Leibniz Algebras and Lie Algebras*

Geoffrey MASON † *and Gaywalee YAMSKULNA* ‡§

† *Department of Mathematics, University of California, Santa Cruz, CA 95064, USA*
E-mail: *gem@ucsc.edu*

‡ *Department of Mathematical Sciences, Illinois State University, Normal, IL 61790, USA*
E-mail: *gyamsku@ilstu.edu*

§ *Institute of Science, Walailak University, Nakon Si Thammarat, Thailand*

Received September 09, 2013, in final form October 19, 2013; Published online October 23, 2013
http://dx.doi.org/10.3842/SIGMA.2013.063

Abstract. This paper concerns the algebraic structure of finite-dimensional complex Leibniz algebras. In particular, we introduce left central and symmetric Leibniz algebras, and study the poset of Lie subalgebras using an associative bilinear pairing taking values in the Leibniz kernel.

Key words: Leibniz algebras; Lie algebras

2010 Mathematics Subject Classification: 17A32

1 Introduction

Throughout the present paper, a *left Leibniz algebra* means a nonassociative $\mathbb{C}$-algebra M with a product (or *bracket*) [ ] satisfying the following identity for all $a, b, c \in M$:

$$[a[bc]] = [[ab]c] + [b[ac]]. \tag{1}$$

(1) says that the left adjoint $\mathrm{ad}_a : b \mapsto [ab]$ $(a, b \in M)$ is a *derivation* of M, so that $\mathrm{ad} : M \to \mathrm{Der}(M)$, $a \mapsto \mathrm{ad}_a$, is a morphism of M to the algebra of derivations of M regarded as a Lie (or left Leibniz) algebra with the usual bracket. The *Leibniz kernel* is the subspace $C(M)$ spanned by $[aa]$ $(a \in M)$; M is a Lie algebra if, and only if, $C(M) = 0$.

Leibniz algebras were first studied for their own sake by Loday [5] (see also [6, Section 10.6]). The rationale for the present work is partially motivated by the triangular decomposition

$$V = (\oplus_{n\leq 0} V_n) \oplus V_1 \oplus (\oplus_{n\geq 2} V_n) \tag{2}$$

of a vertex operator algebra (VOA) V. In the most widely studied case when V is of *CFT-type*, i.e. $V_n = 0$ for $n < 0$ and $V_0 = \mathbb{C}\mathbf{1}$ is spanned by the vacuum vector, the summands in (2) are Lie algebras. In the general case (2) satisfies only the weaker condition of being a decomposition into $\mathbb{Z}$-graded *left Leibniz algebras* with respect to the 0^{th} operation in V (cf. [4]). This decomposition plays a rôle in our work [7] where, among other things, we are interested in the Lie subalgebras of V_1 and the vertex subalgebras of V that they generate. This leads directly to the main theme of the present paper, which is the study of the poset of Lie subalgebras of certain kinds of Leibniz algebras. Readers uninured to VOA theory need not be concerned – the present paper deals solely with Leibniz algebras, and no further mention of VOAs will be made.

The definition of Leibniz algebra is *not* left-right symmetric; a right Leibniz algebra, in which the map $b \mapsto [ba]$ for fixed a is a derivation, is not necessarily a left Leibniz algebra. We call

*This paper is a contribution to the Special Issue on New Directions in Lie Theory. The full collection is available at http://www.emis.de/journals/SIGMA/LieTheory2014.html

an algebra that is both a left and right Leibniz algebra a *symmetric* Leibniz algebra. Notice from (1) that a left Leibniz algebra M satisfies

$$[[aa]b] = 0, \qquad a, b \in M, \tag{3}$$

and dually, a right Leibniz algebra satisfies

$$[a[bb]] = 0, \qquad a, b \in M. \tag{4}$$

We call M a left *central* Leibniz algebra if it is a left Leibniz algebra that also satisfies (4). Equivalently, M is both the left and right centralizer of $C(M)$. There is a hierarchy of algebras

$$\{\text{left Leibniz}\} \supseteq \{\text{left central Leibniz}\} \supseteq \{\text{symmetric Leibniz}\} \supseteq \{\text{Lie}\},$$

and in fact each containment is *strict*.

We now describe the contents of the present paper, which deals solely with *finite-dimensional*, complex, left Leibniz algebras M. We tacitly assume this in everything that follows. In Section 2 we discuss some basic facts, in particular concerning Levi subalgebras and Levi decompositions of Leibniz algebras. Levi subalgebras (i.e., semisimple Lie subalgebras that complement the solvable radical) are readily seen to exist in M (see [3] and Section 2 below). Malcev's theorem for Lie algebras does *not* extend to left Leibniz algebras, though it does for various special classes, including left central Leibniz algebras. Section 3 is devoted to these issues.

