

Multi-Component Integrable Systems and Invariant Curve Flows in Certain Geometries^{*}

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Abstract. In this paper, multi-component generalizations to the Camassa–Holm equation, the modified Camassa–Holm equation with cubic nonlinearity are introduced. Geometric formulations to the dual version of the Schrödinger equation, the complex Camassa–Holm equation and the multi-component modified Camassa–Holm equation are provided. It is shown that these equations arise from non-stretching invariant curve flows respectively in the three-dimensional Euclidean geometry, the two-dimensional Möbius sphere and n -dimensional sphere $S^n(1)$. Integrability to these systems is also studied.

Key words: invariant curve flow; integrable system; Euclidean geometry; Möbius sphere; dual Schrödinger equation; multi-component modified Camassa–Holm equation

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1 Introduction

Integrable systems solved by the inverse scattering method usually arise from shallow water wave, physics, optical communication and applied sciences etc. Integrable systems have many interesting properties, such as Lax-pair, infinite number of conservation laws and Lie–Bäcklund symmetries, multi-solitons, Bäcklund transformations and bi-Hamiltonian structure etc. [1, 44], which are helpful to explore other properties of integrable systems [1, 44, 56].

It is of great interest to study geometric aspects of integrable systems. So far, very few integrable systems were found to have geometric formulations. The relationship between completely integrable systems and the finite-dimensional differential geometry of curves has been studied extensively. It turns out that some integrable systems arise from invariant curve flows in certain geometries [2–7, 9–15, 21, 23, 24, 26, 28–42, 45, 47–52, 55, 57, 59, 60]. The pioneering work on this topic was done by Hasimoto [24]. He showed that the integrable nonlinear Schrödinger equation (NLS)

$$i\phi_t + \phi_{ss} + |\phi|^2\phi = 0$$

is equivalent to the system for the curvature κ and τ of curves γ in \mathbb{R}^3

$$\kappa_t = -2\tau\kappa_s - \kappa\tau_s, \quad \tau_t = \frac{\kappa_{sss}}{\kappa} - \frac{\kappa_s\kappa_{ss}}{\kappa^2} - 2\tau\tau_s + \kappa\kappa_s \quad (1)$$

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via the so-called Hasimoto transformation $\phi = \kappa \exp(i \int^s \tau(t, z) dz)$. Indeed, the system (1) is equivalent to the vortex filament equation

$$\gamma_t = \gamma_s \times \gamma_{ss} = \kappa \mathbf{b}, \quad (2)$$

where \mathbf{b} is the binormal vector of γ . Marí Beffa, Sanders and Wang [39, 51] noticed that Hasimoto transformation is a gauge transformation relating the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ to the parallel frame $\{\mathbf{t}^1, \mathbf{n}^1, \mathbf{b}^1\}$. It is also a Poisson map which takes Hamiltonian structure of the NLS equation to that of the vortex filament flow [29]. The Hasimoto transformation has been generalized in [51] to the Riemannian manifold with constant curvature, which is used to obtain the corresponding integrable equations associated with the invariant non-stretching curve flows. The parallel frames and other kinds of frames are also used to derive bi-Hamiltonian operators and associated hierarchies of multi-component soliton equations from non-stretching curve flows on Lie group manifolds [3, 4, 31, 39]. The KdV equation, the modified KdV equation, the Sawada–Kotera equation and the Kaup–Kuperschmidt equation were shown to arise from the invariant curve flows respectively in centro-equi affine geometry [7, 9, 48], Euclidean geometry [21], special affine geometry [11, 35] and projective geometries [11, 30, 41].

The integrable systems with non-smooth solitary waves have drawn much attention in the last two decades because of their remarkable properties. The celebrated Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m + au_x = 0, \quad m = u - u_{xx}, \quad (3)$$

was proposed as a model for the unidirectional propagation of the shallow water waves over a flat bottom, with $u(x, t)$ representing the water’s free surface in non-dimensional variables [8]. It was also found using the method of recursion operators by Fokas and Fuchssteiner [19] as a bi-Hamiltonian equation with an infinite number of conserved functionals. Geometrically, the Camassa–Holm equation arises from a non-stretching invariant planar curve flow in the centro-equi affine geometry [9], and the periodic CH equation (3) describes geodesic flows on $\text{diff}(\mathbb{S}^1 \times \mathbb{R})$ with respect to right-invariant Sobolev H^1 metric for $a = 0$ [13, 14, 28] and Bott–Virasoro algebra for $a \neq 0$ [40]. A dual version of the Ito system is the two-component Camassa–Holm equation [46], the periodic two-component CH equation also describe geodesic flows on an extended Bott–Virasoro algebra [22].

It is remarked that all nonlinear terms in the CH equation are quadratic. In contrast to the integrable modified KdV equation with a cubic nonlinearity, it is of great interest to find integrable CH-type equations with cubic or higher-order nonlinearity and non-smooth solitary waves. To the best of our knowledge, two scalar integrable CH-type equations with cubic nonlinearity have been discovered. The first equation is [18, 46, 49]

$$m_t + \frac{1}{2} \left((u^2 + \delta u_x^2) m \right)_x = 0, \quad m = u + \delta u_{xx}, \quad (4)$$

where $\delta = \pm 1$, and the second one is the so-called Novikov equation [25, 43]

$$m_t + 3uu_x m + u^2 m_x = 0, \quad m = u - u_{xx},$$

which are completely integrable, and admit peaked solitons. Recently, systems of CH-type equations with cubic nonlinearity were also obtained [20, 55].

The CH equation can also be derived by the tri-Hamiltonian duality approach basing on bi-Hamiltonian structure of the KdV equation. Other examples of dual integrable systems obtained using the method of tri-Hamiltonian duality can be found in [18, 46]. Nonlinear dual integrable systems, such as the CH equation and the modified CH equations, are endowed with

nonlinear dispersion, which in most cases, enables these systems to support non-smooth soliton-like structures. It was remarked in [23] that the modified CH equation (4) can be regarded as a Euclidean-invariant version of the CH equation (3), just as the modified KdV equation is a Euclidean-invariant counterpart to the KdV equation from the viewpoint of curve flows in Klein geometries [9, 10, 21, 48].

The aim of this paper is to provide geometric formulations to multi-component integrable systems admitting non-smooth solitons. We shall show that several multi-component integrable systems with non-smooth solitons, such as a dual version of the Schrödinger equation [17, 46], the complex CH equation and multi-component modified CH equations arise from the invariant curve flows respectively in three-dimensional Euclidean geometry, Möbius sphere and the n -dimensional unit sphere $\mathbb{S}^n(1)$. To obtain integrable systems relating to these geometric flows, we shall use the scale limit technique. The outline of this paper is as follows. In Section 2, a non-stretching invariant binormal curve flow in \mathbb{R}^3 is introduced and studied. Making use of the system for curvature and torsion corresponding to this flow, we obtain a novel integrable Schrödinger equation by a scale limit approach, which is completely integrable system and can be obtained by the so-called tri-Hamiltonian duality approach [18, 46]. In Section 3, we give a brief discussion on Möbius 2-sphere $PO(3, 1)/H$ and the n -dimensional sphere $SO(n+1)/SO(n)$, the Cartan structure equations for curves in both geometries are reviewed, which will be used in subsequent sections to study curve flows in both geometries. In Section 4, we consider the non-stretching curve flows in Möbius 2-sphere. It is shown that the complex Camassa–Holm equation and complex Hunter–Saxton equation describe the non-stretching curve flows in Möbius 2-sphere. The bi-Hamiltonian structure for the complex Camassa–Holm equation is obtained. In Section 5, we study non-stretching curve flows in the n -dimensional sphere $\mathbb{S}^n(1)$. Interestingly, we find that a multi-component modified CH equation (a multi-component generalization of the modified Camassa–Holm equation) is equivalent to a non-stretching curve flow in $\mathbb{S}^n(1)$. Integrability of the system is identified. Finally, Section 6 contains concluding remarks on this work.

