PROPERTIES OF CHARGED PARTICLE MOTION EQUATIONS IN CROSSED FIELDS AND LARMOR'S THEOREM

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Motion equations of charged particle placed in crossed fields observed in the laboratory frame of reference and in rotating one are compared. It is shown that the motion equation in fields *(E, H)* in a plane transverse to the magnetic field in a rotating frame has the same form as the motion equation in fields of another sstrength *(E', H')* in the laboratory frame. The invariant of motion equation under rotation transformation is found. A problem that is more general than the Larmor's one is formulated and studied. There are found out the rotation frequency and the condition under which the particle motion equation in the fields *(E_1, H_1)* in the laboratory frame coincides with the motion equation in the fields *(E_2, H_2)* in the rotating frame.

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INTRODUCTION

In electrodynamics, Larmor's theorem is known, according to which "in the non-relativistic case the behavior of a system of charges all having the same e/m, performing a finite motion in a centrally–symmetric electric field $\vec{E}$ and in a weak uniform magnetic field $\vec{H}$, is equivalent to the behavior of the same system of charges in the same electric field in a coordinate system, rotating uniformly with angular velocity $\vec{\Omega} = (e/2mc)\vec{H}$ " [1, 2]. The author of the present paper has an impression that the mathematical properties of motion equations are studied more intensively by mathematicians than by physicists. In physics of non-neutral plasmas, under the conditions of which Larmor's theorem is realized, the properties of motion equations are studied and used even less.

In the present paper we consider the properties of the particle motion equations in crossed fields without assuming that the magnetic field is necessarily weak, and the motion is finite. We will find out whether the behavior of charges that move in a uniform magnetic field $\vec{H}_1$ and in a centrally–symmetric electric field $\vec{E}_1$ can be "equivalent" in the plane transverse to the magnetic field to the behavior of charges in some other fields $\vec{E}_2$, $\vec{H}_2$, observed in a rotating frame. If it is "equivalent", then under what condition and at what frequency of rotation ω_{rot} does this take place? By "equivalence" we mean the coincidence of motion equations in fields $\vec{E}_1$, $\vec{H}_1$ in the laboratory system, and in fields $\vec{E}_2$, $\vec{H}_2$ in a frame of reference rotating with frequency ω_{rot}. This problem can be considered as a generalization of the problem solved by Larmor.

We compare the particle motion equations in the laboratory system and in a rotating coordinate system (Section 2) and consider the above mentioned problem in a centrally–symmetric and a cylindrically–symmetric electric field (Section 3).

1. MOTION EQUATIONS IN LABORATORY AND IN ROTATING COORDINATE SYSTEMS

Let us consider the motion of a particle placed in a centrally symmetric electric field $\vec{E}$ and in a homogeneous magnetic field $\vec{H}$. The origin of the coordinate system O is compatible with the center of symmetry of the electric field. The axis Oz is directed along the magnetic field. In the laboratory frame, the motion equations have the form

$$\begin{cases} \ddot{x} - \omega_c \dot{y} = (e/m)E_x, \\ \ddot{y} + \omega_c \dot{x} = (e/m)E_y, \\ \ddot{z} = (e/m)E_z. \end{cases} \quad (1)$$

Here $E_x = -d\Phi_0/dx = (E/r)x$, $E_y = -d\Phi_0/dy = (E/r)y$, $E_z = -d\Phi_0/dz = (E/r)z$ – are the components of strength of a radial electric field, $\Phi_0(r)$ – is a field potential, $r = (x^2 + y^2 + z^2)^{1/2}$ – radius, $E = -d\Phi_0/dr$ – is a strength of a radial electric field. The cyclotron frequency $\omega_c = eH/mc$ can be positive or negative depending on the sign of the charge e.