The remainder of the paper revolves about the symmetric bilinear pairing $\psi : M \times M \to C(M)$, $(a, b) \mapsto [ab] + [ba]$. This is a feature of all Leibniz algebras, but for left central Leibniz algebras it is particularly efficacious, because in this case it is *associative*. Then the radical R of ψ is a 2-sided ideal of M, and the poset $\mathcal{L}$ of Lie subalgebras of M coincides with the set of Leibniz subalgebras which are also isotropic subspaces. General properties of this set-up are developed in Section 4, including (Proposition 1) the fact that R coincides with the intersection of the maximal elements of $\mathcal{L}$. We also give (Lemma 7) a general construction of a class of left central Leibniz algebras based solely on Lie-theoretic data. In Sections 5 and 6 we consider left central Leibniz algebras with $\dim C(M) = 1$ (so that ψ is a trace form in the usual sense). Such algebras arise, for example, as quotients M/U whenever U is a hyperplane of $C(M)$. We prove (Theorem 3) that there is a Lie subalgebra $L \subseteq M$ that is also a maximal isotropic subspace of M (and therefore also a maximal element of $\mathcal{L}$) and such that L/R is *nilpotent*. In Section 6 we assume that M is also symmetric, a property that holds precisely when $M' \subseteq R$ (Lemma 8). We prove (Theorem 4) that a symmetric Leibniz algebra with $\dim C(M) = 1$ has a 2-sided ideal of codimension at most 1 that arises from the construction of Lemma 7. In this way, we more-or-less obtain a characterization of such Leibniz algebras in terms of Lie-theoretic data.

2 Basic facts

In this section, M is a left Leibniz algebra. For subsets $A, B \subseteq M$, we define $[AB]$ (or $[A, B]$) to be the subspace spanned by all brackets $[ab]$ ($a \in A$, $b \in B$). If $A = \{a\}$ is a singleton, we write $[\{a\}B] = [a, B]$. Introduce

$$\begin{aligned} &Z(M) := \{z \in M \mid [M, z] = [z, M] = 0\},\\ &M' := [MM],\\ &C(M) := \operatorname{span}\langle [aa] \mid a \in M\rangle = \operatorname{span}\langle [ab] + [ba] \mid a, b \in M\rangle. \end{aligned}$$

$Z(M)$, M' and $C(M)$ are 2-sided ideals of M called the *center*, *derived subalgebra* and *Leibniz kernel* of M respectively. (The second equality defining $C(M)$ is equivalent to the first by polarization. That $C(M)$ is a 2-sided ideal then follows from (3) and (1) with $b = c$. See also [1, 2].) Obviously, M is a Lie algebra if, and only if, $C(M) = 0$.

Set $M^{(0)} := M$ and for $n \geq 0$ inductively define $M^{(n+1)} := (M^{(n)})'$; M is called *solvable* if $M^{(n)} = 0$ for some $n \geq 0$. M has a unique maximal solvable ideal, called the *solvable radical* of M and denoted by $B(M)$. M is called *abelian* if $M' = 0$. By (3), M is the right centralizer of $C(M)$, i.e.,

$$[C(M)M] = 0. \tag{5}$$

Consequently, $C(M)$ is a 2-sided abelian ideal of M and $C(M) \subseteq B(M)$. Because $C(M/C(M)) = 0$, $M/C(M)$ is a Lie algebra and $B(M)/C(M)$ is its solvable radical. M is a left central Leibniz algebra if, and only if, $C(M) \subseteq Z(M)$.

It is not hard to see that the analog of the *Levi decomposition* for Lie algebras holds in M (cf. [3]). Indeed, set $N = C(M)$, $B = B(M)$, and let $N \subseteq T \subseteq M$ with T/N a Levi subalgebra of the Lie algebra M/N. We have a Levi decomposition in M/N,

$$M/N = B/N \oplus T/N.$$

Because $T' \subseteq T$ and N acts trivially on the left of M, T is a left T/N-module. Because T/N is semisimple, Weyl's theorem of complete reducibility tells us that there is a left T-submodule that complement N in T, call it S. Therefore, $S' \subseteq [TS] \subseteq S$, so that S is a Leibniz subalgebra of M. Moreover, $C(S) \subseteq S \cap N = 0$, whence S is, in fact, a Lie subalgebra of M. It is clearly isomorphic to T/N, hence is semisimple. We have a direct sum decomposition

$$M = B \oplus S. \tag{6}$$

We call a Lie subalgebra S of M that complements B as in (6) a *Levi factor*, or *Levi subalgebra*, of M; (6) itself is called a *Levi decomposition* of M. The Levi subalgebras of M can be characterized as the Lie subalgebras of M of maximal dimension subject to being semisimple.

3 On Malcev's theorem for Leibniz algebras

Malcev's theorem for finite-dimensional complex Lie algebras includes the statements that all Levi subalgebras are conjugate by the exponential of an inner derivation, and every semisimple Lie subalgebra is contained in a Levi subalgebra. We shall see that both of these assertions are generally false for Leibniz algebras. On the other hand, versions of the Malcev theorem can be proved for certain classes of Leibniz algebras. This section is concerned with these questions.

