FEATURES OF THE SPECTRA OF NONLINEAR OSCILLATORS IN REGIMES WITH DYNAMIC CHAOS

V.A. Buts[1,2,3], V.V. Kuzmin[1], A.P. Tolstoluzhsky[1]

[1]National Science Center "Kharkov Institute of Physics and Technology", Kharkiv, Ukraine;
[2]Institute of Radio Astronomy of NAS of Ukraine, Kharkiv, Ukraine;
[3]V.N. Karazin Kharkiv National University, Kharkiv, Ukraine

E-mail: tolstoluzhsky@kipt.kharkov.ua

It is shown that the dynamics of particles in the field of a wave packet, whose group velocity is zero, is almost always chaotic. Earlier we showed that in the field of a large number of cyclotron resonances, the higher moments of particle dynamics can be much larger than the lower moments. A kinetic description of such dynamics can be realized only on the basis of the generalized Fokker-Planck equation. Such an equation is obtained. Some features that appear as a result of taking into account the higher moments are described.

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INTRODUCTION

It is known that the regimes with dynamic chaos are characteristic practically for all nonlinear oscillatory systems. The strict proof of the conditions of transition to the regime with dynamic chaos represents rather difficult task. In the vast majority of physical problems there is an opportunity to use simple analytical criterion of transition to dynamic chaos – Chirikov's criterion. This criterion is simple. His physical contents is transparent. It is used in huge number of researches. However this criterion is still phenomenological criterion and in many cases it needs to be used with caution. In particular, in work [1] it is suggested that when the ratio of width of nonlinear resonances to the distance between these resonances it is greater, than number of waves, dynamics can be not chaotic, but regular. This rather transparent physical reason. This question is studied below in the first section.

The second feature of the description of the regimes with dynamic chaos is that for the description of these regimes the equations like Fokker-Planck's equations are often used. However such equations take into account the influence on the dynamics of particles only the first, second moments of this dynamics. In work it has been shown what in many cases of the dynamic of charged particles in the conditions of cyclotron resonances is described by the moments which have such feature that the highest moments are large, than the lowest moments. In this case the equations of type of Fokker-Planck demand generalization on a case of taking note of these highest moments. Such generalized equations are written out below in point.

1. PARTICLE DYNAMICS IN THE FIELD OF THE WAVE PACKET

To determine the main features of the motion of charged particles in the field of a wave packet, we consider the simplest model in which such features can be shown. As such a model can serve as one-dimensional model of the motion of charged particles in the field of a large number of waves:

$$\ddot{z} = \frac{e}{m}\sum_i E_i \sin\left(k_i z - \omega_i t\right). \qquad (1)$$

In order to clarify the conditions for the appearance of regimes with dynamic chaos, we first consider the motion of a particle in one of these waves. From equation (1) we can then obtain the well-known integral:

$$\dot{\varphi}^2\big/2 - \Omega^2 \cos\varphi = H = const\ , \qquad (2)$$

here: $\varphi = kz - \omega t$, $\Omega^2 = \dfrac{|e|Ek}{m\omega^2}$, $\dot{\varphi} = d\varphi / d\tau;\ \tau = \omega t.$

Using the integral (2), we find the width of the nonlinear resonance:

$$\dot{\varphi}_{\max} = +2\Omega\ ,\quad \dot{\varphi}_{\min} = -2\Omega\ . \qquad (3)$$

To determine the distance between the resonances, we note that the effective interaction of particles with the wave of the packet occurs under the conditions of Cerenkov resonance. In this case it is easy to determine the distance between the resonances:

$$\Delta\dot{\varphi} = -\Delta k\left[v_0 - \left(\Delta\omega\big/\Delta k\right)\right]\ . \qquad (4)$$

At obtaining of (4) we took into account that $v = v_{ph} = \omega / \mathrm{k}$. Using expressions (3) and (4), it is easy to find the conditions for the onset of local instability:

$$K = \left(\frac{\omega}{\Delta\omega}\right)\frac{2\sqrt{A}}{\left[1 - v_g / v_{ph}\right]} = N\frac{2\sqrt{A}}{\left[1 - v_g / v_{ph}\right]}\ , \qquad (5)$$

here v_g is group velocity; N is number of waves in the packet.