2 An integrable nonlinear Schrödinger equation

We consider the flows of space curves in \mathbb{R}^3 , given by

$$\gamma_t = U\mathbf{n} + V\mathbf{b} + W\mathbf{t}, \quad (5)$$

where \mathbf{t} , \mathbf{n} and \mathbf{b} denote the tangent, normal and binormal vectors of the curves, respectively. The velocities U , V and W depend on the curvature and torsion as well as their derivatives with respect to arc-length parameter s . The arc-length parameter s is defined implicitly by $ds = hdp$, $h = |\gamma'(p)|$, where p is a free parameter and is independent of time. We denote by κ and τ the curvature and torsion of the curves, respectively. Governed by the flow (5), time evolutions of those geometric invariants are given by [24, 42]

$$\begin{aligned} \dot{\mathbf{t}} &= \left(\frac{\partial U}{\partial s} - \tau V + \kappa W \right) \mathbf{n} + \left(\frac{\partial V}{\partial s} + \tau U \right) \mathbf{b}, \\ \dot{\mathbf{n}} &= - \left(\frac{\partial U}{\partial s} - \tau V + \kappa W \right) \mathbf{t} + \left[\frac{1}{\kappa} \frac{\partial}{\partial s} \left(\frac{\partial V}{\partial s} + \tau U \right) + \frac{\tau}{\kappa} \left(\frac{\partial U}{\partial s} - \tau V + \kappa W \right) \right] \mathbf{b}, \\ \dot{\mathbf{b}} &= - \left(\frac{\partial V}{\partial s} + \tau U \right) \mathbf{t} - \left[\frac{1}{\kappa} \frac{\partial}{\partial s} \left(\frac{\partial V}{\partial s} + \tau U \right) + \frac{\tau}{\kappa} \left(\frac{\partial U}{\partial s} - \tau V + \kappa W \right) \right] \mathbf{n}, \\ \dot{h} &= 2h \left(\frac{\partial W}{\partial s} - \kappa U \right) \end{aligned} \quad (6)$$

and

$$\begin{aligned}\frac{\partial \tau}{\partial t} &= \frac{\partial}{\partial s} \left[\frac{1}{\kappa} \frac{\partial}{\partial s} \left(\frac{\partial V}{\partial s} + \tau U \right) + \frac{\tau}{\kappa} \left(\frac{\partial U}{\partial s} - \tau V \right) + \tau \int^s \kappa U ds' \right] + \kappa \tau U + \kappa \frac{\partial V}{\partial s}, \\ \frac{\partial \kappa}{\partial t} &= \frac{\partial^2 U}{\partial s^2} + (\kappa^2 - \tau^2)U + \frac{\partial \kappa}{\partial s} \int^s \kappa U ds' - 2\tau \frac{\partial V}{\partial s} - \frac{\partial \tau}{\partial s} V.\end{aligned}\quad (7)$$

Assuming that the flow is intrinsic, namely the arc-length does not depend on time, it implies from (6) that

$$W_s = \kappa U. \quad (8)$$

In terms of (7), one finds that the complex function

$$\phi = \kappa \eta, \quad \eta = \exp \left[i \int^s \tau(s', t) ds' \right]$$

satisfies the equation [24, 42]

$$\begin{aligned}\phi_t &= \left(\frac{\partial^2}{\partial s^2} + |\phi|^2 + i\phi \int^s ds' \tau \bar{\phi} + \frac{\partial \phi}{\partial s} \int^s ds' \bar{\phi} \right) (U\eta) \\ &\quad + \left(i \frac{\partial^2}{\partial s^2} + i|\phi|^2 + \phi \int^s ds' \tau \bar{\phi} - i\phi \int^s ds' \frac{\partial \bar{\phi}}{\partial s'} \right) (V\eta),\end{aligned}\quad (9)$$

where $\bar{\phi}$ denotes the complex conjugate of ϕ .

Let $U = 0$, $V = \kappa$, where κ is a real function, then (8) implies that $W = C_1$, where C_1 is a constant. Setting $C_1 = 0$, we derive from (9) the celebrated Schrödinger equation

$$i\phi_t + \phi_{ss} + \frac{1}{2}|\phi|^2\phi = 0. \quad (10)$$

Let $U = -\kappa_s$, $V = -\kappa\tau$, then $W = -\frac{1}{2}\kappa^2 + C_2$, where C_2 is a constant. Letting $C_2 = 0$, we find that ϕ satisfies the mKdV system

$$\phi_t + \phi_{sss} + \frac{3}{2}|\phi|^2\phi_s = 0.$$

In the following, we shall consider the case $U = W = 0$. Denote $\theta(s, t) = \int^s \tau(s', t) ds'$, $g = V\eta$. It follows from (9) that ϕ satisfies the equation

$$i\phi_t + g_{ss} + |\phi|^2g - \phi \int^s g(\cos \theta - i \sin \theta) k_{s'} ds' = 0. \quad (11)$$

Set $\tilde{u} = \kappa \cos \theta$, $\tilde{v} = \kappa \sin \theta$, $g = g_1 + ig_2$, then the equation (11) is separated to two equations

$$\begin{aligned}\tilde{u}_t &= -g_{2,ss} - \tilde{v} \partial_s^{-1} [\kappa (g_1 \cos \theta + g_2 \sin \theta)_s], \\ \tilde{v}_t &= g_{1,ss} + \tilde{u} \partial_s^{-1} [\kappa (g_1 \cos \theta + g_2 \sin \theta)_s].\end{aligned}$$

Furthermore, letting $\tilde{u} = u + v_s$, $\tilde{v} = v - u_s$, and choosing the binormal velocity V to be $V = \partial_s^{-1} [(u^2 + v^2)_s / \kappa]$, we find that u and v satisfy the system

$$\begin{aligned}(u + v_s)_t &= -g_{2,ss} - (v - u_s)(u^2 + v^2), \\ (v - u_s)_t &= g_{1,ss} + (u^2 + v^2)(u + v_s)\end{aligned}\quad (12)$$

with

$$g_1 = \frac{(u + v_s)(u^2 + v^2)_s}{(u + v_s)^2 + (v - u_s)^2}, \quad g_2 = \frac{(v - u_s)(u^2 + v^2)_s}{(u + v_s)^2 + (v - u_s)^2}.$$

Applying the scaling transformations

$$s \mapsto s, \quad t \mapsto \epsilon^2 t, \quad u \mapsto \epsilon^{-1} u, \quad v \mapsto \epsilon^{-1} v$$

to system (12) produces

$$\begin{aligned} (u + v_s)_t &= -\epsilon^2 g_{2,ss} - (v - u_s)(u^2 + v^2), \\ (v - u_s)_t &= \epsilon^2 g_{1,ss} + (u^2 + v^2)(u + v_s). \end{aligned} \quad (13)$$