We begin with a general construction. Let S be a finite-dimensional complex Lie algebra and N a finite-dimensional left S-module with action $S \times N \to N, (a, m) \mapsto a.m$ ($a \in S$, $m \in N$). Let $M = S \oplus N$ with bracket

$$[(a, m)(b, n)] := ([ab], a.n). \tag{7}$$

Here, of course, $[ab]$ denotes the bracket in S. (7) defines the structure of a Leibniz algebra on M; S is a Lie subalgebra of M, and $C(M) = [SN]$.

Now suppose in addition that S is semisimple. Then S is a Levi subalgebra of M and N is the solvable radical. Suppose that S_1 is any Levi subalgebra of M. Then $M = S_1 \oplus N$ and $S_1 = \{(x, c(x)) \,|\, x \in S\}$ for some linear map $c : S \to N$. Moreover $[(x, c(x))(y, c(y))] = ([xy], x.c(y)) = ([xy], c([xy]))$, so that $x.c(y) = c([xy])$. This says that c is a morphism of left S-modules, where S furnishes the left adjoint S-module. Conversely, given $c \in \mathrm{Hom}_S(S, N)$, the set of pairs $\{(x, c(x)) \,|\, x \in S\}$ is a Levi subalgebra of M.

Suppose that S is simple and N a simple S-module that is *not* the adjoint module. Then $\mathrm{Hom}_S(S, N) = 0$, so that S is the *unique* Levi subalgebra of M. In the case of Lie algebras, if there is a unique Levi subalgebra then it is an ideal. However, as our construction shows, this is not necessarily true for Leibniz algebras.

Suppose that $T \subseteq S$ is a simple Lie subalgebra. Replacing S by T in the previous paragraphs, we see that if $d \in \mathrm{Hom}_T(T, N)$ then $T_1 := \{(x, d(x) \,|\, x \in T\}$ is a Lie subalgebra of M isomorphic to T. In order for T_1 to be contained in a Levi subalgebra, it is necessary and sufficient that d is the *restriction* $d = \mathrm{Res}^S_T(c)$ for some $c \in \mathrm{Hom}_S(S, N)$.

Example 1. $S = \mathfrak{sl}_3$, V the natural 3-dimensional module, $N = S^2(V)$ the second symmetric square, T the $\mathfrak{sl}_2$-subalgebra of S corresponding to a simple root. Then $\mathrm{Hom}_S(S, N) = 0$ and $\mathrm{Hom}_T(T, N) \neq 0$. Thus there exists T_1 *not* contained in the unique Levi subalgebra S. The theory of highest weight modules can be used to construct many similar examples.

Example 2. Suppose that S is simple and there is an isomorphism of left S-modules $S \cong N$. Then $\mathrm{Hom}_S(S, N) \neq 0$, and there are at least two distinct Levi subalgebras in M. On the other hand, because $[NM] = 0$, every inner derivation of M leaves every Levi subalgebra invariant, so that conjugacy *a la* Malcev does not hold in M. (This is essentially Example 2 in [3].)

Now we turn to some positive results in the direction of Malcev's theorem.

Theorem 1. *Let M be a left Leibniz algebra with Leibniz kernel N and solvable radical B. Let S be a Levi subalgebra of M, and suppose that $[SN] = 0$. If $T \subseteq M$ is any semisimple Lie subalgebra, there is an automorphism $\alpha = \exp(\mathrm{ad}\, x)$ $(x \in [M, B])$ such that $\alpha(T) \subseteq S$.*

Remark 1. If one Levi subalgebra S satisfies $[SN] = 0$, then *all* Levi subalgebras have the same property.

Proof of Theorem 1. Let $T \subseteq M$ be any semisimple Lie subalgebra. By Malcev's theorem for Lie algebras applied to M/N, we can find $x + N \in [M/N, B/N]$ so that $\alpha = \exp(\mathrm{ad}\,(x + N))$ and $\alpha(T + N/N) \subseteq S + N/N$. Because of (5), we can identify α with $\exp(\mathrm{ad}\, x)$, which we also denote by α. Then $\alpha(T) \subseteq S + N$. Because $[M/N, B/N] = ([M, B] + N)/N$, we may choose $x \in [M, B]$. Because of the assumption that $[SN] = 0$, $S + N$ is a *reductive* Lie subalgebra, and S is the unique maximal semisimple subalgebra. Therefore $\alpha(T) \subseteq S$, and the theorem is proved. ■

Theorem 2. *Let M be a left Leibniz algebra with Leibniz kernel N and solvable radical B. Suppose that $[BN] = 0$. Then given any pair of Levi subalgebras S_1 and S_2 of M, there is a derivation f of M such that $\exp(f)(S_1) = S_2$.*

Remark 2. The conditions of the theorem apply in the situation of Example 2, for example. Thus, while we may not have conjugacy of Levi subalgebras using only exponentials of inner derivations, conjugacy is restored in some cases if more general automorphisms are permitted.