Looking at formulas (4) and (5), it is already possible to make several important conclusions. The first is clear (from formula (4)) that if the group velocity tends to the phase velocity of the wave, then the distance between the resonances tends to zero. This means that all waves of the packet are located on a rectilinear section of the dispersion. In the phase space, the resonances of such waves all coincide. For particles, such resonances are practically indistinguishable. Dynamics should be regular. Second, on the other hand, if the group velocity of the waves tends to zero (for example, Langmuir waves in a plasma), then, as can be seen from formula (5), the resonance overlap criterion turns out to be much smaller than the number of waves participating in the packet. In this case, as for the first time, apparently, it was noted in the work, the dynamics of the particles should be chaotic. We note here that the nonrelativistic dynamics of particles always corresponds to the case $A << 1$.

Further analysis of particle dynamics has been carried out by numerical methods. For this, the right-hand side of equation (1) has been represented in this form:

$$G=\sum_{i=m}^{N}\sin\left[\left(k+i\frac{\Delta k}{N}\right)z-\left(\omega+i\frac{\Delta\omega}{N}\right)t\right] \quad . \qquad (6)$$

Formula (6) describes the structure of the fields that make up the wave packet. A feature of the field of such wave packet is its interference of the fields composing the packet. As an example, Fig. 1 shows the form of the field of this packet at fixed coordinates and time in the case of 25 waves ($\Delta\omega = v_g \Delta k$). As the number of waves in the packet increases, the distance between the field maxima increases, and the amplitude of each of the maxima increases.

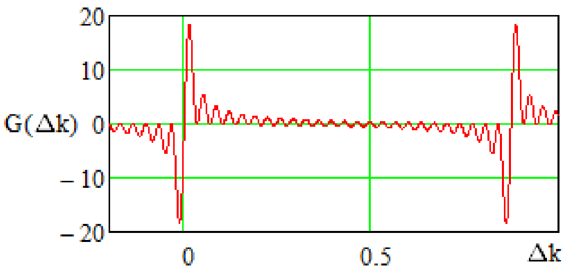


Fig. 1. Form of the field for packet of waves with 25 waves

We will consider dynamics of charged particles in the field of two waves with A=0.1 amplitudes in the beginning. At the same time, for simplicity, frequencies of waves $\omega_1 = 1$, $\omega_2 = 0.99$ and wave number $k_1 = 1$ have been chosen.

An analysis of the dynamics of charged particles in the field of two waves with amplitudes A = 0.1 showed that the chaotic dynamics of particles begins at $K \geq 0.4$. Such dynamics are retained up to values K~31.

Dependences of the particle velocity on time, their spectra and correlation functions are shown in Figs. 2, 3.

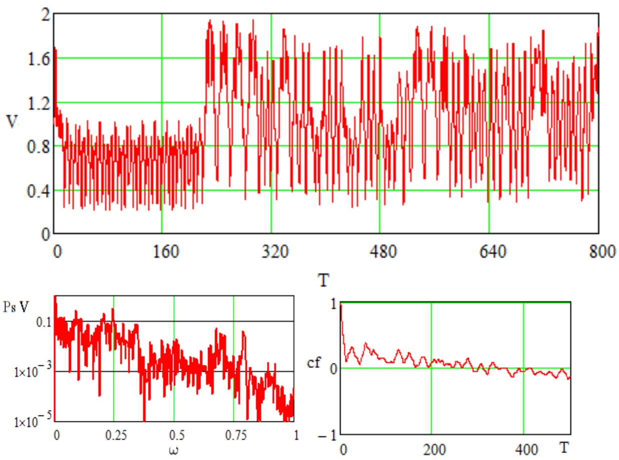


Fig. 2. Particle velocity V, power spectral density PsV and correlation function cf for K=0.46

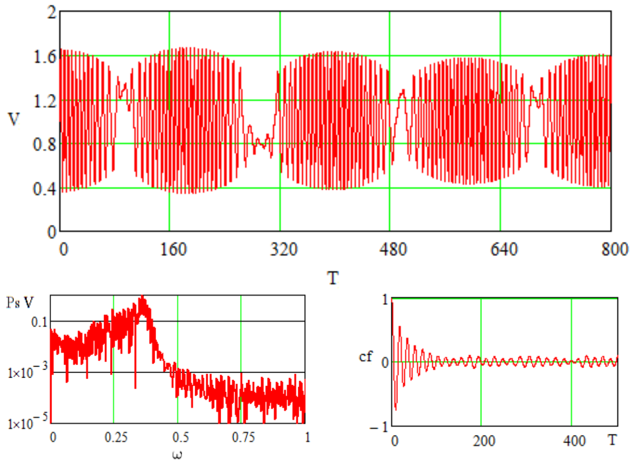


Fig. 3. Particle velocity V, power spectral density PsV and correlation function cf for K~31

And only after this, with further increase in the parameter K, the dynamics becomes regular. Especially this tendency can be shown when the group velocity is zero ($v_g = 0$).