We are interested in the motion of the particle mainly in the plane transverse to the magnetic field xy. Let us write the first two Eqs. (1) in complex form. We introduce a complex radius vector in the plane xy: $u = x + iy$. We multiply the second Eq. (1) by i, add to the first equation and take into account that the electric field component in the plane transverse to the axis Oz is equal to $E_\perp = E_x + iE_y = (E/r)u$. As a result, we obtain the motion equations (1) in the plane xy in complex form

$$\ddot{u} + i\omega_c \dot{u} - (e/m)(E/r)u = 0 . \quad (2)$$

The motion equation in Oz direction (1) can be presented in the form:

$$\ddot{z} - (eE/mr)z = 0 .$$

We transform the motion equation (2) to a coordinate system rotating with frequency ω_{rot} around the axis Oz. We introduce the radius vector in a rotating coordinate system $u' = x' + y'$ according to formula

$$u = u' \exp\left(i\omega_{rot} t\right). \qquad (3)$$

Substituting Eq. (3) in (2), we obtain the motion equation in a rotating system:

$$\ddot{u}' + i\omega_c' \dot{u}' - \left[eE'/(mr)\right] u' = 0. \qquad (4)$$

In the Eq. (4) the following notations are introduced:

$$\omega_c' \equiv 2\omega_{rot} + \omega_c, \qquad (5)$$

$$eE'/(mr) \equiv \omega_{rot}^2 + \omega_c \omega_{rot} + eE/(mr) \qquad (6)$$

(in Eqs. (4), (5) $\omega_c' = eH'/mc$).

Comparison of Eqs. (4) and (2) shows that the particle motion equation in fields E, H in the transverse plane x', y' in a coordinate system rotating with a frequency ω_{rot} looks like the motion equation in the fields E', H' (5), (6) in the laboratory system [3].

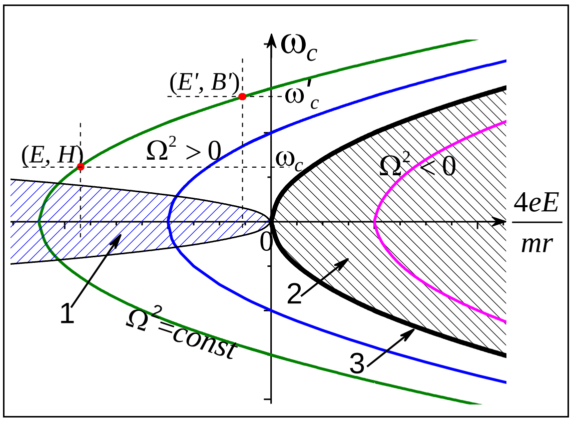
We make a few remarks to the equations (4) – (6). They are valid for an arbitrary rotation frequency ω_{rot}, for an arbitrary dependence on the radius of the electric field E, for a finite and infinite particle motion along the radius. They do not contain an approximation of weak magnetic or electric fields. The quantities E and E', H and H' can be of different signs. The combinations of the fields $eE/(mr)$ and $eE'/(mr)$ (6) differ by a constant that does not depend on the radius r. For particles with different values of the ratio (e/m), the values of E', H' differ.

Using Eqs. (5) and (6), it is not difficult to show that under the rotational transformation (3) the combination of fields $\Omega^2 \equiv \omega_c^2 - 4eE/(mr)$ remains invariant, i. e.

$$\Omega^2 \equiv \omega_c^2 - 4eE/(mr) = \omega_c'^2 - 4eE'/(mr) \equiv \Omega'^2. \qquad (7)$$

The existence of an invariant is a general property of second-order linear ordinary differential equations (see [4], part 1, §25.1; [5], §3.3.2). Strictly speaking the quantity $I = \Omega^2/4$ is called the invariant of equation (2). In the general case ($E/r \neq const$) Eqs. (2), (4) are nonlinear. Moreover, they are a system of nonlinear equations. But, as we see, the invariant also exists in this case.