Proof of Theorem 2. First note that $S_i + N/N$ $(i = 1, 2)$ are two Levi subalgebras of the Lie algebra $L = M/N$. By the Levi–Malcev theorem for M/N, there is a derivation $\delta \in \mathrm{Der}(L)$ such that $\exp(\delta)$ maps $S_1 + N/N$ onto $S_2 + N/N$. Because $[NM] = 0, \delta$ lifts to a derivation of M. So in proving the theorem, we may, and shall, assume that $S_1 + N = S_2 + N$.

Arguing in the same way as the discussion preceding Example 1, we know that $S_2 = \{(x, c(x)) \,|\, x \in S_1\}$ for some $c \in \mathrm{Hom}_{S_1}(S_1, N)$. We will show that the linear map $f : M \to M$ defined by

$$f(x) = \begin{cases} c(x), & x \in S_1, \\ 0, & x \in B, \end{cases}$$

has the desired properties. f is well-defined because $M = B \oplus S_1$, and since $f(M) \subseteq N \subseteq B$ then $f^2 = 0$. To see that f is a derivation, note that for $x, y \in S_1$, $a, b \in B$, we have

$$f([(x+a)(y+b)]) = f([xy]) = c([xy]),$$
$$[(x+a)f(y+b)] + [(f(x+a))(y+b)] = [(x+a)c(y)] = c([xy]),$$

where we used $[ac(y)] \in [BN] = 0$ to ensure the last equality.

Since $f^2 = 0$, we have $\exp(f) = Id + f$, whence $\exp(f)(x) = x + c(x)$ for $x \in S_1$. Consequently, $\exp(f)$ maps S_1 onto S_2. This completes the proof of the theorem. ■

4 Left central Leibniz algebras

In this section, M is a left central Leibniz algebra with Leibniz kernel $N = C(M)$. Recall that this means that $N \subseteq Z(M)$, i.e. $[MN] = 0$. Note that M satisfies the hypotheses of both Theorem 1 and Theorem 2, so that the Levi–Malcev theorem holds in M. This suggests that left central Leibniz algebras are more manageable than general Leibniz algebras.

We introduce and study a very useful bilinear pairing attached to M. For $a, b \in M$, write

$$\psi_M(a, b) = \psi(a, b) := [ab] + [ba].$$

Then $\psi : M \times M \to N$ is a symmetric bilinear map. Even though ψ takes values in N, it still makes sense to use terminology associated with the more familiar situation of a $\mathbb{C}$-valued bilinear form on M. Thus $a \in M$ is *isotropic* if $\psi(a, a) = 0$ ($[aa] = 0$); otherwise, a is *anisotropic*. a and b are *orthogonal* if $\psi(a, b) = 0$. For a subspace $U \subseteq M$, define

$$U^\perp := \{b \in M \mid \psi(a, b) = 0,\ a \in U\},$$
$$\mathrm{rad}(U) := U \cap U^\perp.$$

U is *totally isotropic* if $U \subseteq U^\perp$ (this is equivalent to every element of U being isotropic, and also to $U = \mathrm{rad}\, U$), and *nondegenerate* if $\mathrm{rad}\, U = 0$. $U, V \subseteq M$ are *orthogonal* if $V \subseteq U^\perp$. The *radical* of ψ (or M) is

$$R := \mathrm{rad}(M) = M^\perp = \{x \in M \mid \psi(x, a) = 0,\ a \in M\}.$$

These definitions apply to all Leibniz algebras. ψ is particularly useful in the study of central Leibniz algebras because of the next result.

Lemma 1. *Suppose that M is a left central Leibniz algebra. Then ψ is associative in the sense that for $a, b, c \in M$ we have*

$$\psi([ab], c) = \psi(a, [bc]). \tag{8}$$

In particular, R is a 2-sided ideal of M.

Proof. The identities $([ab] + [ba])c = a([bc] + [cb]) = 0$ hold for all $a, b, c \in M$. Therefore,

$$\begin{aligned}\psi([ab], c) &= [[ab]c] + [c[ab]] = -([[ba]c] + [c[ba]]) = -([b[ac]] - [a[bc]] + [[cb]a] + [b[ca]]) \\ &= [a[bc]] + [[bc]a] = \psi(a, [bc]).\end{aligned}$$

This proves (8). That R is a 2-sided ideal is an immediate consequence. ■

Because M is a left central Leibniz algebra, we have $N \subseteq R$. Thus, M/R is a Lie algebra equipped with the symmetric nondegenerate pairing induced by ψ. Introduce the poset (with respect to containment)

$$\mathcal{L} := \{\text{Lie subalgebras of } M\}.$$

Lemma 2. *Let $L \subseteq M$ be a Leibniz subalgebra. Then the following are equivalent:*

(*a*) $L \in \mathcal{L}$;

(*b*) *L is a totally isotropic subspace*;

(*c*) $L + R \in \mathcal{L}$.

