ДОПОВІДІ
НАЦІОНАЛЬНОЇ
АКАДЕМІЇ НАУК
УКРАЇНИ

doi: https://doi.org/10.15407/dopovidi2019.01.016
UDC 519.6

V.I. Norkin

V.M. Glushkov Institute of Cybernetics of the NAS of Ukraine, Kiev
NTU of Ukraine “Igor Sikorsky Kyiv Polytechnic Institute”
E-mail: vladimir.norkin@gmail.com

B&B method for discrete partial order and quasiorder optimizations

Presented by Corresponding Member of the NAS of Ukraine P.S. Knopov

The paper extends the Branch and Bound (B&B) method to find all nondominated points in a partially or quasiordered space. The B&B method is applied to the so-called constrained partial/quasi order optimization problem, where the feasible set is defined by a family of partial/quasi order constraints. The framework of the generalized B&B method is standard, it includes partition, estimation, and pruning steps, but bounds are different, they are set-valued. For bounding, the method uses a set ordering in the following sense. One set is "less or equal" than the other set if, for any element of the first set, there is a "greater or equal" element in the second one. In the B&B method, the partitioning is applied to the parts of the original space with nondominated upper bounds. Parts with small upper bounds (less than some lower bound) are pruned. Convergence of the method to the set of all nondominated points is established. The acceleration with respect to the enumerative search is achieved through the group evaluation of elements of the original space.

Keywords: *quasiorder, partial order, nondominated solutions, discrete optimization, branch and bound method.*

Partial/quasi order optimization is a research field, which studies optimization problems involving order relations. Classical examples of such problems are given by the multiobjective optimization [1—4]. Various applications of the partial/quasi order optimization are considered in [5]. Related works include the optimization with dominance constraints [6, 7] and set-valued optimization [8].

In the present paper, we consider a problem of finding the optimal (nondominated) elements (with respect to some partial/quasi order) on a discrete feasible set of elements defined by means of some other partial/quasi orders. A similar problem setting was considered in [9]. In practical problems, the feasible set may contain a huge number of elements, so the enumerative search is questionable. We develop a branch and bound (B&B) method for this problem and prove its convergence. The method subdivides the original problem into a sequence of subproblems, selects subproblems containing optimal elements, and proceeds until such subproblems become trivial. Acceleration with respect to the enumerative search is achieved due to the group evaluation of feasible elements, by using lower and upper bounds (with respect to the partial/quasi orders) for groups. In the present work, we use set-valued bounds for the fathoming of subproblems within the B&B framework and provide general convergence results. Remark that these set-valued bounds are particularly simple for the bi-objective linear optimization problems and are just piecewise linear concave curves [10]. The method generalizes particular B&B schemes used in the Pareto optimization [9—11].

1. Quasiorders and partial orders [3, 8].

Definition 1 (Quasiorders and partial orders). A binary relation $\succeq$ on a set $\aleph$ is called a quasiorder iff

for any $x \in \aleph$, it holds $x \succeq x$ (reflectivity);

for any $\{x, y, z \in \aleph\}$, the relations $x \succeq y$ and $y \succeq z$ imply $x \succeq z$ (transitivity).

A quasiorder $\succeq$ on $\aleph$ is called a partial order iff, for any $\{x, y \in \aleph\}$, the relations $x \succeq y$ and $y \succeq x$ imply $x = y$ (antisymmetry).

By definition, the relation $x \succ y$ means $x \succeq y$ and $x \neq y$; the relation $x \preceq y$ means $y \succeq x$; the relation $x \succ y$ means $y \succ x$.

A set $\aleph$ with a quasiorder (partial order) binary relation $\succeq$ on $\aleph$ is called a quasiorder (partial order) set/space.

Definition 2 (Optimality). A subset $X^* \subset X$ of a quasiorder set $X \subseteq \aleph$ is called $\succ$-nondominated ((Pareto) optimal) if, for any $x^* \in X^*$, there is no $x \in X$ such that $x \succ x^*$.

We now consider the space $2^{\aleph}$ of all subsets of $\aleph$ and introduce the order relations $\succeq, \succ$ in $2^{\aleph}$ [1,8,12,13].

Definition 3 (Quasiorders and partial orders in the space of sets). Subsets $A, B \subset \aleph$ satisfy the set relation $A \succeq B$ iff, for any $b \in B$, there is an element $a \in A$ such that $a \succeq b$.