Expanding u and v in powers of the small parameter ϵ

$$\begin{aligned} u(t, s) &= u_0(t, s) + \epsilon u_1(t, s) + \epsilon^2 u_2(t, s) + \dots, \\ v(t, s) &= v_0(t, s) + \epsilon v_1(t, s) + \epsilon^2 v_2(t, s) + \dots, \end{aligned}$$

and plugging them into system (13), we find that the leading order terms $u_0(t, s)$ and $v_0(t, s)$ satisfy the system

$$\begin{aligned} (u_0 + v_{0,s})_t + (u_0^2 + v_0^2)(v_0 - u_{0,s}) &= 0, \\ (v_0 - u_{0,s})_t - (u_0^2 + v_0^2)(u_0 + v_{0,s}) &= 0. \end{aligned} \quad (14)$$

Again we use the notation $\phi(t, s) = u_0(t, s) + iv_0(t, s)$, then it is inferred from (14) that $\phi(t, s)$ satisfies the equation

$$i(\phi_t - i\phi_{ts}) + |\phi|^2(\phi - i\phi_s) = 0, \quad (15)$$

which is a dual version of the Schrödinger equation (10), and can be obtained by the approach of tri-Hamiltonian duality [18, 46]. Equation (15) is formally completely integrable since it admits bi-Hamiltonian structure [46]

$$\rho_t = \mathcal{E}_1 \frac{\delta H_2}{\delta \rho} = \mathcal{E}_2 \frac{\delta H_1}{\delta \rho},$$

where $\rho = \phi - i\phi_s$, \mathcal{E}_1 and \mathcal{E}_2 defined by

$$\mathcal{E}_1 = \partial_s + i \quad \text{and} \quad \mathcal{E}_2(F) = \rho \partial_s^{-1} (\bar{\rho} F - \rho \bar{F})$$

are compatible Hamiltonian operators, while

$$H_1 = \int_{\mathbb{R}} (|\phi|^2 - i\bar{\phi}\phi_s) ds = \int_{\mathbb{R}} \rho \bar{\phi} ds, \quad H_2 = \frac{1}{2} \int_{\mathbb{R}} |\phi|^2 (|\phi|^2 - i\bar{\phi}\phi_s) ds.$$

are the corresponding Hamiltonian functionals.

3 Möbius sphere $\text{PO}(3, 1)/H$ and unit sphere $\text{SO}(n + 1)/\text{SO}(n)$

In this section, we give a brief account of Möbius 2-sphere $\text{PO}(3, 1)/H$ and unit sphere $S^n(1) = \text{SO}(n + 1)/\text{SO}(n)$. Please refer to the book [53] for the details of the two geometries.

3.1 Möbius 2-sphere

Let $(u_0, u_1, u_2, u_3) \in \mathbb{R}^4$, we define the inner product on \mathbb{R}^4 by

$$\langle x, y \rangle = x^T \Lambda_{3,1} y,$$

where $x, y \in \mathbb{R}^4$, and the matrix $\Lambda_{3,1}$ is

$$\Lambda_{3,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

A vector field $x \in \mathbb{R}^4$ is said to be light-like, if it satisfies $\langle x, x \rangle = 0$. All the light-like vector fields form a set L , which is called optical cone, defined by the equation

$$2x_0x_3 - x_1^2 - x_2^2 = 0.$$

Clearly it is homogeneous, namely for any $\lambda \in \mathbb{R}$, if $x \in L$, then $\lambda x \in L$. The projectivisation of L is said to be Möbius 2-sphere, which is isomorphic to \mathbb{S}^2 . Recall that

$$\mathrm{O}(3, 1) = \{g \in \mathrm{GL}(4, \mathbb{R}) : g^T \Lambda_{3,1} g = \Lambda_{3,1}\},$$

and the Möbius group is defined to be $\mathrm{PO}(3, 1) = \mathrm{O}(3, 1)/\pm I$. We denote

$$\begin{aligned} H &= \{h \in \mathrm{PO}(3, 1) : h[e_3] = [e_3]\} \\ &= \left\{ \begin{pmatrix} a^{-1} & 0 & 0 \\ v & A & 0 \\ b & \eta^T & a \end{pmatrix} \in \mathrm{O}(3, 1); A \in \mathrm{O}(2), a \in \mathbb{R}^+, v \in \mathbb{R}^2 \right\}, \end{aligned}$$

where $[e_3]$ denotes the equivalent class of e_3 in $P(\mathbb{R}^4)$, $[e_3] = (0, 0, 0, *)$. A straightforward computation gives

$$\eta = aA^T v, \quad b = \frac{a}{2} v^T v = \frac{a}{2} \|v\|^2.$$

It is easy to verify that the group $\mathrm{PO}(3, 1)$ acts on the Möbius sphere transitively (the group action is the usual conformal transformation).

Definition 1. The Klein pair $(\mathrm{PO}(3, 1), H)$ is called the Möbius 2-sphere.

For any $g \in \mathrm{PO}(3, 1)$, there exists a unique decomposition around the identity of the group

$$g = g_1 g_0 g_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ v & I_2 & 0 \\ \frac{1}{2}\|v\|^2 & v^T & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 & u^T & \frac{1}{2}\|u\|^2 \\ 0 & I_2 & u \\ 0 & 0 & 1 \end{pmatrix},$$

where $a \in \mathbb{R}^+$, $A \in \mathrm{O}(2)$, $u, v \in \mathbb{R}^2$, $h = g_1 g_0 \in H$, u in g_{-1} part represent a local coordinate of the point in $\mathrm{PO}(3, 1)/H$. For the corresponding Lie algebra \mathfrak{g} , there exists also a decomposition

$$\mathfrak{o}(3, 1) = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1},$$

where

$$\begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & p^T & 0 \end{pmatrix} \in \mathfrak{g}_1, \quad \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \in \mathfrak{g}_0, \quad \begin{pmatrix} 0 & q^T & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_{-1},$$

$p, q \in \mathbb{R}^2$, $\epsilon \in \mathbb{R}$, $S \in \mathrm{O}(2)$. The Lie algebra of the isotropy group H is $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$ while $\mathfrak{g}/\mathfrak{h} = \mathfrak{g}_{-1}$ is identified to the tangent space of the conformal sphere $\mathrm{PO}(3, 1)/H$.

3.2 n -dimensional sphere $\mathbb{S}^n(\mathbf{1}) = \mathbf{SO}(n+1)/\mathbf{SO}(n)$

The n -dimensional unit-sphere is also a homogeneous space $M = G/H = \mathbf{SO}(n+1)/\mathbf{SO}(n)$. The corresponding Lie algebra has the following Cartan–Killing decomposition

$$\mathfrak{so}(n+1) = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{so}(n) \oplus \mathbb{R}^n,$$

with

$$\begin{pmatrix} 0 & -p^T \\ p & 0 \end{pmatrix} \in \mathfrak{m}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \Theta \end{pmatrix} \in \mathfrak{h},$$

where $p \in \mathbb{R}^n$, $\Theta \in \mathfrak{so}(n)$, and the decomposition satisfies

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h},$$

where \mathfrak{m} is identified to the tangent space $T_x M \cong \mathbb{R}^n$ of $M = \mathbf{SO}(n+1)/\mathbf{SO}(n)$. The flat Cartan connection of principle $\mathbf{SO}(n)$ bundle $\mathbf{SO}(n+1) \rightarrow \mathbb{S}^n$ is given by the Maurer–Cartan form of Lie group $\mathbf{SO}(n+1)$. The Cartan structure equation reads as

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] = 0,$$

where $\mathfrak{so}(n+1)$ -valued one-form ω is decomposed to

$$\omega = \omega_H + \theta, \quad \omega_H \in \Lambda^1(P, \mathfrak{h}), \quad \theta \in \Lambda^1(P, \mathfrak{g}/\mathfrak{h}),$$

where $\mathfrak{g}/\mathfrak{h}$ -valued θ represents a linear coframe, \mathfrak{h} -valued ω_H represents a linear connection on \mathbb{S}^n . The corresponding Cartan structure equation is separated to

$$\mathfrak{J} \equiv d\theta + \frac{1}{2}[\omega_H, \theta] + \frac{1}{2}[\theta, \omega_H] = 0$$

and

$$\mathfrak{R} \equiv d\omega_H + \frac{1}{2}[\omega_H, \omega_H] = -\frac{1}{2}[\theta, \theta],$$

where \mathfrak{J} and \mathfrak{R} are called torsion and curvature forms, respectively.

