LIMITS OF APPLICABILITY OF THE WEAKLY RELATIVISTIC APPROXIMATION IN THE THEORY OF PLASMA WAVES

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The more general weakly-relativistic plasma dispersion functions are introduced and discussed in the frame of the fully relativistic approach for the theory of plasma waves.
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INTRODUCTION

Presently, the analytical and numerical investigation of the excitation, propagation and absorption of electron cyclotron waves in thermonuclear plasma is performed as a rule in the frame of the weakly relativistic approximation, when longitudinal (to the magnetic field line) spatial dispersion of plasma is expressed in the terms of the weakly relativistic plasma dispersion functions (PDFs)

$$F_{n+3/2}(z,a)=\frac{e^{-a}}{\left(\sqrt{a}\right)^{n+1/2}}\int_0^{+\infty} du\frac{\left(\sqrt{u}\right)^{n+1/2} I_{n+1/2}\left(2\sqrt{au}\right)e^{-u}}{u+z-a},$$

$$(n=0,\pm1,\pm2...),\qquad(1)$$

where n is the number of electron cyclotron harmonic, $a=\mu N_{\parallel}^2/2$, $\mu=(c/V_{Te})^2$, c is speed of light in the vacuum, V_{Te} is the thermal velocity of electrons, $N_{\parallel}$ is the longitudinal refractive index of plasma, $I_{n+1/2}(x)$ is modified Bessel function of the half-integer order [1]. Applicability limits of such approximation is not clear enough, since they could be defined exactly only on the basis of somewhat limit transitions in the frame of the exact fully relativistic approach, when plasma dielectric tensor elements are expressed in the terms of the Cauchy type integrals, named by the exact relativistic or fully relativistic PDFs [2]: for the case $0\le N_{\parallel}<1$

$$Z_{n+3/2}(z,a,\mu)=\frac{\sqrt{\pi\beta}e^{\beta(a^*-2a)}}{\sqrt{2\mu}K_2(\mu)e^{\mu}\left(\sqrt{a}\right)^{n+1/2}}\times$$

$$\int_0^{+\infty}\frac{\left(\sqrt{u(u/(2\mu)+1/\sqrt{\beta})}\right)^{n+1/2} I_{n+1/2}\left(2\beta\sqrt{a}\sqrt{u(u/(2\mu)+1/\sqrt{\beta})}\right)e^{-\beta u}du}{u+z-a^*},\qquad(2)$$

where $\beta=1/(1-N_{\parallel}^2)$, $K_2(x)$ is the MacDonald function of the 2nd order, $a^*=\mu(1-\sqrt{1-N_{\parallel}^2})$, $I_{n+1/2}(x)$ is modified Bessel function with half-integer index, in the contrary case $N_{\parallel}>1$

$$Z_{n+3/2}(z,a,\mu)=-\sqrt{-\beta}\,\frac{\exp(-2\beta a-\mu)}{\sqrt{2\pi\mu}K_2(\mu)\left(\sqrt{a}\right)^{n+1/2}}\times$$

$$\int_{-\infty}^{+\infty}\frac{\left(\sqrt{a-t+t^2/(2\mu)}\right)^{n+1/2} K_{n+1/2}\left(-2\beta a^{1/2}\sqrt{a-t+t^2/(2\mu)}\right)\exp(\beta t)dt}{t-z},$$

$$(n=0,\pm1,\pm2...),\qquad(3)$$

where $K_{n+1/2}(x)$ is MacDonald function with half-integer index

The main scope of present work is the definition of applicability limits of the weakly relativistic approximation from fully relativistic PDFs (2) and (3) on the base limit transition $\mu\to\infty$.

1. THE WEAKLY RELATIVISTIC PDFs WITHOUT TAKING INTO ACCOUNT THE TRANSVERSE SPATIAL DISPERSION

From comparison of the weakly relativistic PDFs (1) and fully relativistic ones (2) and taking into account the asymptotic relation $K_2(\mu)e^{\mu}\sim\sqrt{\pi/(2\mu)}$ when $\mu\to\infty$ [3] it is easy to see, that $Z_{n+3/2}(z,a,\mu)\to F_{n+3/2}(z,a)$ when $\mu\to\infty$ and $a^*\to a$. The first limit transition corresponds to the transition into the cold plasma approximation, the second one corresponds to the transition into the nearly perpendicular (to the magnetic field) wave propagation case ($N_{\parallel}^2<<1$), when $a^*\to\mu N_{\parallel}^2/2=a$ and $\beta\approx1$.