When $E/r = const$ the coefficients in Eqs. (2), (4) do not depend on the radius r and the equations are linear. The quantity Ω^2 (7) also does not depend on the radius and in the plane of the parameters "electric field-magnetic field" parameters E, B and E', B' are located on the line of equal value $\Omega^2 = const$. In the axes $\left[4eE/(mr),\ \omega_c^2\right]$ isolines $\Omega^2 = const$ are straight lines. In the axes $\left[4eE/(mr),\ \omega_c\right]$ they have the form of parabolas. Their behavior is shown in Figure in the axes $\left[4eE/(mr),\ \omega_c\right]$, as well as the arrangement of points corresponding to fields E, H and quantities E', H'.



The behavior of isolines $\Omega^2 = const$ in the parameter plane $\left[4eE/(mr),\ \omega_c\right]$ when $E/r = const$. The shaded area (1) in the left part of the figure indicates the region of a weak magnetic field in which the Larmor's theorem is applicable. The shaded area (2) on the right-hand side of the figure indicates the region of infinite motion of the particles, $\Omega^2 < 0$; the line (3) is the boundary of the region $\Omega^2 = 0$. In the region $\Omega^2 > 0$, the motion of the particle is finite on radius. The locations of the points E, H and E', H' correspond to the rotation frequency $\omega_{rot} > 0$

The value Ω figures in the theory of non-neutral plasmas. For a radius-independent expression Ω^2 (7) ($E/r = const$) and a finite motion of the particle along the radius ($\Omega^2 > 0$), the quantity Ω is called the "vortex frequency" [6], the "modified" cyclotron frequency. It determines the frequency of the particle's oscillations along the radius r. Strictly speaking, the frequency Ω was introduced in problems with the cylindrical symmetry of the electric field ($E_z = 0$). As we see, the same combination of fields (7) also arises when a particle moves in a spherically symmetric electric field and a homogeneous magnetic field.

The motion equation in the direction Oz in the rotating coordinate system has the same form as in the laboratory system ($z = z'$). The three-dimensional motion equation in a rotating frame looks like a motion equation in an axially symmetric field with components $E_\perp' = (E'/r)|u'|$ and $E_z' = E_z = (E/r)z$. The multipliers (E'/r) and (E/r) are not equal to each other.

We consider several special cases for Eqs. (5), (6) when $E/r = const$.

a) Let in the laboratory frame $\omega_c = 0$, $E \neq 0$. In a rotating coordinate system, the particle motion equation in the transverse plane looks like the motion equation in fields equal to

$$\omega_c' = 2\omega_{rot}, \qquad (8)$$

$$eE'/(mr) = \omega_{rot}^2 + eE/(mr) = \omega_c'^2/4 + eE/(mr). \qquad (9)$$

This case is considered in the Larmor's theorem in the weak magnetic field approximation. Outside this approximation, the electric fields E and E' (9) do not coincide. It should be specially noted that the equality

(8) , which is the essence of Larmor's theorem, is valid in a magnetic field of any strength.

b) In the laboratory system $\omega_c = 0$, $E = 0$. In a rotating frame we have

$$\omega'_c = 2\omega_{rot}, \quad eE'/(mr) = \omega^2_{rot} > 0. \qquad (10)$$

c) In the laboratory system $\omega_c \neq 0$, $E \neq 0$. A rotating system can always be found in which the motion equation has the form as it has in the absence of a magnetic field ($\omega'_c = 0$). In this case, the rotation frequency ω_{rot} and the electric field E' are equal

$$\omega_{rot} = -\omega_c/2, \quad eE'/(mr) \equiv -\omega^2_c/4 + eE/(mr), \qquad (11)$$

and the motion equation in the plane $x'y'$ has the form of a reduced or normal form of equation (2) ([4], part 1, §25.1, [5], §3.3.2): $\ddot{u}' - eE'/(mr)u' = 0$.