4 Curve flows in $\mathbf{PO}(3,1)/H$ and the complex CH equation

For the Möbius geometry $\mathbf{PO}(3,1)/H$, its Cartan connection takes values on $\mathfrak{g} = \mathfrak{o}(3,1)$, with the form

$$\omega = \begin{pmatrix} -\epsilon & \xi^T & 0 \\ \eta & \Theta & \xi \\ 0 & \eta^T & \epsilon \end{pmatrix},$$

and the Cartan structure equation reads as

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Let's consider the invariant curve flows for curves $\gamma(x, t) = (u_1(x, t), u_2(x, t))$ on the conformal sphere $M^2 = \mathbf{PO}(3,1)/H$, where x denotes the parameter of the curves, t is the time variable, u_1 and u_2 denote the local coordinates on M^2 . Let $\gamma_t = \gamma_* \frac{\partial}{\partial t}$ be the evolutionary vector field of

the curves, $\gamma_x = \gamma_* \frac{\partial}{\partial x}$ denotes the tangent vector of the curves. Assuming that the curve flow is intrinsic, namely the parameter x for the curve does not depend on time t , we have

$$[\gamma_x, \gamma_t] = 0.$$

It was shown in [16, 36] that there exists a conformally equivariant moving frame (the Frenet frame) $\rho = \rho(x, t) \in \text{PO}(3, 1)$ along the curve $\gamma(x, t) \subset M^2$. Let D_x and D_t denote respectively the vector field $\frac{d}{dx}$ and $\frac{d}{dt}$ along the curves ρ in $\text{PO}(3, 1)$, then the Frenet formulae for the conformally parametric curves is

$$\rho_x = \rho \hat{\omega}(D_x),$$

with

$$\hat{\omega}(D_x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k_1 & 0 & 0 & 1 \\ k_2 & 0 & 0 & 0 \\ 0 & k_1 & k_2 & 0 \end{pmatrix}, \quad (16)$$

where k_1 and k_2 are the conformally differential invariants for the curve $\gamma(x, t)$. The time evolution for the frame $\rho(x, t)$ can be written as

$$\rho_t = \rho \hat{\omega}(D_t),$$

where

$$\hat{\omega}(D_t) = \begin{pmatrix} -\epsilon & h_1 & h_2 & 0 \\ f_1 & 0 & -\alpha & h_1 \\ f_2 & \alpha & 0 & h_2 \\ 0 & f_1 & f_2 & \epsilon \end{pmatrix}, \quad (17)$$

and $\epsilon, \alpha, f_i, h_i$ ($i = 1, 2$) are some conformal differential invariants related to k_1 and k_2 , to be determined. By the Cartan structure equation, one gets

$$\Omega(D_x, D_t) = \frac{d}{dt} \hat{\omega}(D_x) - \frac{d}{dx} \hat{\omega}(D_t) - [\hat{\omega}(D_x), \hat{\omega}(D_t)] = 0. \quad (18)$$

Plugging (16) and (17) into (18) results in the following equations

$$\epsilon = -h_{1,x}, \quad \alpha = h_{2,x}, \quad (19)$$

$$f_1 = \epsilon_x + k_1 h_1 + k_2 h_2, \quad (20)$$

$$f_2 = \alpha_x + k_2 h_1 - k_1 h_2, \quad (21)$$

$$k_{1,t} = f_{1,x} - k_1 \epsilon + \alpha k_2, \quad (22)$$

$$k_{2,t} = f_{2,x} - \epsilon k_2 - \alpha k_1, \quad (23)$$

where (19) is the torsion part (i.e., the \mathfrak{g}_{-1} part) of the Cartan structure equation (18), which can be written as

$$\begin{pmatrix} \epsilon \\ \alpha \end{pmatrix} = \begin{pmatrix} -\partial_x \\ \partial_x \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \equiv J_1 \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Inserting (19) into (20) and (21) gives

$$f_1 = -h_{1,xx} + k_1 h_1 + k_2 h_2, \quad f_2 = h_{2,xx} + k_2 h_1 - k_1 h_2. \quad (24)$$

Substituting (19) and (24) into (22) and (23), we obtain the evolution equations for the curvatures k_1 and k_2 [32, 36]

$$\begin{aligned} k_{1,t} &= -h_{1,xxx} + 2k_1h_{1,x} + k_{1,x}h_1 + 2k_2h_{2,x} + k_{2,x}h_2, \\ k_{2,t} &= h_{2,xxx} - 2k_1h_{2,x} - k_{1,x}h_2 + 2k_2h_{1,x} + k_{2,x}h_1, \end{aligned}$$

which is equivalent to

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \begin{pmatrix} -\partial^3 + k_1\partial + \partial k_1 & k_2\partial + \partial k_2 \\ k_2\partial + \partial k_2 & \partial^3 - k_1\partial - \partial k_1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \equiv J_2 \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \quad (25)$$

The following cases are considered.

Case 1. Setting $k_1 = 1/2 + m \equiv 1/2 + u - u_{xx}$, $k_2 = n \equiv v - v_{xx}$, $h_1 = 1 - u$ and $h_2 = v$ in (25), we obtain the new two-component CH equation

$$\begin{aligned} m_t + 2u_xm + um_x - 2v_xn - vn_x &= 0, \\ n_t + 2u_xn + un_x + 2v_xm + vm_x &= 0. \end{aligned} \quad (26)$$

The above system admits the following Lax-pair

$$\phi_x = U\phi, \quad \phi_t = V\phi, \quad (27)$$

with

$$U = \begin{pmatrix} 0 & \frac{1}{2} + \lambda m & \lambda n & 0 \\ 1 & 0 & 0 & \frac{1}{2} + \lambda m \\ 0 & 0 & 0 & \lambda n \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -u_x & f_1 & f_2 & 0 \\ \lambda^{-1} - u & 0 & v_x & f_1 \\ v & -v_x & 0 & f_2 \\ 0 & \lambda^{-1} - u & v & u_x \end{pmatrix},$$

where $f_1 = \frac{1}{2}(\lambda^{-1} + u) + \lambda(vn - um)$, $f_2 = \frac{1}{2}v - \lambda(un + vm)$.

