On Asymptotic Regimes of Orthogonal Polynomials with Complex Varying Quartic Exponential Weight[*]

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Abstract. We study the asymptotics of recurrence coefficients for monic orthogonal polynomials $\pi_n(z)$ with the quartic exponential weight $\exp[-N(\frac{1}{2}z^2 + \frac{1}{4}tz^4)]$, where $t \in \mathbb{C}$ and $N \in \mathbb{N}$, $N \to \infty$. Our goal is to describe these asymptotic behaviors globally for $t \in \mathbb{C}$ in different regions. We also describe the "breaking" curves separating these regions, and discuss their special (critical) points. All these pieces of information combined provide the global asymptotic "phase portrait" of the recurrence coefficients of $\pi_n(z)$, which was studied numerically in [*Constr. Approx.* **41** (2015), 529–587, arXiv:1108.0321]. The main goal of the present paper is to provide a rigorous framework for the global asymptotic portrait through the nonlinear steepest descent analysis (with the g-function mechanism) of the corresponding Riemann–Hilbert problem (RHP) and the continuation in the parameter space principle. The latter allows to extend the nonlinear steepest descent analysis from some parts of the complex t-plane to all noncritical values of t. We also provide explicit solutions for recurrence coefficients in terms of the Riemann theta functions. The leading order behaviour of the recurrence coefficients in the full scaling neighbourhoods the critical points (double and triple scaling limits) was obtained in [*Constr. Approx.* **41** (2015), 529–587, arXiv:1108.0321] and [Asymptotics of complex orthogonal polynomials on the cross with varying quartic weight: critical point behaviour and the second Painlevé transcendents, in preparation].

Key words: double scaling limit of orthogonal polynomials; asymptotics of recurrence coefficients; method of Riemann–Hilbert problem; nonlinear steepest descent analysis

2010 Mathematics Subject Classification: 33D45; 33E17; 15B52

Contents

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1 Introduction and main results

In this paper we continue (see [4]) our study of monic polynomials $\pi_n(z)$, orthogonal with respect to the quartic exponential weight $e^{-Nf(z,t)}$, where $f(z,t) = \frac{1}{2}z^2 + \frac{1}{4}tz^4$, $t \in \mathbb{C}$ and $N \in \mathbb{N}$, and the contour of integration consists of four rays in the directions of the exponential decay of $e^{-Nf(z,t)}$. The exponent $f(z,t)$ is also known as the potential.

In [4] we obtained local (in t) asymptotics of the recurrence coefficients $\alpha_n(N,t), \beta_n(N,t)$, associated with $\pi_n(z)$, in the limit $n = N \to \infty$, in full neighborhoods of critical points $t_0 = -\frac{1}{12}$ and $t_1 = \frac{1}{15}$ through certain Painlevé I transcendents (note that we will not encounter t_1 in the particular settings considered in this paper). Those results were a significant development of [13, 15], where these asymptotics were studied only "away from the poles" of the Painlevé I transcendents, that is, on certain subsets of full neighborhoods. The (only) remaining critical point $t_2 = \frac{1}{4}$ for $f(z,t) = \frac{1}{2}z^2 + \frac{1}{4}tz^4$ was studied in [7], where the Painlevé II transcendents played the same role as Painlevé I transcendents in the case of t_0, t_1 (see [8] for analysis of t_2 in the case of real orthogonal polynomials on $\mathbb{R}$). In order to understand the role of the critical points t_j, $j = 0, 1, 2$, in the global "asymptotic phase portrait", in [4] we studied numerically the regions of different asymptotic regimes (in the limit $n = N \to \infty$) in the complex t-plane (in fact, on the Riemann surface of $t^{\frac{1}{4}}$), see, for example, Fig. 2. We found there that there are six possible topologically different cases of global "asymptotic phase portraits", which are determined by the constant weights ϱ_j, see (1.1), associated with each ray.