Proof. The equivalence of (a) and (b) follows directly from the definitions. Next, if $L+R$ is a Lie algebra then L is necessarily a Lie subalgebra, so that (c) $\Rightarrow$ (a). Conversely, if (a) holds then L is a Lie subalgebra of M. In this case, $L+R$ is itself a Leibniz subalgebra because R is a 2-sided ideal. Moreover, if $a \in R$, $b \in L$, then

$$\psi(a+b, a+b) = \psi(a,a) + 2[bb] + 2\psi(a,b) = 0.$$

Therefore, $R+L$ is totally isotropic, and the equivalence of (a) and (b) applied to $R+L$ shows that (c) holds. This completes the proof of the lemma. ∎

Lemma 3. *Suppose that $U \subseteq M$ is a totally isotropic 2-sided ideal of M. Then $U \in \mathcal{L}$ and $U' \subseteq R$.*

Proof. Because U is a totally isotropic Leibniz subalgebra of M, Lemma 2 tells us that $U \in \mathcal{L}$. Now let $a, b \in U$, $x \in M$. Then $[bx] \in U$, and the associativity of ψ (Lemma 8) together with the fact that U is totally isotropic implies that $\psi([ab], x) = \psi(a, [bx]) = 0$. Therefore $[ab] \in R$ for all $a, b \in U$, i.e. $U' \subseteq R$. This completes the proof of the lemma. ∎

Lemma 4. *Suppose that $R \subseteq L \subseteq M$ such that $L/R \subseteq M/R$ is a semisimple Lie subalgebra. Then $L \in \mathcal{L}$.*

Proof. L is clearly a Leibniz subalgebra of M. Let $S \subseteq L$ be a Levi subalgebra. Because L/R is semisimple we have $L = R + S$, and this is a Lie subalgebra thanks to the equivalence of (a) and (c) in Lemma 2. ∎

Lemma 5. *M/R has no nonzero semisimple ideals. In particular, every minimal ideal of M/R is abelian.*

Proof. Suppose that U/R is an ideal of M/R that is also a semisimple Lie algebra. By Lemma 4, $U \in \mathcal{L}$, and in particular it is totally isotropic. Then $U' \subseteq R$ by Lemma 3. Therefore U/R is both abelian and semisimple, whence it reduces to 0. This proves the first assertion of the lemma. Because every minimal ideal of the Lie algebra M/R is either abelian or semisimple, the second assertion follows. This completes the proof of the lemma. ∎

Proposition 1. *Let $\mathcal{L}^*$ be the set of maximal elements of $\mathcal{L}$. Then*

$$R = \bigcap_{L \in \mathcal{L}^*} L.$$

Proof. Let $L \in \mathcal{L}^*$. Then $R+L$ is also a Lie subalgebra by Lemma 2, so that $R \subseteq L$ because L is maximal. This shows that R is contained in every $L \in \mathcal{L}^*$, and hence also in their intersection.

To prove the opposite containment, we use induction on $\dim N$. There is nothing to prove if M is a Lie algebra, so we may, and shall, assume that this is not the case. Therefore, $N \neq 0$. Suppose that $\dim N \geq 2$. Then every hyperplane $N_0 \subseteq N$ is a 2-sided ideal in M, and is itself contained in every maximal Lie subalgebra L. Then by induction we obtain

$$\bigcap_{L \in \mathcal{L}^*} L/N_0 = R(M/N_0),$$

which says that if $a \in \cap L$ then $\psi(a,x) \in N_0$ for all $x \in M$. Then $\psi(a,x) \in \cap N_0 = 0$ (the last intersection ranging over hyperplanes of N), and thus $a \in R$, as required This reduces us to the case when $\dim N = 1$, so that we can think of ψ as a $\mathbb{C}$-valued bilinear form. We assume this for the remainder of the proof.

Because M is not a Lie algebra, $R \neq M$. Suppose that R has codimension 1. Then every nonzero element of M/R is anisotropic, so that if $a \in M \setminus R$ then a cannot be contained in a Lie subalgebra of M. Thus in this case, R is the *unique* element in $\mathcal{L}^*$, and the desired result is clear.

Finally, suppose that M/R has dimension at least 2. Because ψ is a nondegenerate $\mathbb{C}$-valued bilinear form on M/R, M/R is *spanned* by *isotropic* vectors, i.e., elements $a+R$ with $[aa]=0$. Such elements a are contained in some maximal Lie subalgebra; therefore, if $b \in \cap_{L\in\mathcal{L}^*} L$, a and b generate a Lie subalgebra of M, so that $[ab]+[ba]=0$. Because the isotropic vectors $a+R$ span, we can conclude that $[ab]+[ba]=0$ for all $a \in M$, whence $b \in R$. This completes the proof of the proposition. ■

We will need the next lemma in Section 5.