Case 2. Setting $k_1 = m = -u_{xx}$, $k_2 = n = -v_{xx}$, $h_1 = 1 - u$, $h_2 = v$, we arrive at the complex Hunter–Saxton equation

$$\begin{aligned} m_t + 2u_xm + um_x - 2v_xn - vn_x &= 0, \\ n_t + 2u_xn + un_x + 2v_xm + vm_x &= 0, \end{aligned}$$

which admits the Lax-pair (27) with

$$U = \begin{pmatrix} 0 & -\lambda u_{xx} & -\lambda v_{xx} & 0 \\ 1 & 0 & 0 & -\lambda u_{xx} \\ 0 & 0 & 0 & -\lambda v_{xx} \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} -u_x & \lambda(vn - um) & -\lambda(un + vm) & 0 \\ \lambda^{-1} - u & 0 & v_x & \lambda(vn - um) \\ v & -v_x & 0 & -\lambda(un + vm) \\ 0 & \lambda^{-1} - u & v & u_x \end{pmatrix}.$$

Case 3. In (25), letting h_1 and h_2 satisfy

$$k_1 = \frac{1}{2} \frac{h_1^2 - h_2^2}{(h_1^2 + h_2^2)^2}, \quad k_2 = -\frac{h_1 h_2}{(h_1^2 + h_2^2)^2},$$

we get the two-component Harry–Dym equation

$$\begin{aligned} h_{1,t} &= (h_1^3 - 3h_1h_2^2)h_{1,xxx} + (h_2^3 - 3h_1^2h_2)h_{2,xxx}, \\ h_{2,t} &= (h_2^3 - 3h_1^2h_2)h_{1,xxx} - (h_1^3 - 3h_1h_2^2)h_{2,xxx}. \end{aligned}$$

Remark 1. Marí Beffa [32, 36] showed that the complex KdV equation arises from the invariant curve motion in Möbius 2-sphere. Indeed, taking $h_1 = -k_1$, $h_2 = -k_2$ in (25) yields the complex KdV equation

$$k_{1,t} = k_{1,xxx} - 3k_1k_{1,x} + 3k_2k_{2,x}, \quad k_{2,t} = k_{2,xxx} - 3k_1k_{2,x} - 3k_2k_{1,x}.$$

Its Hamiltonian structure J_2 , see (25), was originally derived in [32]. One can see that the bi-Hamiltonian structure J_1 and J_2 of the complex KdV equation comes from the Cartan structure equation for the conformal invariant curve flow. According to the decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1},$$

the Cartan curvature form Ω is decomposed to $\Omega = \Omega_1 + \Omega_0 + \Omega_{-1}$, where J_1 comes from torsion part of the structure equation, i.e., the Ω_{-1} part, and J_2 arises from the Ω_1 part. In the sequel, we will show that the complex CH equation admits a bi-Hamiltonian structure $\hat{\mathcal{J}}_1$ and $\hat{\mathcal{J}}_2$. It turns out that the complex CH equation is a dual version of the complex KdV equation (in the sense of [46]).

It is well-known that the CH equation is a bi-Hamiltonian system [8]

$$m_t = \mathcal{J} \frac{\delta H_1}{\delta m} = \mathcal{D} \frac{\delta H_2}{\delta m},$$

where $\mathcal{J} = -(m\partial + \partial m)$ and $\mathcal{D} = -(\partial - \partial^3)$ are Hamiltonian operators, $H_1 = -\frac{1}{2} \int um dx$ and $H_2 = -\frac{1}{2} \int u(u^2 + u_x^2) dx$ are the corresponding Hamiltonian functionals. As for the CH equation, the complex CH equation can be obtained by the approach of tri-Hamiltonian duality [46]. Indeed, we have the following result.

Theorem 1. *The complex CH equation (26) is a bi-Hamiltonian system, which can be written as*

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \hat{\mathcal{J}}_1 \begin{pmatrix} \frac{\delta \hat{H}_2}{\delta m} \\ \frac{\delta \hat{H}_2}{\delta n} \end{pmatrix} = \hat{\mathcal{J}}_2 \begin{pmatrix} \frac{\delta \hat{H}_1}{\delta m} \\ \frac{\delta \hat{H}_1}{\delta n} \end{pmatrix}$$

with

$$\hat{\mathcal{J}}_1 = \begin{pmatrix} \partial^3 - \partial & 0 \\ 0 & \partial - \partial^3 \end{pmatrix}, \quad \hat{\mathcal{J}}_2 = \begin{pmatrix} m\partial + \partial m & n\partial + \partial n \\ n\partial + \partial n & -m\partial - \partial m \end{pmatrix}$$

and

$$\hat{H}_1 = \frac{1}{2} \int (vn - um) dx, \quad \hat{H}_2 = \frac{1}{2} \int [u(u^2 + u_x^2) - u(3v^2 + v_x^2) - 2vu_x v_x] dx.$$

Proof. Clearly, $\hat{\mathcal{J}}_1$ and $\hat{\mathcal{J}}_2$ are skew symmetric. To prove they are Hamiltonian operators, it suffices to prove that the Poisson bracket defined by $\hat{\mathcal{J}}_2$ satisfies the Jacobi identity.

The bi-vector associated with $\hat{\mathcal{J}}_2$ is defined by [44]

$$\Theta_{\hat{\mathcal{J}}_2} = \frac{1}{2} \int_{\mathbb{R}} (\vartheta \wedge \hat{\mathcal{J}}_2 \vartheta) dx = \int_{\mathbb{R}} (m\theta \wedge \theta_x + n\theta \wedge \zeta_x + n\zeta \wedge \theta_x - m\zeta \wedge \zeta_x),$$

where $\vartheta = (\theta, \zeta)$, θ and ζ denote the basic unit vectors corresponding to m and n , respectively, the notation \wedge denotes the usual inner product between ϑ and $\hat{\mathcal{J}}_2 \vartheta$. It suffices to show that the Schouten bracket vanishes, namely $[\hat{\mathcal{J}}_2, \hat{\mathcal{J}}_2] = 0$. In terms of

$$\text{pr } v_{\hat{\mathcal{J}}_2 \vartheta}(m) = 2m\theta_x + m_x\theta + 2n\zeta_x + n_x\zeta, \quad \text{pr } v_{\hat{\mathcal{J}}_2 \vartheta}(n) = 2n\theta_x + n_x\theta - 2m\zeta_x - m_x\zeta,$$

a direct computation shows

$$\begin{aligned} [\hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2] &= \text{pr } v_{\hat{\mathcal{J}}_2 v}(\Theta_{\hat{\mathcal{J}}_2}) \\ &= \int (2n\zeta_x \wedge \theta \wedge \theta_x + n_x \zeta \wedge \theta \wedge \theta_x + 2n\theta_x \wedge \theta \wedge \zeta_x - m_x \zeta \wedge \theta \wedge \zeta_x \\ &\quad + n_x \theta \wedge \zeta \wedge \theta_x - 2m\zeta_x \wedge \zeta \wedge \theta_x - 2m\theta_x \wedge \zeta \wedge \zeta_x - m_x \theta \wedge \zeta \wedge \theta_x) dx = 0, \end{aligned}$$

where the skew-symmetric property for the wedge product is used.