When considering the dynamics of particles in a packet the amplitude of all the packet waves are A=0.1, and the wave frequencies of the packet were chosen in the range $\omega_1 = 0.99$, $\omega_2 = 1.0$, wave number $k_2 = 1$. The wave of the packet was uniformly distributed in the interval between these two fixed waves.

The analysis of particle dynamics in the packet shows that practically always when $v_g = 0$ the dynamics of particles is chaotic Fig. 4.

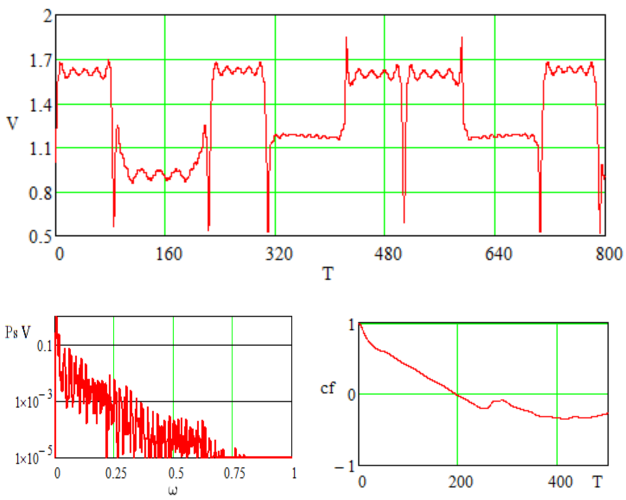


Fig. 4. Particle velocity V, power spectral density PsV and correlation function cf for K=1, number of waves in the packet N=10

If, however, $\Delta\omega = \Delta k \cdot v_g$; $v_g = v_{ph}$, that corresponds to the linear section of the dispersion of the waves composing the packet, then the dynamics for a sufficiently large number of waves turns out to be regular Fig. 5.

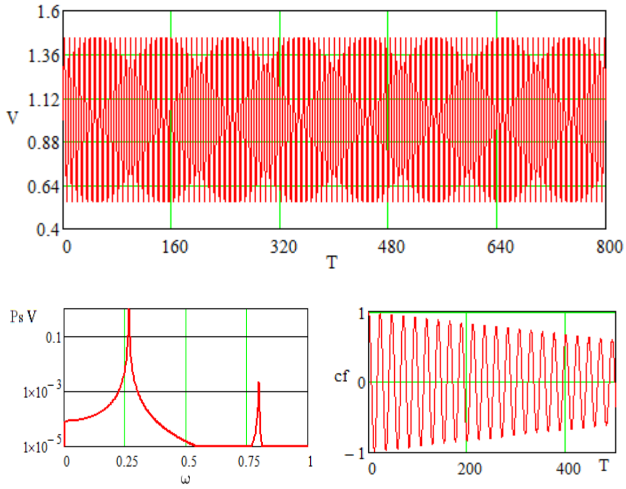
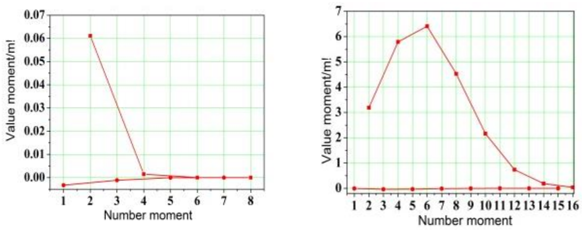


Fig. 5. Particle velocity V, power spectral density PsV and correlation function cf for K=1000, number of waves in the packet N=10

This corresponds to the case that in Fig. 1 the next field pulse goes to infinity. Therefore, the effective interaction of particles with the field occurs only in the region of one maximum of this field. The dynamics is regular.