The objective of the current paper is to give a rigorous foundation for the numerical results, obtained in [4]. We will restrict our attention only to one of the above mentioned six cases, called Generic case, which will be defined below. Our goal is to rigorously justify the asymptotic phase portrait for the Generic case (case **(a1)** in [4]) using the *principle of continuation in the parameter space*, developed in [1] and [24]. As it was stated in [1], this technique will work for any k-th degree polynomial $f(z, \vec{t})$ that depends on k complex parameters $(t_1, \ldots, t_k) = \vec{t}$ that are the coefficients of f (it is obvious that the free term in f is irrelevant). Because of that consideration, a number of statements in this paper will be formulated and proved for general polynomial potentials $f(z; \vec{t})$. We also obtain explicit expressions for the recurrence coefficients and for the pseudonorms of $\pi_n(z)$ in terms of the Riemann theta functions. For the benefit of the reader, we repeat here some facts from [4] that are required for the exposition.
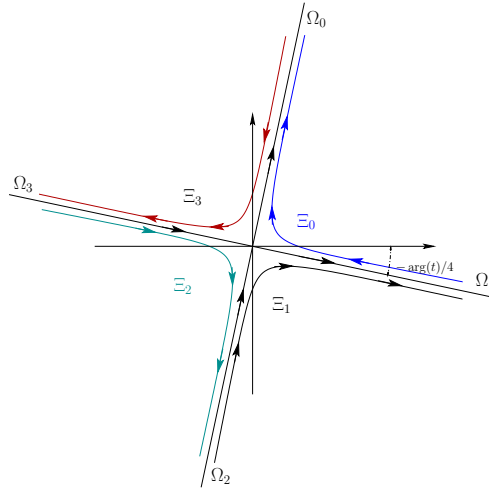
Figure 1. The contours of integration and the asymptotic directions. The contour Ξ_0 is homologically equal to $-\Xi_1 - \Xi_2 - \Xi_3$ and, thus, can be excluded from the definition of the pairing (1.3).

As $z \to \infty$, the weight function is exponentially decaying in four sectors S_j of width $\pi/4$, centered around the rays $\Omega_j = \left\{ z \colon \arg z = -\frac{\arg t}{4} + \frac{\pi(1-j)}{2} \right\}$, $j = 0, 1, 2, 3$. Since the argument of t enters in this formula divided by four, we shall understand $\arg(t) \in [-\pi, 7\pi)$, so that, effectively, t lives on a four-sheeted Riemann surface. We consider the polynomials $\pi_n(z)$ to be integrated on the "cross" formed by the rays Ω_j, where the rays $\Omega_{0,1}$ are oriented outwards (away from the origin) and the rays $\Omega_{2,3}$ – inwards. The corresponding bilinear form is

$$\langle p, q \rangle_{\vec{\varrho}} = \sum_{j=0}^{3} \varrho_j \int_{\Omega_j} p(z) q(z) e^{-Nf(z,t)} dz, \qquad f(z,t) := \frac{1}{2} z^2 + \frac{1}{4} t z^4, \tag{1.1}$$

where $\vec{\varrho} = \{\varrho_0, \varrho_1, \varrho_2, \varrho_3\}$ is a vector of fixed complex numbers chosen to satisfy the so-called "traffic condition"[1]

$$\varrho_1 + \varrho_0 = \varrho_3 + \varrho_2. \tag{1.2}$$

Moreover, since multiplying all the ϱ_j's by a common nonzero constant does not affect the families of orthogonal polynomials, these parameters are only defined modulo multiplication by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Interpreting the coefficients ϱ_j as the traffics along the rays Ω_j, $j = 0, 1, 2, 3$, we observe that the condition (1.2) implies that there is no source/sink of traffic at the origin. Equivalently, this condition means that all the contours of integration can be viewed as starting and ending at $z = \infty$.

Indeed [2], the bilinear form (1.1) can be represented as

$$\langle p, q \rangle_{\vec{\nu}} = \sum_{k=1}^{3} \nu_k \int_{\Xi_k} p(z) q(z) e^{-Nf(z,t)} dz, \tag{1.3}$$

$$\nu_1 = \varrho_1, \qquad \nu_2 = \varrho_3 - \varrho_0, \qquad \nu_3 = -\varrho_0,$$

where Ξ_j, $j = 1, 2, 3$, are simple contours emanating from ∞ along Ω_{j+1} and returning to ∞ along Ω_j with the indices taken modulo 4, see Fig. 1. The weights ν_j on the corresponding contours Ξ_j, $j = 1, 2, 3$, will be also called traffics. Contours with zero traffic are not included in the $\Omega = \cup_{j=1}^3 \Xi_j$ for the bilinear form (1.3).