d) When $E/r = const$ one can always find a rotating frame in which the motion equation has the form as in the absence of an electric field ($E' = 0$): $\ddot{u}' + i\omega'_c\dot{u}' = 0$. The rotation frequencies of such a system (ω_{rot}) coincide with the "slow" or "fast" frequencies of particle rotation in crossed fields [6], and the cyclotron frequency (ω'_c) – with the "modified" cyclotron frequency Ω taken with the corresponding sign:

$$\omega_{rot} = (1/2)(-\omega_c \pm \Omega), \qquad \omega'_c = \pm\Omega . \qquad (12)$$

e) It is always possible to find a rotating coordinate system in which the particle moves both in a magnetic field directed opposite to the original ($\omega'_c = -\omega_c$). The frequency of rotation of such a system (ω_{rot}) and the electric field in it (E') are equal

$$\omega_{rot} = -\omega_c, \quad eE'/(mr) = eE/(mr). \qquad (13)$$

f) If the fields E', H' act in the laboratory system, then in the coordinate system rotating with frequency $-\omega_{rot}$ from (5), the motion equation looks like an equation in the fields E, H.

2. GENERALIZED LARMOR'S PROBLEM

Two equations (5), (6) contain five parameters – the fields E, H and E', H' and frequency ω_{rot}. Depending on parameters that we consider to be known and parameters that we want to determine, different problems originate. In the previous section, two values E', H' were uniquely determined from two equations (5), (6) for the given fields E, H and the rotation frequency of the coordinate system ω_{rot}.

In this section we consider another formulation of the problem, somewhat more general than that considered by Larmor [1]. Let us compare the particle motion equations in two cases: 1) the particle moves in a centrally symmetric electric field E_1 and a homogeneous magnetic field H_1, and 2) the particle moves in the fields E_2 and H_2. The fields H_1 and H_2 are not supposed to be weak. Let us determine whether there is a rotating coordinate system in which the particle motion equation in fields E_2, H_2 in the plane transverse to the magnetic field coincides with the particle motion equation in the fields E_1, H_1 written in the laboratory system. What is the rotation frequency of such a system ω_{rot}, if this system exists?

In this formulation of the problem, the fields E_1, H_1 and E_2, H_2 are assumed to be given and only the frequency ω_{rot} is to be found. To determine one value ω_{rot}, we have two equations (5), (6), i.e. the system is overdetermined. In the notations of the problem under consideration, the equations take the form:

$$\begin{cases} \omega_{c1} = 2\omega_{rot} + \omega_{c2}, & (14) \\ eE_1/(mr) = \omega^2_{rot} + \omega_{c2}\omega_{rot} + eE_2/(mr), & (15) \end{cases}$$

where $\omega_{c1,2}$ are the particle cyclotron frequencies in the fields $H_{1,2}$. From (14) we find the required rotation frequency of the coordinate system in the fields E_2, H_2:

$$\omega_{rot} = (1/2)(\omega_{c1} - \omega_{c2}) . \qquad (16)$$

It is determined only by magnetic fields. This fact corresponds to the spirit of Larmor's theorem, according to which rotation is equivalent to a magnetic field. Substituting (16) into (15), we find the relation to which the fields E_1, H_1 and E_2, H_2 must satisfy, so that equality (15) is fulfilled simultaneously with (14):

$$\Omega^2_1 = \Omega^2_2 , \qquad (17)$$

where $\Omega^2_{1,2} \equiv \omega^2_{c1,2} - 4eE_{1,2}/(mr)$. Thus, the particle motion equation in fields E_2, H_2 in the plane transverse to the magnetic field coincides in a rotating coordinate system with the motion equation in fields E_1, H_1 in the laboratory system only at a rotation frequency (16) and only if the "modified" cyclotron frequencies $\Omega^2_{1,2}$ (17) coincide. Or , which is the same, the invariants of motion equations coincide, $I_1 = I_2$. For fields E_1, H_1 and E_2, H_2 that do not satisfy the relation (17), there is no rotating coordinate system in which the particle motion equations coincide.

These conclusions are consistent with the property of "equivalence in function" of second-order linear ordinary differential equations of the form (2), (4) ([4] part 1, §25.1; [5] §3.3.2). However, in the case under consideration, the equations are in general nonlinear. The remarks listed in section 2 relate to this section too.