Subsets $A, B \subset \aleph$ satisfy the set relation $A \succ B$ iff, for any $b \in B$, there is an element $a \in A$ such that $a \succ b$.

The notation $A \preceq B$, $A \prec B$ means $B \succeq A$, $B \succ A$, respectively.

Lemma 1. *Let the relation $\succeq$ be a quasiorder in $\aleph$. Then the corresponding set relation $\succeq$ is a quasiorder in the space of subsets of $\aleph$.*

Let the relation $\succeq$ be a partial order in $\aleph$. Then the corresponding set relation $\succeq$ is a partial order in the space of subsets of $\aleph$, consisting of mutually nondominated elements.

Example 1 (Violation of the "antisymmetry" property for the set relation $\succeq$). Let $a_1 \prec a_2$ and $A = \{a_1, a_2\}$, $B = \{a_2\}$. Then $A \succeq B$ and $A \preceq B$, but $A \neq B$.

Example 2 ({0,1}-string ordering). Let us consider a collection of $\{0, 1\}$-strings S and the following majority ordering "$\succeq$" in it:

$S \ni s_1 \succeq s_2 \in S$, if the number of ones in s_1 is not less than the number of ones in s_2;

$s_1 \succ s_2$, if the number of ones in s_1 is greater than the number of ones in s_2.

Such kind of ordering appears in maximum satisfiability problems, where the quality of a solution is measured by the number of conditions (e.g., inequalities) satisfied. This ordering is a quasiorder, but not a partial order (the antisymmetry requirement is not fulfilled). However, the reinforced transitivity property is fulfilled: if $(s_1 \succ s_2, s_2 \succ s_3)$, or $(s_1 \succeq s_2, s_2 \succ s_3)$, or $(s_1 \succ s_2, s_2 \succeq s_3)$, then $s_1 \succ s_3$.

Remark 1. If the underlying quasiorder $\succeq$ in $\aleph$ has the extended transitivity property, i.e., from $(x \succeq y, y \succ z)$ or $(x \succ y, y \succeq z)$, it follows $x \succ z$, then the induced set relation $\succeq$ in $2^{\aleph}$

has a similar property. Namely, for any subsets $A, B, C \subset \aleph$, it holds true: if $(A \succ B, B \succ C)$, or $(A \succeq B, B \succ C)$, or $(A \succ B, B \succeq C)$, then $A \succ C$. A similar reinforced transitivity property holds for the {0,1}-string ordering of Example 2.

2. Branch and Bound (B&B) method for discrete partial/quasi order optimization. In this section, we consider a branch and bound algorithm for finding all $\succ$-nondominated elements X_c^* of a subset $X_c \subset \aleph$ defined by some other quasiorders in the space $\aleph$. The B&B method treats elements of the optimization space $\aleph$ in groups by the lower and upper boundings of jointly all elements in the groups. Thus, it works with a smaller number of objects than the total number of elements in the space. This B&B method generalizes a number of specific B&B algorithms (see [9—11]) for solving the vector (Pareto) optimization problems, $\mathrm{R}^m \ni f(v) \to \mathrm{Max}_{v \in V}$, to a more general objective (quasiorder) space $\aleph \ni f(v)$.

Assumption A. *Let* $\succeq$ *be a quasiorder in a space* $\aleph$ *with the following extended transitivity property: for* $x, y, z \in \aleph$, *if* $(x \succeq y, y \succ z)$ *or* $(x \succ y, y \succeq z)$, *then* $x \succ z$.

If the relation $\succeq$ is a partial order, then Assumption A is fulfilled automatically.

Assumption B. *Let* $\succeq$ *be the quasiorder relation on subsets of* $\aleph$ *induced (in the sense of Definition* 3) *by a quasiorder relation* $\succeq$ *on elements of* $\aleph$. *Assume that there are mappings* $L, U : 2^{\aleph} \to 2^{\aleph}$ *such that*

B1: *For any* $X \subseteq \aleph$, *it holds true* $L(X) \preceq X \preceq U(X)$;

B2: *If a set* $X \subset \aleph$ *is a singleton, then* $L(X) = X = U(X)$.

Remark 2. Standard (and often poor) bounds in the vector optimization are the so-called ideal and nadir points [10, 11], i.e., single-valued bounds. The bounds $L(X)$ and $U(X)$ in assumption (B1) may be sets, i.e., for any element $l \in L(X)$, there is an element $x \in X$ such that $l \preceq x$, and, for any element $x' \in X$, there is an element $u \in U(X)$ such that $x' \preceq u$. Due to the reflexivity of the relation $\succeq$, it is admissible that $L(X) \subset X$.