Next, we prove that the Hamiltonian operators $\hat{\mathcal{J}}_1$ and $\hat{\mathcal{J}}_2$ are compatible, i.e.

$$[\hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2] + [\hat{\mathcal{J}}_2, \hat{\mathcal{J}}_1] = \text{pr } v_{\hat{\mathcal{J}}_1 \vartheta}(\Theta_{\hat{\mathcal{J}}_2}) + \text{pr } v_{\hat{\mathcal{J}}_2 \vartheta}(\Theta_{\hat{\mathcal{J}}_1}) = 0.$$

Note that

$$\text{pr } v_{\hat{\mathcal{J}}_2 \vartheta}(\Theta_{\hat{\mathcal{J}}_1}) = 0, \quad \text{pr } v_{\hat{\mathcal{J}}_1 \vartheta}(m) = \theta_{xxx} - \theta_x, \quad \text{pr } v_{\hat{\mathcal{J}}_1 \vartheta}(n) = \zeta_x - \zeta_{xxx}.$$

Through integration by parts, we get

$$\begin{aligned} [\hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2] + [\hat{\mathcal{J}}_2, \hat{\mathcal{J}}_1] &= \text{pr } v_{\hat{\mathcal{J}}_1 \vartheta}(\Theta_{\hat{\mathcal{J}}_2}) + \text{pr } v_{\hat{\mathcal{J}}_2 \vartheta}(\Theta_{\hat{\mathcal{J}}_1}) \\ &= \int (\theta_{xxx} \wedge \theta \wedge \theta_x - \zeta_{xxx} \wedge \theta \wedge \zeta_x + \zeta_x \wedge \zeta \wedge \theta_x - \zeta_{xxx} \wedge \zeta \wedge \theta_x \\ &\quad - \theta_{xxx} \wedge \zeta \wedge \zeta_x + \theta_x \wedge \zeta \wedge \zeta_x) dx \\ &= \int (-\zeta_{xxx} \wedge \theta \wedge \zeta_x - \zeta_{xxx} \wedge \zeta \wedge \theta_x - \theta_{xxx} \wedge \zeta \wedge \zeta_x) dx \\ &= \int (-\zeta_{xxx} \wedge \theta \wedge \zeta_x - \zeta_{xxx} \wedge \zeta \wedge \theta_x - \theta_x \wedge \zeta_x \wedge \zeta_{xx} - \theta_x \wedge \zeta \wedge \zeta_{xxx}) dx \\ &= \int (-\zeta_{xxx} \wedge \theta \wedge \zeta_x - \theta_x \wedge \zeta_x \wedge \zeta_{xx}) dx = 0. \end{aligned}$$

Thus $\hat{\mathcal{J}}_1$ and $\hat{\mathcal{J}}_2$ are a Hamiltonian-pair. Let's write the complex CH equation (26) as follows

$$\begin{pmatrix} m \\ n \end{pmatrix}_t = \hat{\mathcal{J}}_2 \begin{pmatrix} \delta \hat{H}_1 \\ \delta m \\ \delta \hat{H}_1 \\ \delta n \end{pmatrix}.$$

It is easy to get

$$\frac{\delta \hat{H}_1}{\delta m} = -u, \quad \frac{\delta \hat{H}_1}{\delta n} = v.$$

Thus

$$\frac{\delta \hat{H}_1}{\delta u} = -m, \quad \frac{\delta \hat{H}_1}{\delta v} = n.$$

Hence we deduce that

$$\hat{H}_1 = \frac{1}{2} \int (vn - um) dx.$$

To compute \hat{H}_2 , we write (26) as

$$\begin{aligned} m_t &= -\partial \left(\frac{3}{2} u^2 - \frac{1}{2} u_x^2 - uu_{xx} - \frac{3}{2} v^2 + \frac{1}{2} v_x^2 + vv_{xx} \right), \\ n_t &= \partial (uv_{xx} + vu_{xx} + u_x v_x - 3uv). \end{aligned}$$

Since \hat{H}_2 satisfies

$$m_t = -\partial(1 - \partial^2) \frac{\delta \hat{H}_2}{\delta m} = -\partial \frac{\delta \hat{H}_2}{\delta u}, \quad n_t = \partial(1 - \partial^2) \frac{\delta \hat{H}_2}{\delta n} = \partial \frac{\delta \hat{H}_2}{\delta v}.$$

It follows that

$$\begin{aligned} \frac{\delta \hat{H}_2}{\delta u} &= \frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx} - \frac{3}{2}v^2 + \frac{1}{2}v_x^2 + vv_{xx}, \\ \frac{\delta \hat{H}_2}{\delta v} &= uv_{xx} + vu_{xx} + u_x v_x - 3uv. \end{aligned}$$

Note that

$$\begin{aligned} \frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx} &= \frac{\delta}{\delta u} \frac{1}{2} \int (u^3 + uu_x^2) dx, \\ \frac{\delta}{\delta u} \int uv^2 dx &= v^2, \\ \frac{\delta}{\delta v} \int uv^2 dx &= 2uv, \\ \frac{\delta}{\delta u} \int \left(-\frac{1}{2}uv_x^2 - u_x vv_x \right) dx &= \frac{1}{2}v_x^2 + vv_{xx}. \end{aligned}$$

Hence we arrive at

$$\hat{H}_2 = \frac{1}{2} \int (u^3 + uu_x^2 - 3uv^2 - uv_x^2 - 2u_x vv_x) dx. \quad \blacksquare$$

5 Curve flows on $\mathbb{S}^n(1)$ and multi-component modified CH equations

Assume that $\gamma(x, t)$ is a curve flow on unit sphere $\mathbb{S}^n(1) = \text{SO}(n+1)/\text{SO}(n)$, which satisfies $\|\gamma\| = 1$, where x is the invariant arc-length parameter, t is the time. The natural frame of the curve $\gamma \in \mathbb{S}^n(1)$ is $\{e_1 = \gamma_x, e_2, \dots, e_n\}$. Let $\rho = (e_0 = \gamma, e_1, \dots, e_n) \in \text{SO}(n+1)$ be the lift from $\mathbb{S}^n(1)$ to bundle space $\text{SO}(n+1)$, and D_x and D_t denote respectively the tangent and evolutionary vector field. It follows that

$$\rho_x = \rho \hat{\omega}(D_x), \quad (28)$$

where $\hat{\omega}$ is the Cartan connection

$$\hat{\omega}(D_x) = \begin{pmatrix} 0 & -1 & \vec{0}^T \\ 1 & 0 & -\vec{k}^T \\ \vec{0} & \vec{k} & O \end{pmatrix}, \quad O \in \mathfrak{so}(n-1),$$

(the natural frame formulae (28) for curves on the sphere comes out from the Frenet formulae [15] through the Hasimoto transformation). Here $\vec{k} = (k_1, k_2, \dots, k_{n-1})$ is the natural curvature vector of γ .

Assume that the curve flow is governed by

$$\gamma_t = f e_1 + h_1 e_2 + h_2 e_3 + \dots + h_{n-1} e_n,$$

where the tangent velocity f and normal velocities h_i ($i = 1, 2, \dots, n-1$) depend on the curvatures and their derivatives with respect to arc-length x .

The induced time evolution for the frame is

$$\rho_t = \rho \hat{\omega}(D_t),$$

with

$$\hat{\omega}(D_t) = \begin{pmatrix} 0 & -f & -\vec{h}^T \\ f & 0 & -\vec{\xi}^T \\ \vec{h} & \vec{\xi} & \Theta \end{pmatrix}, \quad \Theta \in \mathfrak{so}(n-1),$$

where $\vec{h}, \vec{\xi} \in \mathbb{R}^{n-1}$, $\vec{\xi}$ is a unknown vector, which will be determined later by the structure equations.

First, we assume that the flow is intrinsic, namely, the distribution $\{D_x, D_t\}$ satisfies $[D_x, D_t] = 0$ so that the integral submanifold is a smooth two-dimensional surface on Lie group $\text{SO}(n+1)$. Making use of the Cartan structure equation

$$\frac{d}{dt} \hat{\omega}(D_x) - \frac{d}{dx} \hat{\omega}(D_t) - [\hat{\omega}(D_x), \hat{\omega}(D_t)] = 0,$$

one gets the following equations

$$f_x = \langle \vec{k}, \vec{h} \rangle, \tag{29}$$

$$\vec{\xi} = \vec{h}_x + f \vec{k}, \tag{30}$$

$$\Theta_x = \vec{k} \otimes \vec{\xi} - \vec{\xi} \otimes \vec{k}, \tag{31}$$

$$\vec{k}_t = \vec{\xi}_x - \Theta \vec{k} + \vec{h}, \tag{32}$$

where (29) is the arc-length preserving condition.