The behavior of the isolines $\Omega^2 = const$ is shown in Figure in the whole plane of the values of the fields E, H. A coincidence of the motion equations is possible if both points (E_1, H_1 and E_2, H_2) lie on the single isoline. The region of a weak magnetic field in which Larmor's theorem is approximately valid (the shaded region 1 in Figure), constitutes a small part of the entire plane of the parameters E, H. In this region, the exact expression for the isoline (7) is approximated by an expression $\Omega^2 \approx -4eE/(mr) = const$. The isoline corresponding to this approximation is a vertical line intersecting the horizontal coordinate axis at a point $4eE/(mr) < 0$.

In the remaining part of the plane, it is necessary to use the exact equalities (16), (17), according to which the isolines $\Omega^2 = const$ have the form of a parabola.

The motion equation in the direction Oz does not changes under the transition from the laboratory to the rotating coordinate system, so the motion equations in the direction Oz in the fields E_1, H_1 and E_2, H_2 always differ. The coincidence of the equations of motion in all three dimensions is impossible.

The obtained results (5), (6) and (16), (17) are also valid for the electric field of the cylindrical symmetry ($E_{z1,2} = 0$). The patterns of the behavior of the isolines $\Omega^2 = const$ in Figure and all comments to them are transferred without change to this case. The remarks presented in Section 2 are valid too. The motion equations in the direction Oz have the form $\ddot{z} = 0$. They coincide in any radial fields E_1, E_2. When the motion equation in the transverse plane in fields E_2, H_2 in a rotating frame (16) coincides with the equation in the fields E_1, H_1 in the laboratory system, the equations coincide in all three directions. With an appropriate choice of the initial conditions, the solutions of the equations also coincide.

CONCLUSIONS

Larmor's theorem and the results of Sections 2 and 3 can be useful in finding solutions of the motion equations. Knowing the solution in single point of isoline $\Omega^2 = const$, one can find a solution in any other point of the same isoline without solving the problem, and using only the transformation (3). For example, knowing the solution in some negative electric field in which the ion trajectory is a hypocycloid, find a solution in a positive field in which the trajectory is an epicycloid. It is also necessary to take into account that along with the transformation of motion equation the initial conditions must also be transformed.

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СВОЙСТВА УРАВНЕНИЙ ДВИЖЕНИЯ ЗАРЯЖЕННОЙ ЧАСТИЦЫ В СКРЕЩЕННЫХ ПОЛЯХ И ТЕОРЕМА ЛАРМОРА

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Проведено сравнение уравнений движения заряженной частицы в скрещенных полях в лабораторной и во вращающейся системах координат. Показано, что уравнение движения в полях (E, H) в поперечной к магнитному полю плоскости во вращающейся системе имеет такой же вид, что и уравнение движения в полях другой напряженности (E', H') в лабораторной системе. Определен инвариант уравнения движения при преобразовании вращения. Рассмотрена задача, обобщающая задачу, рассмотренную Лармором. Определены частота вращения и условие, при которых уравнение движения частицы в полях (E_1, H_1) в лабораторной системе совпадает с уравнением движения в полях (E_2, H_2) во вращающейся системе.

ВЛАСТИВОСТІ РІВНЯНЬ РУХУ ЗАРЯДЖЕНОЇ ЧАСТКИ В СХРЕЩЕНИХ ПОЛЯХ І ТЕОРЕМА ЛАРМОРА

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Проведено зіставлення рівнянь руху зарядженої частки в схрещених полях у лабораторній і обертовій системах координат. Показано, що рівняння руху в полях (E, H) у поперечній до магнітного поля площині в обертовій системі має такий же вигляд, що й рівняння руху в полях іншої напруженості (E', H') в лабораторній системі. Визначено інваріант рівняння руху при трансформації обертання. Розглянута задача, що узагальнює задачу, розглянуту Лармором. Визначені частота обертання й умова, при яких рівняння руху частки в полях (E_1, H_1) у лабораторній системі збігається з рівнянням руху в полях (E_2, H_2) в обертовій системі.