Example 3 (Lower set-valued bounds in vector optimization). For a vector (Pareto) optimization problem: $f(v) = [f_i(v)]_{i=1}^m \to \mathrm{Max}_{v \in V}$, as an upper bound $U(X)$ of the image set $X = \{f(v) : v \in V\}$, one can take the ideal point $I(X) = \{\max_{v \in V} f_i(v)\}_{i=1}^m$. As a lower bound $L(X)$, one can take the set of values $\{f(v_\lambda), \lambda \in \Lambda\}$ for a number of solutions $\{v_\lambda, \lambda \in \Lambda\}$ of the scalarized problems: $\sum_{i=1}^m \lambda_i f_i(v) \to \max_{v \in V}$, $0 \neq \lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda \subset R_+^m$.

Example 4 (Set-valued bounds in discrete problems). Consider a vector discrete optimization problem: $f(v) = [f_i(v)]_{i=1}^m \to \mathrm{Max}_{v \in V \subset \{0,1\}^n}$, Standard vector bounds for the image set $f(v) = \{f(v), v \in V\}$ are nadir and ideal points: $N(f(v)) = \{\min_{v \in V} f_i(v)\}_{i=1}^m$, $I(f(v)) = \{\max_{v \in V} f_i(v)\}_{i=1}^m$.

Set-valued bounds can be constructed in the following way. Let us fix some components v_{j_1} and v_{j_2} of v at their possible values 0 or 1 and construct lower $L(f(V))$ and upper $U(f(V))$ bounds satisfying assumption B as follows. The set-valued bound $L(f(V))$ consists of two points: $\{\min_{\{v \in V : v_{j_1} = 0\}} f_i(v)\}_{i=1}^m$, $\{\min_{\{v \in V : v_{j_1} = 1\}} f_i(v)\}_{i=1}^m$. (If only one set $\{v \in V : v_{j_1} = 0\}$ or $\{v \in V : v_{j_1} = 1\}$ is nonempty, then only one point is used).

Similarly, the set-valued bound $U(f(V))$ consists of the following points: $\{\max_{\{v \in V : v_{j_2} = 0\}} f_i(v)\}_{i=1}^m$, $\{\max_{\{v \in V : v_{j_2} = 1\}} f_i(v)\}_{i=1}^m$.

To get better bounds, one can fix more components $v_j, j \in J_1$ and $v_j, j \in J_2$ of v on their all possible values. In this case, the bounds $L(f(V))$ and $U(f(V))$ consist of $2^{|J_1|}$ and $2^{|J_2|}$ points, respectively. What variables to fix to get good bounds is a (research) question.

Example 5 (Multiattribute optimization). The problem is to find nondominated multiattribute entries in big data sets. It is assumed that separate attributes take on values in completely ordered sets. This problem can be solved by both an enumerative pairwise comparison algorithm and by the B&B algorithm. The latter algorithm firstly finds an interval for the data, e.g., nadir and ideal points, subdivides it into smaller subintervals and finds nadir/feasible and ideal points for these subintervals, refines the family of subintervals from empty and dominated ones, and continues the subdivision and refinement procedures until the nondominated subintervals become singletons.

Example 6. Let $F_M(v)=\{f_j(v), j\in M\}\to \mathrm{Max}_{\cup_{k\in K}\{v\in V:\, g_k(v)\leqslant c_k\}}$ be a multicriteria optimization problem with finite or infinite sets M, K of criteria and constraints. By defining $G_K(v)=$ $=\{g_k(v), k\in K\}$, $C_K=\cup_{k\in K}\{c_k\}$, the latter problem can be rewritten in the form $F_M(v)\to$ $\to \mathrm{Max}_{\{v\in V:\, G_K(v)\preceq C_K\}}$, i.e., by means of the quasiorder set relation "$\preceq$".

Example 7 (Multilevel multicriteria and multilevel partial/quasi order optimization problems). A multicriteria (or partial order) optimization problem $\mathrm{Max}_{v\in V}F_1(v)$ usually singles out a whole Pareto-optimal set of nondominated solutions $V^*\subset V$ and the corresponding efficient frontier $F_1(V^*)$, e.g., an efficient frontier in the financial portfolio analysis. The second step in analyzing the problem is to define the second mapping (partial order relation) $F_2: V\to F_2(V)$ to select a narrower subset of nondominated solutions, and so on.