Lemma 6. *Let $R \subseteq B \subseteq M$ satisfy $B/R := B(M/R)$. Then $B^\perp \subseteq B$.*

Proof. Write $B^\perp = S_0 \oplus B_0$, where S_0 and B_0 are a Levi factor and the solvable radical respectively for $B^\perp$. Because B is an ideal in M then so too is B_0. Therefore, $B_0 \subseteq B \cap B^\perp$. Thus $B_0 = \mathrm{rad}(B)$, and in particular B_0 is totally isotropic. So we see that there is an orthogonal decomposition $B^\perp = S_0 \perp B_0$, and since both summands are totally isotropic then $B^\perp$ is a totally isotropic ideal of M. Now we can apply Lemma 3, with $B^\perp$ playing the rôle of U, to conclude that $S_0 \subseteq R$. Therefore, $B^\perp \subseteq B$, and the lemma is proved. ■

We now describe the construction of a class of left central Leibniz algebras that depends only on Lie-theoretic data $(L_1, L_2, R, \alpha, \pi)$ satisfying (a)–(c) below. The set-up is as follows:

(a) a pair of Lie algebras L_1, L_2 with a common ideal R and L_2/R abelian:

$$\begin{array}{l} 0 \to R \to L_1 \to L_1/R \to 0 \\ \qquad\ \ \| \\ 0 \to R \to L_2 \to L_2/R \to 0 \end{array}$$

Set $Z := R \cap Z(L_1) \cap Z(L_2)$.

(b) a morphism of Lie algebras $\alpha : L_1 \to \mathrm{Der}(L_2)$ with $\alpha|_R = \mathrm{ad}_R$, $\alpha(L_1)|_R = \mathrm{ad}_{L_1}|_R$. Setting $\alpha(x_1)(y_1) = x_1.y_1$, these assumptions amount to

$$\begin{aligned} &[x_1x_2].y_1 = x_1.(x_2.y_1) - x_2.(x_1.y_1), \\ &x_1.[y_1y_2] = [(x_1.y_1)y_2] + [y_1(x_1.y_2)], \\ &x_1.y_1 = [x_1y_1] \qquad (x_1 \in R \text{ or } y_1 \in R), \end{aligned}$$

for $x_1, x_2 \in L_1$, $y_1, y_2 \in L_2$, and where [ ] is the bracket in L_1 or L_2.

(c) an *injective* morphism of left L_1-modules $\pi : L_1/R \to \mathrm{Hom}_{\mathbb{C}}(L_2/R, Z)$ such that $\mathrm{im}\,\pi$ annihilates no nonzero elements of L_2/R. (L_1 acts on L_2/R via α, trivially on Z, and with the induced left action on $\mathrm{Hom}_{\mathbb{C}}(L_2, Z)$). Lifting π to a morphism of left L_1-modules $\pi : L_1 \to \mathrm{Hom}_{\mathbb{C}}(L_2, Z)$ and setting $\psi' : L_1 \times L_2 \to Z, (x_1, y_1) \mapsto \pi(x_1)(y_1)$, these assumptions mean that

$$\begin{aligned} &\psi'([x_1x_2], y) = \psi'(x_1, x_2.y), \\ &R = \{x_1 \in L_1 \,|\, \psi'(x_1, L_2) = 0\} = \{y_1 \in L_2 \,|\, \psi'(L_1, y_1) = 0\}. \end{aligned}$$

Define $M = (L_1 \oplus L_2, [\,])$, where

$$[(x_1, y_1)(x_2, y_2)] := ([x_1 x_2], x_1.y_2 - x_2.y_1 + [y_1 y_2] + \psi'(x_2, y_1)). \tag{9}$$

Thus L_1, L_2 are naturally Lie subalgebras of M. One calculates that M is a left Leibniz algebra, L_2 is a 2-sided ideal, and $Z \oplus Z \subseteq Z(M)$. Moreover, $\psi((x_1, y_1), (x_2, y_2)) = (0, \psi'(x_2, y_1) + \psi'(x_1, y_2)) \in Z \oplus Z$. So M is a left central Leibniz algebra with radical $R \oplus R$. Furthermore, $D := \{(a, -a) \,|\, a \in R\}$ is a 2-sided ideal. So $\tilde{M} := M/D$ is itself a central Leibniz algebra, and the radical $R \oplus R/D$ of the induced bilinear form $\tilde{\psi}$ on $\tilde{M}$ is naturally identified with R. We state these conclusions as

Lemma 7. *Suppose that $(L_1, L_2, R, \alpha, \pi)$ satisfies assumptions (a)–(c). Then $\tilde{M} = \tilde{M}(L_1, L_2, R, \alpha, \pi)$ is a left central Leibniz algebra with radical R.*

5 Left central Leibniz algebras of rank 1

For a Leibniz algebra M, we define the *rank* of M to be the *dimension* of $N = C(M)$, and denote it by $\mathrm{rk}(M)$. This is a useful invariant for left central Leibniz algebras, because in this case any subspace $N_0 \subseteq N$ is a 2-sided ideal of M and M/N_0 is a left central Leibniz algebra with $\mathrm{rk}(M/N_0) = \dim N - \dim N_0$. In this way, we can try to reduce questions about left central Leibniz algebras to those of rank 1. These are easier to deal with, because the form ψ may be considered to be a *trace form* in the usual sense, i.e. a $\mathbb{C}$-valued associative bilinear form. This method was already used in the proof of Proposition 1.