Here it is necessary specially to note that the weakly relativistic PDFs (1) and fully relativistic ones (2) are fair only for the case $N_{\parallel}<1$. In the contrary case $N_{\parallel}>1$ the exact relativistic PDFs are expressed by the Cauchy integrals, defined at the real axis in the form (3). Performing in the definition (3) the limit transition $\mu\to\infty$ and making the change $-\beta=1/(N_{\parallel}^2-1)$ we will obtain the weakly relativistic PDFs of the new form, corresponding to the condition $N_{\parallel}^2>1$ [2].

$$F_{n+3/2}(z,a)=-\sqrt{1/(N_{\parallel}^2-1)}\,\frac{\exp(2a/(N_{\parallel}^2-1))}{\sqrt{2\pi}\left(\sqrt{a}\right)^{n+1/2}}\times$$

$$\int_{-\infty}^{+\infty}\frac{\left(\sqrt{a-t}\right)^{n+1/2} K_{n+1/2}\left(2\sqrt{a}\sqrt{a-t}/(N_{\parallel}^2-1)\right)\exp\left(-t/(N_{\parallel}^2-1)\right)dt}{t-z}.\qquad(4)$$

Now making in the definition (2) only the limit transition $\mu \to \infty$ we will obtain the weakly relativistic PDFs also of the new form, corresponding to the condition $N_{\parallel}^2 < 1$.

$$Z_{n+3/2}(z,a) = \frac{e^{(a^* - 2a)/(1-N_{\parallel}^2)}}{\left(\sqrt{(1-N_{\parallel}^2)}\right)\left(\sqrt{a}\right)^{n+1/2}} \times$$

$$\int_0^{+\infty} \frac{\left(\sqrt{u\sqrt{1-N_{\parallel}^2}}\right)^{n+1/2} I_{n+1/2}\left(2\frac{\sqrt{a}}{1-N_{\parallel}^2}\sqrt{u\sqrt{1-N_{\parallel}^2}}\right) e^{-\beta u} du}{u - a^* + z}. \qquad (5)$$

It is easy to see that the form (4) goes to the form (1) when $N_{\parallel} = 0$ (perpendicular wave propagation) and $a^* = a$ (almost perpendicular wave propagation). It is easy to see, that $a^* = 2a / \left(1 + \sqrt{1 - N_{\parallel}^2}\right)$ and consequently for a^* is true condition $a \le a^* \le 2a$ when $0 \le N_{\parallel} \le 1$.

Methods of evaluation of PDFs (4) and (5) are given in [2].

2. ACCOUNT OF THE TRANSVERSE SPATIAL DISPERSION

One-dimensional integral forms for fully relativistic plasma dielectric tensor, taking into account the transverse spatial dispersion and longitudinal one for the case $N_{\parallel}^2 < 1$, were given in [4]

$$\operatorname{Im}\varepsilon_{11} = \Delta \sum_{n=-\infty}^{+\infty} \frac{n^2}{\lambda} \varphi(z) \int_{-1}^{1} du e^{-2\sqrt{a}Ku} J_n^2\{\Theta\}, \qquad (6)$$

$$\operatorname{Im}\varepsilon_{12} = i\Delta \sum_{n=-\infty}^{+\infty} \frac{n}{\lambda} \varphi(z) \int_{-1}^{1} du e^{-2\sqrt{a}Ku} \Delta J_n\{\Theta\} J_n'\{\Theta\},$$

$$\operatorname{Im}\varepsilon_{22} = \Delta \sum_{n=-\infty}^{+\infty} \frac{1}{\lambda} \varphi(z) \int_{-1}^{1} du e^{-2\sqrt{a}Ku} (\Delta)^2 J_n\{\Theta\} J_n'^2\{\Theta\},$$

$$\operatorname{Im}\varepsilon_{13} = \Delta \sum_{n=-\infty}^{+\infty} \frac{n}{\sqrt{\lambda}} K\varphi(z) \int_{-1}^{1} u du e^{-2\sqrt{a}Ku} J_n^2\{\Theta\},$$

$$\operatorname{Im}\varepsilon_{23} = i\Delta \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{\lambda}} K\varphi(z) \int_{-1}^{1} u du e^{-2\sqrt{a}Ku} \Delta J_n\{\Theta\} J_n'\{\Theta\},$$

$$\operatorname{Im}\varepsilon_{33} = \Delta \sum_{n=-\infty}^{+\infty} 2K^2 \varphi(z) \int_{-1}^{1} u^2 du e^{-2\sqrt{a}Ku} J_n^2\{\Theta\},$$

$$\varphi(z) = e^z K e^{-2\beta a(1-z/\mu)},$$

$$\Delta = \left(\frac{\omega_{p0}}{\omega}\right)^2 \frac{\pi e^{-\mu}}{K_2(\mu)} \sqrt{\frac{\mu}{2}}, \quad \Theta = K\sqrt{2\lambda(1-u^2)/\beta},$$

$$\operatorname{Re}\varepsilon_{ij}(a,z,\mu) = \delta_{ij} + \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\operatorname{Im}\varepsilon_{ij}(a,t,\mu)dt}{t-z}.$$

After the weakly relativistic limit transition $\mu \to \infty$ in the integral forms (6) one will have functions, generating elements of plasma dielectric tensor for the weakly relativistic case with taking exactly into account the transverse and longitudinal spatial dispersion of plasma and suitable for numerical evaluations for arbitrary ECRF waves in the weakly relativistic plasma.