Suppose we have found an approximation C of the Pareto-efficient set $F_1(V^*)$, for example, consisting of a finite collection of elements, $C=\{c_1,\ldots,c_n\}$. Then the second stage problem may have the form: $\mathrm{Max}_{\{v\in V,\, F_1(v)\succeq C\}}F_2(v)$, i.e., is given by means of the quasiorder set relation "$\succeq$".

Problem setting. Suppose there are several quasiorders $\succeq_i$, $i=0,\ldots,n,$ defined on the same (objective) space $\aleph$. Define the relations $x\succ_i y$ as $x\succeq_i y$ and $x\neq y$. Corresponding set relations $\succeq_i$ and $\succ_i$ for subsets of $\aleph$ are defined in Definition 3.

Define the feasible set $X_c=\{x\in X\subseteq\aleph: C_i\preceq_i x\preceq_i D_i,\quad i=1,\ldots,n\}$, where $C_i\subset\aleph$, $D_i\subset\aleph$, $i=1,\ldots,n,$. Remark that the condition $C_i\preceq_i x$ means $x\in\bigcap_{c\in C_i}\{x'\in\aleph: x'\succeq_i c\}$ and $x\preceq_i D_i$ means $x\in\bigcup_{d\in D_i}\{x'\in\aleph: x'\succeq_i d\}$.

The problem is to check if $X_c\neq\varnothing$ and, in the latter case, to find the set X_c^* of $\succ_0$-nondominated elements in X_c. The set X_c can be (very) large, so the enumerative search may be questionable. The following B&B algorithm solves the problem by means of lower $L_i(\cdot)$ and upper $U_i(\cdot)$ bounds of subsets of $\aleph$. We assume that these bounds satisfy Assumptions A, B. The acceleration of the search is due to a group evaluation of elements of $\aleph$. The following B&B algorithm solves the set problem.

Constrained Branch and Bound algorithm (CBB-algorithm).

Step 0 (Initialization). Form an initial finite partition $P_0=\{X^p\subset\aleph: p=1,2,\ldots\}$ such that $X\subseteq\bigcup_p X^p$. Calculate bounds $L_i(X^p)$ and $U_i(X^p)$ for all $X^p\in P_0$, $i=0,1,\ldots,n,$. Set $k=0$.

Step 1 (Remove infeasible partition sets). Clean the partitions P_k from certainly infeasible sets, i.e., put $P_k := P_k\setminus\{X^p\in P_k: X^p\bigcap X=\varnothing,\quad \text{or } C_i\not\preceq U_i(X^p),\quad \text{or } L_i(X^p)\not\preceq D_i \text{ for some } i\}$.

Step 2 (Remove non-optimal partition sets). Clean the partitions P_k from sets not containing optimal points, i.e., put $P_k := P_k\setminus\{X^p\in P_k: U_0(X^p)\prec_0 L_0(X^q) \text{ for some singleton } X^q\in P_k\}$.

Step 3 (Look for partition sets with nondominated upper bounds). Find all U_0-nondominated partition sets $Y^p\in P_k$ such that there is no other partition set $X^q\in P_k$ with $U_0(X^q)\succ U_0(Y^p)$.

Step 4 (Check for the stopping conditions). If the partition P_k is empty or all U_0-nondominated partition sets $Y^p \in P_k$ are singletons, then Stop.

Step 5 (Partitioning). If there is an U_0-nondominated non-singleton partition set $Y^p \in P_k$, then form a partition of this set $P_k''(Y^k) = \{Y_i^k \neq \varnothing,\ i = 1, 2, \ldots\}$ such that $Y^k = \bigcup_i Y_i^k$ and $Y_i^k \bigcap_i Y_j^k = \varnothing$ for $Y_i^k, Y_j^k \in P_k''(Y^k)$, $i \neq j$. Define the new full partition $P_k := (P_k \setminus Y^k) \bigcup P_k''(Y^k)$. Elements of P_k will also be denoted as X^p.

Step 6 (Estimation of bounds). For all new subsets $X^p \in P_k'' \subset P_k$, calculate lower $L_i(X^p)$ and upper $U_i(X^p)$ bounds, $i = 0, 1, \ldots, n,$; for other subsets $Y^p \in P_k$, bounds remain the same. Put $k := k + 1$ and go to Step 1.