If M_1, M_2 are two Leibniz algebras, their orthogonal sum $M_1 \perp M_2$ is the direct sum $M_1 \oplus M_2$ with product $[(a_1, b_1)(a_2, b_2)] = ([a_1 a_2], [b_1 b_2])$. Then $M_1 \perp M_2$ is a Leibniz algebra, M_1 and M_2 are orthogonal ideals (with respect to $\psi_{M_1 \perp M_2}$), and $\mathrm{rk}(M_1 \perp M_2) = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$. Thus rk is additive over orthogonal sums.

If M has index $r \geq 1$, we can find r hyperplanes $N_1, \ldots, N_r$ of N such that $\cap_{i=1}^r N_i = 0$. Then there is an injective morphism of Leibniz algebras

$$M \to \perp_{i=1}^r M/N_i, \qquad a \mapsto (a + N_1, \ldots, a + N_r).$$

In this way, any left central Leibniz algebra of rank $r \geq 1$ is isomorphic to a subalgebra of an orthogonal sum of r left central Leibniz algebras of rank 1.

Theorem 3. *Suppose that M is a left central Leibniz algebra of rank 1, and let B be as in Lemma 6. Then there is at least one maximal Lie algebra $L \in \mathcal{L}^*$ with the following properties:*

- *(a) L is a maximal isotropic subspace of M;*
- *(b) L/R is nilpotent;*
- *(c) $L \subseteq B$.*

Proof. If $L \in \mathcal{L}^*$ then $R \subseteq L$ (cf. Proposition 1). So part (b) makes sense. To prove the theorem, we use induction on $\dim M$. If every nonzero element of M/R is anisotropic, then R is both the unique maximal Lie subalgebra of M and the unique maximal totally isotropic subspace, in which case all parts of the theorem are obvious. So we may, and shall, assume that M/R contains nonzero isotropic elements. As in the proof of Proposition 1, this implies that $\dim(M/R) \geq 2$.

We assume first that there is a (nonzero) ideal U/R in M/R which is totally isotropic. If M/R is solvable, we choose any such ideal U/R; if M/R is nonsolvable we take $U = B^{\perp}$. Note that in the second case, B is a proper ideal of M, so that $B^{\perp}/R$ is nonzero thanks to the assumption that M has rank 1. Moreover, $B^\perp/R$ is totally isotropic in the second case because of Lemma 6.

With U/R chosen in this way, $U^\perp$ is a proper ideal of $M, U^\perp \subseteq B$, and $U = \mathrm{rad}(U^\perp)$. By induction there is a Lie subalgebra $L \subseteq U^\perp$ which is a maximal isotropic subspace of $U^\perp$ and such that L/U is nilpotent. If L_1 is a maximal isotropic subspace of M that contains L, then $L_1 \subseteq U^\perp$, and this implies that $L = L_1$ since L is maximal isotropic in $U^\perp$. So L is maximal isotropic in M.

It remains to show that L/R is nilpotent. Let $a \in M, u \in U, x \in L$. Then $\psi(a, [ux]) = \psi([au], x) = 0$, which shows that $[LU] \subseteq R$. Another way to say this is $U/R \subseteq Z(L/R)$. Since we already know that L/U is nilpotent, we can conclude that L/R is too. This completes the proof of the theorem in this case. Thus we may, and now shall, suppose that *no* nonzero ideal U/R of M/R is totally isotropic.

Now take any minimal nonzero ideal $U/R \subseteq M/R$. Then U/R is nondegenerate, and because M has rank 1 there is an orthogonal decomposition $M/R = U/R \perp U^\perp/R$. Continuing in this way, we obtain an orthogonal decomposition

$$M/R = U_1/R \perp \cdots \perp U_k/R$$

with each U_j/R a minimal, nondegenerate ideal of M/R. By Lemma 5, each U_j/R is *abelian*. Then M/R is itself an abelian Lie algebra, and every nonzero element of M/R generates a 1-dimensional ideal. In particular, since M/R contains nonzero isotropic elements, it also has a nonzero isotropic ideal. This contradiction completes the proof of the theorem. ■

6 Symmetric Leibniz algebras

Lemma 8. *Suppose that M is a left Leibniz algebra. Then M is a symmetric Leibniz algebra if, and only if, $M' \subseteq R$.*

Proof. In any left Leibniz algebra we have

$$[a[bc]] + [[ac]b] = [[ab]c] + [b[ac]] + [[ac]b] = [[ab]c] + \psi([ac], b).$$

Therefore, M is a symmetric Leibniz algebra if, and only if, $\psi([ab], c) = 0$ for all $a, b, c \in M$. The assertion of the lemma is just a restatement of this. ■