For convenience, the following notations are used. For any $\vec{a}, \vec{b} \in \mathbb{R}^{n-1}$, $\langle \vec{a}, \vec{b} \rangle$ denotes the usual Euclidean inner product, i.e., $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b}$, $\vec{a} \otimes \vec{b}$ denotes the tensor product, namely

$$\vec{a} \otimes \vec{b} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_{n-1} \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1} b_1 & a_{n-1} b_2 & \cdots & a_{n-1} b_{n-1} \end{pmatrix}.$$

Define $\vec{a} \wedge \vec{b} = \vec{a} \otimes \vec{b} - \vec{b} \otimes \vec{a}$. From (29) and (30), it follows that

$$f = \partial_x^{-1} \langle \vec{k}, \vec{h} \rangle, \quad \vec{\xi} = \vec{h}_x + (\partial_x^{-1} \langle \vec{k}, \vec{h} \rangle) \vec{k}. \tag{33}$$

In view of (31), we have

$$\Theta = \partial_x^{-1} (\vec{k} \wedge \vec{\xi}). \tag{34}$$

Plugging (34) and (33) into (32) leads to the equation for the curvature vector

$$\vec{k}_t = \vec{h}_{xx} + \langle \vec{k}, \vec{k} \rangle \vec{h} + (\partial_x^{-1} \langle \vec{k}, \vec{h} \rangle) \vec{k}_x + (\partial_x^{-1} (\vec{k}_x \wedge \vec{h})) \vec{k} + \vec{h}, \tag{35}$$

where the identity for vectors

$$(\vec{a} \wedge \vec{b}) \cdot \vec{c} = \langle \vec{b}, \vec{c} \rangle \vec{a} - \langle \vec{a}, \vec{c} \rangle \vec{b}$$

was used.

Analogous to the derivation for the CH equation [18] and the modified CH equation [46], we restrict our attention to the following cases.

Case 1. $\vec{h} = \vec{u}_x$, $\vec{k} = \vec{m} = \vec{u} + \vec{u}_{xx}$. In this case, the tangent velocity f is determined by

$$f = \partial_x^{-1} \langle \vec{u} + \vec{u}_{xx}, \vec{u}_x \rangle = \frac{1}{2} (|\vec{u}|^2 + |\vec{u}_x|^2) + c_0 \equiv \frac{1}{2} \tilde{Q} + c_0, \quad (36)$$

where c_0 is an integration constant. Substituting (36) with $c_0 = -1$ together with the expressions for \vec{k} and \vec{h} into (35), and noting that

$$\begin{aligned} (\partial_x^{-1}(\vec{k}_x \wedge \vec{h}))\vec{k} &= (\partial_x^{-1}(\vec{u}_{xxx} \wedge \vec{u}_x))\vec{m} = (\vec{u}_{xx} \wedge \vec{u}_x)\vec{m} \\ &= -(\vec{u} \wedge \vec{u}_x)\vec{m} - (\vec{u}_x \wedge \vec{m})\vec{m} = -(\vec{u} \wedge \vec{u}_x)\vec{m} - \langle \vec{m}, \vec{m} \rangle \vec{u}_x + \frac{1}{2} \tilde{Q}_x \vec{m}, \end{aligned}$$

we arrive at the multi-component modified CH equation

$$\vec{m}_t = \frac{1}{2} (\vec{m} \tilde{Q})_x - (\vec{u} \wedge \vec{u}_x) \vec{m}. \quad (37)$$

Thus we have established the following result.

Theorem 2. *Assume that curves $\gamma(x, t)$ on the sphere $\mathbb{S}^n(1)$ ($n \geq 1$) are governed by the flow*

$$\gamma_t = \left(\frac{1}{2} \tilde{Q} - 1 \right) e_1 + \sum_{j=1}^{n-1} u_{j,x} e_{j+1}, \quad (38)$$

where $\{e_1, e_2, \dots, e_n\}$ is the natural frame of the curve $\gamma(x, t)$, $(u_1, u_2, \dots, u_{n-1})$ is defined by the curvatures $\vec{k} = \vec{m} = \vec{u} + \vec{u}_{xx}$, $\tilde{Q} = |\vec{u}|^2 + |\vec{u}_x|^2$. Then the flow (38) is intrinsic and the curvature vector \vec{m} fulfills the equation (37).

Remark 2. In the case of $n = 2$, i.e., the case of $\mathbb{S}^2(1)$, equation (37) reduces to the scalar modified CH equation (4) with $\delta = -1$, which is completely integrable. In the case of $n = 3$, let $u_1 = u$, $u_2 = v$, $m = u + u_{xx}$, $n = v + v_{xx}$, then system (37) reduces to

$$\begin{aligned} m_t &= \frac{1}{2} [(u^2 + v^2 + u_x^2 + v_x^2)m]_x - (uv_x - vu_x)n, \\ n_t &= \frac{1}{2} [(u^2 + v^2 + u_x^2 + v_x^2)n]_x + (uv_x - vu_x)m. \end{aligned}$$

In general, the multi-component system (37) can be written as

$$m_{i,t} = \frac{1}{2} \sum_{j=1}^{n-1} [(u_j^2 + u_{j,x}^2)m_i]_x - \sum_{j=1}^{n-1} (u_i u_{j,x} - u_j u_{i,x}) m_j, \quad 1 \leq i \leq n-1, \quad m_i = u_i + u_{i,xx}.$$

Case 2. $\vec{h} = \vec{u}_x$, $\vec{k} = \vec{m} = \vec{u} - \vec{u}_{xx}$. In this case, the tangent velocity f is given by

$$f = \partial_x^{-1} \langle \vec{u} - \vec{u}_{xx}, \vec{u}_x \rangle = \frac{1}{2} \sum_{i=1}^{n-1} (u_i^2 - u_{i,x}^2) + c_1 \equiv \frac{1}{2} Q + c_1, \quad (39)$$

where c_1 is an integration constant. Substituting (39) with $c_1 = 1$ into (35) and noting that

$$(\partial_x^{-1}(\vec{k}_x \wedge \vec{h}))\vec{k} = -(\vec{u} \wedge \vec{u}_x)\vec{m} - (\vec{u}_x \wedge \vec{m})\vec{m} = -(\vec{u} \wedge \vec{u}_x)\vec{m} - \langle \vec{m}, \vec{m} \rangle \vec{u}_x + \frac{1}{2} Q_x \vec{m},$$

we obtain the multi-component modified CH equation

$$\vec{m}_t = \frac{1}{2} (\vec{m} Q)_x - (\vec{u} \wedge \vec{u}_x) \vec{m} + 2\vec{u}_x, \quad \vec{m} = \vec{u} - \vec{u}_{xx}. \quad (40)$$

Thus we have proved the following result.

Theorem 3. Assume that curves $\gamma(x, t)$ on the sphere $\mathbb{S}^n(1)$ ($n \geq 1$) are governed by the flow

$$\gamma_t = \left(\frac{1}{2}Q + 1 \right) e_1 + \sum_{j=1}^{n-1} u_{j,x} e_{j+1}, \quad (41)$$

where $\{e_1, e_2, \dots, e_n\}$ is the natural frame of the curves $\gamma(x, t)$, $(u_1, u_2, \dots, u_{n-1})$ is defined by the curvatures $\vec{k} = \vec{m} = \vec{u} - \vec{u}_{xx}$, $Q = |\vec{u}|^2 - |\vec{u}_x|^2$. Then the flow (41) is intrinsic and the curvature vector \vec{m} satisfies (40).