Theorem (Convergence of the CBB-algorithm). *Assume that the lower and upper bounds L_i, U_i satisfy conditions A, B for each $i = 1, \ldots, n,$. Then the following statements regarding the CBB-algorithm hold true.*

(a) If the space $\aleph$ is finite and the feasible set X_c is empty, then the CBB-algorithm stops after a finite number k' of iterations with the empty partition $P_{k'} \neq \varnothing$.

(b) If $X_c \neq \varnothing$ and X_c is finite, then the optimal ($\succ_0$-nondominated) set $X_c^* \neq \varnothing$, and no element of the X_c^* is deleted in the course of iterations of the CBB-algorithm.

(c) If in the course of iterations the CBB-algorithm generates a singleton U_0-nondiminated partition set $Y^p \in P_k$, i.e., such a singleton set that there is no other partition set $X^q \in P_k$ such that $U_0(X^q) \succ_0 U_0(Y^p)$, then $Y^p \in X_c^*$.

(d) If the space $\aleph$ is finite, then the CBB-algorithm stops after a finite number of iterations k'. In this case, the set of singleton U_0-nondominated partition sets $Y^p \in P_{k'}$ generated by the CBB-algorithm coincides with the optimal set X_c^*.

Proof. (a) If the space $\aleph$ is finite, then the CBB-algorithm stops after a finite number of iterations k', since there may be only a finite number of partition steps. Suppose $X_c = \varnothing$, but $P_{k'} \neq \varnothing$. The partition $P_{k'}$ contains singleton sets $Y^p = y$; otherwise, the algorithm did not stop at iteration k'. Since the algorithm passed Step 1 before stopping, $Y^p = y \in X$, $C_i \preceq_i U_i(Y^p)$, and $L_i(Y^p) \preceq_i D_i$ for all i. By assumption (B2), $y = L_i(Y^p) = U_i(Y^p)$ for all i. Hence, $y \in X_c \neq \varnothing$. We get a contradiction.

(b) Suppose the opposite: at some iteration k', a point $x^* \in X_c^*$ is deleted. It can be deleted only at Step 2 of the CBB-algorithm. This means that there are partition sets $X^p, X^q \in P_{k'}$ such that $x^* \in X^p$ and $U_0(X^p) \succ_0 L(X^q)$ for some singleton $X^q = x^q \in P_{k'}$. Since the algorithm passed Step 1, $x^q \in X$, $C_i \preceq_i U_i(X^q) = x^q$, $x^q = L_i(X^q) \preceq D_i$ for all i, and, hence, $x^q \in X_c$. Since, by (B1), $X^p \preceq U_0(X^p)$, for $x^* \in X^p$, there is an element $u^p \in U_0(X^p)$ such that $x^* \preceq u^p$. For $u^p \in U_0(X^p)$, by the definition of the dominance relation $\prec_0$ and (B2), from $U_0(X^p) \prec_0 L_0(X^q) = x^q$, it follows that $u^p \prec_0 x^q$. By the transitivity of the relations $\prec_0$, $\preceq_0$, we obtain $x^* \preceq_0 u^p \prec_0 x^q$. Hence, $x^* \prec_0 x^q \in X_c$, which means that x^* is not an $\succ_0$-nondominated element of X_c, a contradiction.

(c) Suppose the opposite that, at some iteration k, there is a singleton U_0-nondiminated partition set $Y^k = y \in P_k$ such that $Y^k \notin X_c^*$. Then there is an element $x \in X_c$ such that $x \succ_0 y$. Moreover, due to the theorem assumptions, there is an element $x^* \in X_c^*$ such that $x^* \succeq_0 x \succ_0 y$. By (b), the element x^* is not deleted, so there is a partition set $X^p \in P_k$ such that $x^* \in X^p$. By (B1), (B2), the following relation holds true: $U_0(X^p) \succeq_0 x^* \succ_0 y = U_0(Y^k)$, i.e., $U_0(X^p) \succ y = U_0(Y^k)$, which means that Y^k is not an U_0-nondominated partition set in P_k, a contradiction.

(d) Due to the finiteness of the number of elements in X, there can be only a finite number of iterations with the partitioning of sets. So, there exists an iteration k' such that all U_0-nondominated partition sets become singletons, $Y^{k'} = y^{k'}$, and the CBB-algorithm stops. Then, by (c), all such sets $Y^{k'} = y^{k'} \in X_c^*$.