The alternative case $N_{\parallel} > 1$ was studied in [5], where were obtained plasma dielectric tensor elements for fully relativistic plasma with taking into account the transverse spatial dispersion and longitudinal one

$$\operatorname{Im}\varepsilon_{11} = -\mu\Delta \sum_{n=-\infty}^{+\infty} \frac{n^2}{\lambda} K^* \int_1^{+\infty} du e^{-2\sqrt{a}K^* u} J_n^2\{\Theta^*\}, \qquad (7)$$

$$\operatorname{Im}\varepsilon_{12} = -\mu i\Delta \sum_{n=-\infty}^{+\infty} \frac{n}{\lambda} K^* \int_1^{+\infty} du e^{-2\sqrt{a}K^* u} \Theta J_n\{\Theta^*\} J_n'\{\Theta^*\},$$

$$\operatorname{Im}\varepsilon_{22} = -\mu\Delta \sum_{n=-\infty}^{+\infty} \frac{1}{\lambda} K^* \int_1^{+\infty} du e^{-2\sqrt{a}K^* u} \Theta^2 J_n'^2\{\Theta^*\},$$

$$\operatorname{Im}\varepsilon_{13} = -\mu\Delta \sum_{n=-\infty}^{+\infty} \frac{\sqrt{2}n}{\sqrt{\lambda}} K^* \int_1^{+\infty} du \left[K^* u + \beta\sqrt{a}(1 - z/\mu)\right] \times$$
$$e^{-2\sqrt{a}K^* u} J_n^2\{\Theta^*\},$$

$$\operatorname{Im}\varepsilon_{23} = -i\mu\Delta \sum_{n=-\infty}^{+\infty} \frac{\sqrt{2}}{\sqrt{\lambda}} K^* \int_1^{+\infty} du \left[K^* u + \beta\sqrt{a}(1 - z/\mu)\right] \times$$
$$e^{-2\sqrt{a}K^* u} \Theta J_n\{\Theta^*\} J_n'\{\Theta^*\},$$

$$\operatorname{Im}\varepsilon_{33} = -\mu\Delta \sum_{n=-\infty}^{+\infty} 2K^* \int_1^{+\infty} du \left[K^* u + \beta\sqrt{a}(1 - z/\mu)\right]^2 \times$$
$$e^{-2\sqrt{a}K^* u} J_n^2\{\Theta^*\},$$

$$\operatorname{Re}\varepsilon_{ij}(a,z,\mu) = \delta_{ij} + \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\operatorname{Im}\varepsilon_{ij}(a,t,\mu)dt}{t-z}.$$

After the same weakly relativistic transition $\mu \to \infty$ in formulae (7), one will obtain the functions, generating plasma dielectric tensor elements in the weakly relativistic case, taking into account the perpendicular and longitudinal dispersion of plasma and suitable for numerical investigation of fast and slow plasma EC waves in the thermonuclear plasma.

CONCLUSIONS

The next main conclusions can be extracted from this work:

1. In the frame of fully relativistic approach by means of limit transition $\mu \to \infty$ were derived the two sets of the weakly relativistic PDFs.

2. One PDFs generalize Shkarofsky functions from quasi-perpendicular case into much more wide case $0 \le N_{II} < 1$.

3. Another PDFs give the weakly relativistic PDFs for the case $N_{II} > 1$.

4. Method evaluating the weakly relativistic plasma dielectric tensor elements for arbitrary wave numbers or with taking into account transverse and longitudinal spatial dispersion of plasma is given as well.

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ПРЕДЕЛЫ ПРИМЕНИМОСТИ СЛАБОРЕЛЯТИВИСТСКОГО ПРИБЛИЖЕНИЯ В ТЕОРИИ ПЛАЗМЕННЫХ ВОЛН

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Более общие слаборелятивистские функции вводятся и обсуждаются в рамках точного полностью релятивистского подхода в теории плазменных волн.

МЕЖІ ЗАСТОСУВАННЯ СЛАБОРЕЛЯТИВІСТСЬКОГО НАБЛИЖЕННЯ В ТЕОРІЇ ПЛАЗМОВИХ ХВИЛЬ

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Більш загальні слаборелятивістські функції вводяться і обговорюються в рамках точного повністю релятивістського підходу в теорії плазмових хвиль.