2. ROLE OF THE MOMENTS IN DYNAMICS OF PARTICLES

In paper [2] analysis of the moments at overlapping of cyclotron resonances shows that there are conditions under which the higher moments describing the dynamics of transverse momenta can be greater than the previous ones. The analysis of particle dynamics cannot be carried out by means of equations of the Fokker-Planck type, since only the second moments are taken into account in such equations [3]. The illustration of this feature is shown in the Fig. 6.



a b

Fig. 6. Dependences of the magnitudes of the moments p_x divided by the factorial of their number for the field amplitude: a) ε₀=0.1; b) ε₀=0.19

In these figures the dependence of the magnitude of the moments on their number is presented. And the magnitude of each moment is divided by the factorial of its number (by $m!$). It can be seen from these figures that, for low external field strength ($\varepsilon_0 = eE/mc\omega = 0.1$), the moments rapidly fall with increasing number (see Fig. 6,a). However, for higher strengths (for $\varepsilon_0 = 0.19$), the higher moments turn out to be larger than the moments with smaller numbers. In these cases, to describe the dynamics of particles, it is necessary to generalize the Fokker-Planck equations to the case of taking into account the role of higher moments. To do this, let us write down the relationship between the particle density at the instant of time $\tau + \Delta\tau$ and the particle density at the instant of time τ:

$$n(p,\tau+\Delta\tau) = \int_{-\infty}^{\infty} \left[n(p-p',\tau) \right] \mathrm{f}(p')dp' . \qquad (7)$$

Expression (7) is a mathematical reflection of the fact that the density of particles having a momentum p at a time $\tau + \Delta\tau$ will be determined by all other particles (with other energies) and which, with probability $f(p')$, acquire momentum p' after a time interval $\Delta\tau$. It is convenient to rewrite equation (7) in the form:

$$n(p,\tau+\Delta\tau) - n(p,\tau) = \int_{-\infty}^{\infty} \left[n(p-p',\tau) - n(p,\tau) \right] \mathrm{f}(p')dp' .$$

We decompose the integrand into a series of relatively small displacements:

$$\frac{\partial n}{\partial \tau} = \sum_m \frac{\left\langle (p)^m \right\rangle}{m!} \frac{\partial^m n}{\partial p^m}, \quad m = 2j;\ j = \{1,2,3...\} \qquad (8)$$

where $\left\langle (p)^m \right\rangle$ are the moments of the transverse particle momentum at cyclotron resonances. If to be limited to accounting only of the second moments, then we obtain the usual diffusion equation for the particle density with the diffusion coefficient $D = \left\langle p^2 \right\rangle / 2$:

$$\frac{\partial n}{\partial \tau} = D \frac{\partial^2 n}{\partial p^2} . \qquad (9)$$

For the case presented in Fig. 2, it is necessary to take into account 4-5 terms in the sum (8). In order to clarify the role of higher moments, it suffices to analyze the solutions of equation (8) taking into account only the second and fourth moments:

$$\frac{\partial n}{\partial \tau} = \alpha^2 \frac{\partial^2 n}{\partial p^2} + \beta^2 \frac{\partial^4 n}{\partial p^4} . \qquad (10)$$

If the parameter β is small ($\beta << 1$), then the solution of equation (10) can be sought in the form of a series in this parameter:

$$n = n_0 + \beta n_1 + \beta^2 n_2 + ... \qquad (11)$$

Substituting this series into equation (10), we will find the equations for finding the terms of this series. For example, to find the second term, we can get the following sequence

$$\hat{L}n_0 = \frac{\partial n_0}{\partial \tau} - \alpha \frac{\partial^2 n_0}{\partial p^2} = 0 ;\ \hat{L}n_1 = \beta \frac{\partial^4 n_0}{\partial p^4} ;\ \hat{L}\hat{L}n_1 = \beta \frac{\partial^4 \hat{L}n_0}{\partial p^4} = 0$$

since $\hat{L}n_0 = 0$, then and $\hat{L}n_1 = 0$. The equations for the other terms of the series (11) will have an analogous form. Finally, the series (11) can be written in the form of a series of geometric progression:

$$n(p,t) = n_0 \left[1 + \beta + \beta^2 + ... \right] = n_0(p,t) / (1-\beta) . \quad (12)$$

This expression shows that the solutions of the Fokker-Planck equation are stable with respect to the influence of small higher moments.