Let the algorithm stoped at the iteration k' and let some $\hat{x} \in X_c^*$. By (b), $\hat{x}$ was not removed in the course of iterations. Hence, there exists a partition set $\hat{X}^{k'} \in P_{k'}$ such that $\hat{x} \in \hat{X}^{k'}$ and, by (B1), $\hat{x} \preceq_0 U_0(\hat{X}^{k'})$. Let us show that the set $\hat{X}^{k'}$ cannot be U_0-dominated. Suppose the opposite. Then there is a finite sequence of partition sets $\{X_i^{k'} \in P_{k'}\}_{i=1}^{I}$ such that $U_0(\hat{X}^{k'}) \prec_0 U_0(X_1^{k'})$, $U_0(\hat{X}^{k'}) \prec_0 U_0(X_{i+1}^{k'})$, $U_0(X_I^{k'}) \prec_0 U_0(\hat{Y}^{k'})$, and $\hat{Y}^{k'} \in P_{k'}$ is U_0-nondominated. Hence, the set $\hat{y}^{k'} = \hat{Y}^{k'}$ is a singleton and, by (c), it belongs to X_c^*. Thus, $\hat{x} \preceq_0 U_0(\hat{X}^{k'}) \prec_0 U_0(\hat{Y}^{k'}) = \hat{y}^{k'}$ and, hence, $\hat{x} \prec_0 \hat{y}^{k'} \in X_c^*$, which means that $\hat{x} \notin X_c^*$, a contradiction. Hence, the set $\hat{X}^{k'} \ni \hat{x}$ is U_0-nondominated. Then $\hat{X}^{k'}$ is a singleton and $\hat{x} = \hat{X}^{k'} \in P_{k'}$. On the other hand, by (c), all U_0-nondominated singleton partition sets $X^p \in P_{k'}$ belong to X_c^*. The proof is completed.

3. Conclusions. We have analyzed a general framework for the discrete branch and bound (B&B) methods designed to find optimal (i.e. nondominated) elements in a partial order or quasiorder (with certain extended transitivity property) space. The framework generalizes particular B&B schemes from the vector optimization to more general objective spaces. For example, the space may be an infinite-dimensional vector space, space of strings with ordered components, space of sets with a defined partial or quasiorder relation. We have also considered the so-called constrained quasiorder optimization problems involving several partial/quasi orders. Solutions of the problems are understood as nondominated points of the feasible or objective set. A nonstandard element of the considered B&B framework is that it exploits set-valued (including single-valued) lower and upper bounds for subsets generated by the algorithm. As a lower bound, any subset of feasible points may be used. For a finite discrete feasible set, the B&B algorithm either finds all optimal elements or discover that the feasible set is empty.

Thus, the paper extends horizons of optimization theory to general spaces, not necessary linear or metric, only the order is important. The further research may be devoted to the exploration of the efficiency of the developed B&B method on particular classes of problems and to its extension to the continuous case.

REFERENCES

1. Eichfelder, G. & Jahn, J. (2012). Vector optimization problems and their solution concepts. In Ansari, Q.H. & Yao, J.-C. (Eds.). Recent Developments in Vector Optimization (pp. 1-27). Berlin: Springer.
2. Ehrgott, M. (2005). Multicriteria Optimization. 2nd ed. Berlin, Heidelberg: Springer.
3. Gorohovik, V.V. (1990). Convex and nonsmooth problems of vector optimization. Minsk: Navuka i tehnika (in Russian).
4. Sawaragi, S., Nakayama, H. & Tanino, T. (1985). Theory of multiobjective optimization. Orlando: Academic Press.
5. Fattore, M. & Bruggemann, R. (Eds.) (2017). Partial order concepts in applied sciences. Cham: Springer International Publishing AG.
6. Dentcheva, D. & Ruszczyński, A. (2003). Optimization with stochastic dominance constraints. SIAM J. Optim., 14, pp. 548-566. doi: https://doi.org/10.1137/S1052623402420528
7. De Simone V., Marino, M. & Toraldo, G. (2009). Isotonic regression problems. In Floudas, C.A. & Pardalos, P.M. (Eds.). Encyclopedia of optimization. 2nd ed. (pp. 1774-1777). Boston, MA: Springer.
8. Khan, A. A., Tammer, C. & Zălinescu, C. (2015). Set-valued optimization. An introduction with applications. Berlin, Heidelberg: Springer.