Case 3. $\vec{h} = \vec{u}_x$, $\vec{k} = \vec{m} = -\vec{u}_{xx}$. In this case, the tangent velocity f is determined by

$$f = \partial_x^{-1} \langle -\vec{u}_{xx}, \vec{u}_x \rangle = -\frac{1}{2} |\vec{u}_x|^2 + c_2,$$

where c_2 is an integration constant. It is inferred from (30) and (31) that

$$\vec{\xi} = \left(\frac{1}{2} |\vec{u}_x|^2 + 1 - c_2 \right) \vec{u}_{xx}, \quad \Theta_x = 0.$$

Setting $\Theta = 0$, $c_2 = 1$ and $\vec{v} = \vec{u}_x$, we arrive at the multi-component short-pulse equation

$$\vec{v}_{xt} + \frac{1}{2} (|\vec{v}|^2 \vec{v}_x)_x + \vec{v} = 0. \quad (42)$$

Hence we could prove the following result.

Theorem 4. Assume that curves $\gamma(x, t)$ on the sphere $\mathbb{S}^n(1)$ ($n \geq 1$) are governed by the flow

$$\gamma_t = \left(1 - \frac{1}{2} |\vec{u}_x|^2 \right) e_1 + \sum_{j=1}^{n-1} u_{j,x} e_{j+1}, \quad (43)$$

where $\{e_1, e_2, \dots, e_n\}$ is the natural frame of the curves $\gamma(x, t)$, $(u_1, u_2, \dots, u_{n-1})$ is defined by the curvatures $\vec{k} = \vec{m} = -\vec{u}_{xx}$. Then the flow (43) is intrinsic and the curvature vector $\vec{v} = \vec{u}_x$ satisfies (42).

The scalar equation of (42) was derived by Schäfer and Wayne [52] as a model for the propagation of ultra-short light pulses in silica optical fibers, which is also an approximation of nonlinear wave packets in dispersive media in the limit of few cycles on the ultra-short pulse scale.

It is worth noting that, analogous to the system (37), there is another version of multi-component generalization of the modified Camassa–Holm equation (4) with $\delta = 1$

$$\vec{m}_t = \frac{1}{2} (\vec{m}Q)_x - (\vec{u} \wedge \vec{u}_x) \vec{m}, \quad \vec{m} = \vec{u} - \vec{u}_{xx}, \quad (44)$$

(where $Q = |\vec{u}|^2 - |\vec{u}_x|^2$). However, in contrast to the system (37) arising from the compact Riemannian symmetric space $\mathbb{S}^n(1) = \text{SO}(n+1)/\text{SO}(n)$ with positive constant curvature, the system (44) arises from the noncompact Riemannian symmetric space with negative constant curvature, i.e., the hyperbolic space $\mathbb{H}^n = \text{SO}(n,1)/\text{SO}(n)$. The derivation to integrable curve flows is similar to that for the $\mathbb{S}^n(1)$ case. In fact, the Lie algebra structure corresponding to the hyperbolic space $\mathbb{H}^n = \text{SO}(n,1)/\text{SO}(n)$ is described by

$$\mathfrak{so}(n,1) = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{so}(n) \oplus \mathbb{R}^n,$$

with

$$\begin{pmatrix} 0 & p^T \\ p & 0 \end{pmatrix} \in \mathfrak{m}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \Theta \end{pmatrix} \in \mathfrak{h}$$

where $p \in \mathbb{R}^n$, $\Theta \in \mathfrak{so}(n)$. The Cartan connection matrices $\hat{\omega}(D_x)$ and $\hat{\omega}(D_t)$ of the natural frame are replaced with

$$\hat{\omega}(D_x) = \begin{pmatrix} 0 & 1 & \vec{0}^T \\ 1 & 0 & -\vec{k}^T \\ \vec{0} & \vec{k} & O \end{pmatrix}, \quad O \in \mathfrak{so}(n-1)$$

and

$$\hat{\omega}(D_t) = \begin{pmatrix} 0 & f & \vec{h}^T \\ f & 0 & -\vec{\xi}^T \\ \vec{h} & \vec{\xi} & \Psi \end{pmatrix}, \quad \Psi \in \mathfrak{so}(n-1),$$

for curves on $\mathbb{H}^n = \text{SO}(n, 1)/\text{SO}(n)$. Similar results can be derived for the Hyperbolic space \mathbb{H}^n . Integrability of the equations (37) and (44) are guaranteed by the following results.

Theorem 5. *The systems (44) and (37) are Lax integrable, namely they admit the following $(n+1) \times (n+1)$ Lax-pair*

$$\phi_x = U\phi, \quad \phi_t = V\phi,$$

where for (44),

$$U = \begin{pmatrix} 0 & 1 & \vec{0}^T \\ 1 & 0 & \lambda \vec{m}^T \\ \vec{0} & -\lambda \vec{m} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \frac{1}{2}Q + \lambda^{-2} & \lambda^{-1} \vec{u}_x^T \\ \frac{1}{2}Q + \lambda^{-2} & 0 & \lambda^{-1} \vec{u}^T + \frac{1}{2} \lambda Q \vec{m}^T \\ \lambda^{-1} \vec{u}_x & -\lambda^{-1} \vec{u} - \frac{1}{2} \lambda Q \vec{m} & -\vec{u} \wedge \vec{u}_x \end{pmatrix},$$

and for (37),

$$U = \begin{pmatrix} 0 & 1 & \vec{0}^T \\ -1 & 0 & \lambda \vec{m}^T \\ \vec{0} & -\lambda \vec{m} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \frac{1}{2} \tilde{Q} - \lambda^{-2} & \lambda^{-1} \vec{u}_x^T \\ -\frac{1}{2} \tilde{Q} + \lambda^{-2} & 0 & -\lambda^{-1} \vec{u}^T + \frac{1}{2} \lambda \tilde{Q} \vec{m}^T \\ -\lambda^{-1} \vec{u}_x & \lambda^{-1} \vec{u} - \frac{1}{2} \lambda \tilde{Q} \vec{m} & -\vec{u} \wedge \vec{u}_x \end{pmatrix}.$$

6 Concluding remarks

In this paper, geometrical formulations to several multi-component integrable systems are provided. These systems are regarded as multi-component generalizations of the CH equation and the modified CH equation, which can be obtained through the tri-Hamiltonian duality approach. We showed that an integrable generalization to the nonlinear Schrödinger equation arises from a non-stretching invariant curve flow in the three-dimensional Euclidean geometry. The integrable complex CH equation comes from an invariant curve flow on the Möbius 2-sphere. Furthermore, we verified that multi-component generalizations to the modified CH equation arise naturally from the curve flows in n -dimensional sphere $\mathbb{S}^n(1)$ and the hyperbolic space $\mathbb{H}^n = \text{SO}(n, 1)/\text{SO}(n)$.

In [27], Olver, Kamran and Tenenblat have established the theory for curves in affine symplectic geometry. The curve flows in four-dimensional affine symplectic geometry were studied in [58], and an integrable three-component equation with bi-Hamiltonian structure was obtained. The theory for curves in the centro-equiaffine symplectic geometry was established in [54]. It was shown that certain invariant curve flows in the centro-equiaffine symplectic geometry yield noncommutative KdV equations [47]. It is still not clear that what are the dual version of these integrable equations arising from curve flows in the affine and centro-equiaffine symplectic geometries.

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