In the general case, the solution of equation (10) can be sought in the self-similar form:

$$n(p,t)=\frac{1}{t^a}v(y)\ ,\ \ y=\frac{p}{t^b}\ . \qquad (13)$$

Substituting this solution into initial equation, we obtain

$$\frac{\beta^2}{t^{4b-1}}\frac{\partial^4 v}{\partial^4 y}+\frac{\alpha^2}{t^{2b-1}}\frac{\partial^2 v}{\partial^2 y}+b\frac{\partial v}{\partial y}y+av=0\ . \qquad (14)$$

The parameters a and b are arbitrary. If the parameter β is small ($\beta^2 << 1$), then we obtain at $b=1/2$ the Fokker-Planck equation. It is known that its solution has the form:

$$n(p,t)=\frac{n_0}{2\alpha\sqrt{\pi t}}e^{-\frac{p^2}{4\alpha^2 t}}\ . \qquad (15)$$

As we saw above (see (12)), such function changes a little when considering higher moments, if they are small. Below we consider special cases that allow us to see the role of higher moments in the dynamics of particles. For this, first of all, we will assume that the parameter α is very small ($\alpha^2 << 1$). In this case, equation (14) is simplified:

$$\frac{\beta^2}{t^{4b-1}}\frac{\partial^4 v}{\partial^4 y}+b\frac{\partial v}{\partial y}y+av=0\ . \qquad (16)$$

Equation (16) is also quite complicated for analysis. Therefore, we will be limited only to the asymptotic solution for large values of time ($t\to\infty$).

Choose values $a=1$, $b=1$. In this case, for large values of time, equation (15) can be rewritten in the form:

$$\frac{\partial(vy)}{\partial y}=0\ . \qquad (17)$$

The solution equation (16) has the form:

$$v=C(\mathrm{p})/\mathrm{y}\ . \qquad (18)$$

Here, the "constant" C is an arbitrary function of the momentum. In this case, the solution of the original equation (16) at $t\to\infty$ will have the form:

$$n(p,\infty)=C(\mathrm{p})/\mathrm{p}\ . \qquad (19)$$

"Constant" $C(\mathrm{p})$ is determined from the condition of conservation of the total number of particles (N):

$$N=\int_0^\infty\frac{C(\mathrm{p})}{p}dp\ . \qquad (20)$$

It is visible that this relationship will be valid if we choose the "constant" $C(\mathrm{p})$ in the form:

$$C(\mathrm{p})=\mathrm{p}\cdot\exp(-\varepsilon p)\ ,$$

here – $\varepsilon=1/N$. Then the expression for the particle density at large times will be described by the formula:

$$n(p,\infty)=\exp\left(-p/N\right)\ . \qquad (21)$$

Comparing formula (21) with formula (15) for large times, the qualitative difference in the dependence of the particle density on the momentum is clearly visible.

CONCLUSIONS

We note the most important results of the paper. Analysis of the motion of charged particles in wave packets, whose group velocity is zero, has shown that practically always this dynamics is chaotic. On the other hand, if the packet is formed by waves that are located on a rectilinear section of the dispersion, the dynamics of the particles in such a packet remains regular.

The equation is obtained that generalizes an equation of the Fokker-Planck type to the case of the influence of higher moments of chaotic particle dynamics. It is shown that if the higher moments are small, then solutions of equations of the Fokker-Planck type also remain practically unchanged. However, if, for example, the fourth moment is significantly larger than the second moment, then the differences can be both quantitative and qualitative.

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ОСОБЕННОСТИ СПЕКТРОВ НЕЛИНЕЙНЫХ ОСЦИЛЛЯТОРОВ В РЕЖИМАХ С ДИНАМИЧЕСКИМ ХАОСОМ

В.А. Буц, В.В. Кузьмин, А.П. Толстолужский

Показано, что динамика частиц в поле волнового пакета, групповая скорость которых равна нулю, практически всегда хаотична. Ранее нами показано, что в поле большого числа циклотронных резонансов высшие моменты динамики частиц могут быть значительно большими, чем низшие моменты. Кинетическое описание такой динамики может быть осуществлено только на основе обобщенного уравнения Фоккера-Планка. Такое уравнение получено. Описаны некоторые особенности, появляющиеся в результате учета высших моментов.

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В.О. Буц, В.В. Кузьмін, О.П. Толстолужський

Показано, що динаміка частинок у полі хвильового пакета, групова швидкість яких дорівнює нулю, практично завжди хаотична. Раніше нами показано, що в полі великої кількості циклотронних резонансів вищі моменти динаміки частинок можуть бути значно більшими, ніж нижчі моменти. Кінетичний опис такої динаміки може бути здійснено тільки на основі узагальненого рівняння Фоккера-Планка. Таке рівняння отримано. Описано деякі особливості, що з'являються в результаті обліку вищих моментів