9. Gavanelli, M. (2001). Partially ordered constraint optimization problems. In: Walsh, T. (eds.). Principles and practice of constraint programming — CP 2001. CP 2001. Lecture Notes in Computer Science, Vol. 2239. Berlin, Heidelberg: Springer. doi: https://doi.org/10.1007/3-540-45578-7_64
10. Przybylski, P. & Gandibleux, X. (2017). Multi-objective Branch and Bound. Eur. J. Oper. Res. 260, Iss. 3, pp. 856-872. doi: https://doi.org/10.1016/j.ejor.2017.01.032
11. Norkin, V. I. (2017). B&B solution technique for multicriteria stochastic optimization problems. In Butenko, S., Pardalos, P. & Shylo V. (Eds.). Optimization methods and applications (pp. 345-378). Cham: Springer. doi: https://doi.org/10.1007/978-3-319-68640-0_17
12. Nishnianidze, Z. G. (1984). Fixed points of monotone multivalued operators. Soobshch. Akad. Nauk Gruzin. SSR, 114, No. 3, pp. 489-491.
13. Yang, X. Q. (1992). A Hahn—Banach theorem in ordered linear spaces and its applications. Optimization. 25, No. 1, pp. 1-9.

Received 18.10.2018

В.І.Норкін

Інститут кібернетики ім. В.М. Глушкова НАН України, Київ
НТУ України “Київський політехнічний інститут ім. Ігоря Сікорського”
E-mail: vladimir.norkin@gmail.com

МЕТОД ГІЛОК ТА МЕЖ ДЛЯ ДИСКРЕТНОЇ ОПТИМІЗАЦІЇ В ЧАСТКОВИХ АБО КВАЗІПОРЯДКАХ

У роботі метод гілок і меж/оцінок (B&B-метод) поширюється на задачі пошуку недомінованих елементів у частково або квазіупорядкованій множині. B&B-метод застосовується до задач оптимізації, де допустима множина сама визначається сімейством квазіпорядків. Структура узагальненого B&B-методу є стандартною: він включає в себе розбиття на підзадачі, оцінювання підзадач і відбраковування підзадач, але оцінки підзадач відрізняються, вони можуть бути множинами. Для оцінювання підзадач метод використовує впорядкування множин у такому сенсі. Одна множина “менша або дорівнює” іншій, якщо для будь-якого елемента першої множини існує “більший або рівний” елемент у другій. У B&B-методі розбиття застосовується до підзадач з недомінованими верхніми оцінками. Підзадачі з малими верхніми оцінками (менше деякої нижньої оцінки) видаляються. Встановлено збіжність методу до множини всіх недомінованих елементів. Прискорення по відношенню до переборного пошуку досягається за рахунок групової оцінки елементів вихідного простору.

Ключові слова: *квазіпорядок, частковий порядок, недоміновані розв’язки, дискретна оптимізація, метод гілок та меж.*

В.И.Норкин

Институт кибернетики им. В.М. Глушкова НАН Украины, Киев
НТУ Украины “Киевский политехнический институт им. Игоря Сикорского”
E-mail: vladimir.norkin@gmail.com

МЕТОД ВЕТВЕЙ И ГРАНИЦ ДЛЯ ДИСКРЕТНОЙ ОПТИМИЗАЦИИ В ЧАСТИЧНЫХ ИЛИ КВАЗИПОРЯДКАХ

В работе метод ветвей и границ (B&B-метод) распространяется на задачи поиска недоминируемых точек в частично или квазиупорядоченном множестве. B&B-метод применяется к задачам оптимизации, где допустимое множество само определяется семейством квазипорядков. Структура обобщенного B&B-метода является стандартной: он включает в себя разбиение на подзадачи, оценки подзадач и отбраковку подзадач, но оценочные границы отличаются, они могут быть множествами. Для оценивания подзадач метод использует упорядочение множеств в следующем смысле. Одно множество “меньше или равно” другому, если для любого элемента первого множества существует “больший или равный” элемент во втором. В B&B-методе разбиение применяется к подзадачам с недоминируемыми верхними границами. Подзадачи с малыми верхними границами (меньше некоторой нижней границы) удаляются. Установлена сходимость метода к множеству всех недоминированных точек. Ускорение по отношению к переборному поиску достигается за счет групповой оценки элементов исходного пространства.

Ключевые слова: *квазипорядок, частичный порядок, недоминируемые решения, дискретная оптимизация, метод ветвей и границ.*