ and hence $(R, +, \cdot)$ is a nearring with identity element b.

To complete the proof it remains to show that the set of all non-invertible elements of $(R, +, \cdot)$ coincides with $L = <a> + <b2>$. As in the previous case, it suffices to prove that an element $x = ax_1 + bx_2$ is right invertible if and only if $x_2 \equiv 1 \pmod 2$.

Really, if there exists an element $y = ay_1 + by_2$ such that $x \cdot y = b$, then $\beta(x)y_1 + x_2y_2 \equiv 1 \pmod{2^n}$ by our definition of $x \cdot y$. But then $x_2y_2 \equiv 1 \pmod{2^{m-1}}$ because $\beta(x) \equiv 0 \pmod{2^{n-m+1}}$ by condition (2) of Corollary 3, whence $x_2 \equiv 1 \pmod 2$.

Conversely, if the last congruence holds, then $\alpha(x) \equiv 1 \pmod 2$ by condition (3) of Corollary 3 and so there exist natural numbers y_1, y_2 such that $x_2y_2 \equiv 1 \pmod{2^n}$ and $\alpha(x)y_1 + x_1y_2 \equiv 0 \pmod{2^m}$. Therefore for $y = ay_1 + by_2$ and $z_2 = 1 + \beta(x)y_1$ we have $x \cdot y = bz_2$. Since $\beta(x) \equiv 0 \pmod{2^{n-m+1}}$, there is a natural number s such that $z_2s \equiv 1 \pmod{2^n}$. But then $x \cdot (ys) = (x \cdot y)s = (bz_2)s = b$ and so the element x is right invertible. Thus the nearring R is local and this completes the proof. □

References

[1] B. Amberg, P. Hubert, Ya. Sysak, *Local nearrings with dihedral multiplicative group*, Journal of Algebra, **273**, 2004, pp. 700-717.

Journal Algebra Discrete Math.

[2] J. R. Clay, C.J. Maxson *The near-rings with identities on generalized quaternion groups*, Ist. Lombardo Accad. Sci. Lett. Rend., **A 104**, 1970, pp. 525–530.

[3] J. R. Clay, J.J. Malone, Jr. *The near-rings with identities on certain finite groups*, Math. Scand., **19**, 1966, pp. 146-150.

[4] S. Feigelstock, *Additive Groups of Local Near-Rings*, Communications in Algebra, **34**, 2006, pp.743 - 747.

[5] C. J. Maxson, *On local near-rings*, Math. Zeitschr., **106**, 1968, pp. 197-205.

[6] C. J. Maxson, *Local nearrings of cardinality* p^2, Canad. Math. Bull., **11**, 1968, pp. 555-561.

[7] C. J. Maxson, *On the construction of finite local near-rings (I): on non-cyclic abelian p-groups*, Quart. J. Math. Oxford (2), **21**, 1970, pp. 449-457.

[8] C. J. Maxson, *On the construction of finite local near-rings (II): on non-abelian p-groups*, Quart. J. Math. Oxford (2), **22**, 1971, pp. 65-72.

[9] J. D. P. Meldrum, *Near-rings and their links with groups*, Pitman, Boston, 1985, 275 p.

[10] V. T. Nagrebeckii, *The automorphism groups of finite Miller-Moreno groups* (Russian), Dokl. Akad. Nauk Ukrain. SSR Ser. A, No.**5**, 1980, pp. 75-77.

[11] G. Pilz, *Near-rings. The theory and its applications* (Second edition), North-Holland, Amsterdam, 1983, 470 p.

[12] L. Redei, *Das "schiefe Produkt" in der Gruppentheorie mit Anwendung auf die endlichen nichtkommutativen Gruppen mit lauter kommutativen echten Untergruppen und die Ordnungszahlen, zu denen nur kommutative Gruppen gehören* (German), Comment. Math. Helv., **20**, 1947, pp. 225–264.

Contact information

I. Yu. Raievska Institute of Mathematics Ukrainian National Academy of Science, Tereshchenkivska st. 3, Kiev, 01601, Ukraine
E-Mail: raemarina@rambler.ru

Ya. P. Sysak Institute of Mathematics Ukrainian National Academy of Science, Tereshchenkivska st. 3, Kiev, 01601 Ukraine
E-Mail: sysak@imath.kiev.ua

Received by the editors: 23.09.2011
and in final form 30.03.2012.

Journal Algebra Discrete Math.