Consider the left central Leibniz algebra $M = M(L_1, L_2, R, \alpha, \pi)$ with product (9) and its quotient algebra $\tilde{M} = M/D$ (cf. Lemma 7) introduced at the end of Section 4. Since D is contained in the radical of M, it follows from Lemma 8 that $\tilde{M}$ is symmetric if, and only if, M is symmetric. From (9) and Lemma 8 once more, this holds if, and only if, L_1/R is abelian (recall that L_2/R is abelian by construction) and $x_1.y_2 - x_2.y_1 \in R$ ($x_1, x_2 \in L_1, y_1, y_2 \in L_2$). We assert that the second condition is a consequence of the first. Indeed, because $\pi : L_1/R \to \mathrm{Hom}_{\mathbb{C}}(L_2/R, Z)$ is an injection of L_1-modules then L_1 acts trivially on $\mathrm{im}\,\pi$, i.e., $\mathrm{im}\,\pi$ annihilates each element $x_1.y_1$. Since $\mathrm{im}\,\pi$ has no nonzero invariants in its action on L_2/R, it follows that $x_1.y_1 \in R$, and since this holds for any $x_1 \in L_1$, $y_1 \in L_2$ then the second condition indeed follows, as asserted. Thus we have proved

Lemma 9. *The left central Leibniz algebras $M(L_1, L_2, R, \alpha, \pi)$ and $\tilde{M} = M/D$ are symmetric Leibniz algebras if, and only if, L_1/R and L_2/R are both abelian Lie algebras.*

Now suppose that M_0 is a symmetric Leibniz algebra of rank 1. Thus $N = C(M_0)$ has dimension 1 and ψ defines a trace form on M_0 with radical R. Consequently, we can find a pair of maximal, totally isotropic subspaces $L_1, L_2 \subseteq M_0$ such that $L_1 \cap L_2 = R$ and $L_1 + L_2$ has codimension at most 1 in M_0. ($L_1 + L_2 = M_0$ if, and only if, $\dim(M_0/R)$ is *even*.) Because M_0/R is an abelian Lie algebra by Lemma 8, it follows that $H := L_1 + L_2$ is a 2-sided ideal with Lie subalgebras L_1, L_2; indeed, $L_1, L_2 \in \mathcal{L}^*$.

We will show that $H \cong \tilde{M}(L_1, L_2, R, \alpha, \pi)$ for suitably defined maps α, π. Both L_1, L_2 are ideals of M_0, in particular the left adjoint defines a morphism of Lie algebras $\alpha : L_1 \to \operatorname{Der}(L_2)$ satisfying the assumptions of (b) at the end of Section 4. Similarly, $R = \operatorname{rad}(M)$ and we define the morphism of L_1-modules $\pi : L_1/R \to \operatorname{Hom}_{\mathbb{C}}(L_2/R, N)$ as $\pi(x_1, y_1) := \psi(x_1, y_1)$ ($x_1 \in L_1$, $y_1 \in L_2$). It is easily seen that the assumptions of (c) at the end of Section 4 also hold. (The main point is the associativity of ψ.) Thus the set-up discussed in Section 4 holds, and we can apply Lemma 7 to obtain the left central Leibniz algebras $M(L_1, L_2, R, \alpha, \pi)$ and $\tilde{M}$. We assert that $H \cong \tilde{M}$. To see this, one checks that the map

$$\varphi : \ M(L_1, L_2, R, \alpha, \pi) \to H, \qquad (x_1, y_1) \mapsto x_1 + y_1,$$

is a surjective morphism of Leibniz algebras with kernel $\{(x_1, y_1) \,|\, x_1 + y_1 = 0\} = \{(a, -a) \,|\, a \in R\}$. Now our assertion follows from the very construction of $\tilde{M}$. We have proved

Theorem 4. *A symmetric Leibniz algebra of rank* 1 *has an ideal of codimension at most* 1 *isomorphic to* $\tilde{M}(L_1, L_2, R, \alpha, \pi)$.

Acknowledgements

This work was supported by the NSF (G.M.) and the Simons Foundation (G.Y.). The authors thank the (anonymous) referees for helpful comments.

References

[1] Ayupov Sh.A., Omirov B.A., On Leibniz algebras, in Algebra and Operator Theory (Tashkent, 1997), Kluwer Acad. Publ., Dordrecht, 1998, 1–12.

[2] Barnes D.W., On Engel's theorem for Leibniz algebras, *Comm. Algebra* **40** (2012), 1388–1389, arXiv:1012.0608.

[3] Barnes D.W., On Levi's theorem for Leibniz algebras, *Bull. Aust. Math. Soc.* **86** (2012), 184–185, arXiv:1109.1060.

[4] Lepowsky J., Li H., Introduction to vertex operator algebras and their representations, *Progress in Mathematics*, Vol. 227, Birkhäuser Boston Inc., Boston, MA, 2004.

[5] Loday J.L., Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.* **39** (1993), 269–293.

[6] Loday J.L., Cyclic homology, *Grundlehren der Mathematischen Wissenschaften*, Vol. 301, 2nd ed., Springer-Verlag, Berlin, 1998.

[7] Mason G., Yamskulna G., On the structure of $\mathbb{N}$-graded vertex operator algebras, arXiv:1310.0545.