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[1]The naming is suggested by the fact that the sums of the ϱ_j associated with the incoming and with the outgoing rays are the same.

The generic configuration of the traffic that we consider in this paper is determined by

$$\varrho_1 + \varrho_0 = \varrho_3 + \varrho_2 \neq 0 \qquad \text{and} \qquad \varrho_j \neq 0 \qquad \text{for any} \quad j = 0, 1, 2, 3. \tag{1.4}$$

(The "asymptotic phase portrait" of the generic case is shown on Fig. 2.) However, our methods are equally applicable to any of the five remaining cases. Because of (1.4), without any loss of generality we will assume that

$$\varrho_1 + \varrho_0 = \varrho_3 + \varrho_2 = 1. \tag{1.5}$$

To obtain the asymptotics of the recurrence coefficients (and of the pseudonorm), associated with the polynomials π_n, we use the nonlinear steepest descent analysis of the associated Riemann–Hilbert problem (RHP), see Section 2.1. Our goal is to show that this method is applicable at any $t \in \mathbb{C}^* \setminus \{t_0, t_2\}$. (In this paper, we restrict our attention to the first sheet of the Riemann surface of $t^{\frac{1}{4}}$ only; the remaining sheets can be analyzed similarly.) As it was mentioned earlier, our approach is based on the method of continuation in the (external) parameter space: once the validity of the nonlinear steepest descent method is established for some (regular) point $t = t_*$, say, for some $t_* \in (-\frac{1}{12}, 0)$, it will remain in place as t is continuously deformed in $\mathbb{C}^* \setminus \{t_0, t_2\}$.

We now introduce the recurrence coefficients and the pseudonorms. The orthogonality condition for the monic polynomials $\pi_n(z)$ can be written as

$$\langle \pi_n, z^k \rangle_{\vec{\nu}} = \mathbf{h}_n \delta_{nk}, \qquad k = 0, 1, 2, \ldots, n, \qquad \vec{\nu} = (\nu_1, \nu_2, \nu_3), \tag{1.6}$$

where δ_{ij} denotes the Kronecker delta. The coefficient $\mathbf{h}_n$ can also be written as $\mathbf{h}_n = \langle \pi_n, \pi_n \rangle_{\vec{\nu}}$ and hence is the equivalent of the "square norm" of π_n (in general, it is a complex number). We call $\mathbf{h}_n$ the pseudonorms. The existence of orthogonal polynomials $\pi_n(z)$ is not *a priori* clear. However, if three consecutive monic polynomials exists, then they are related by a three-term recurrence relation

$$\pi_{n+1}(z) = (z - \beta_n)\pi_n(z) - \alpha_n \pi_{n-1}(z), \tag{1.7}$$

where α_n, β_n are called recurrence coefficients. These recurrence coefficients depend on t, N.

If the bilinear pairing is invariant under the map $z \mapsto -z$ then it follows immediately that the orthogonal polynomials are even or odd according to their degree and thus $\beta_n = 0$, $\forall n \in \mathbb{N}$. The remaining recurrence coefficients α_n satisfy

$$\alpha_n[1 + t(\alpha_{n+1} + \alpha_n + \alpha_{n-1})] = \frac{n}{N},$$

which is known in literature as the string equation or the Freud equation [16]. We are interested in the asymptotic limit of α_n, β_n as $N \to \infty$ and $\frac{n}{N} = 1$, so we will use notations $\alpha_n = \alpha_n(t)$, $\beta_n = \beta_n(t)$.

In the case of $t \in (-\frac{1}{12}, 0)$, the asymptotics of α_n, β_n was obtained in [13] as

$$\alpha_n(t) = \frac{\sqrt{1 + 12t} - 1}{6t} + O(n^{-1})$$

and β_n decaying exponentially as $n \to \infty$. In fact, Theorem 1.1 from [13] states that there exists some $n_0 = n_0(t)$, such that $\alpha_n(t)$, $\beta_n(t)$ exist for all $n \geq n_0$ and have the above mentioned asymptotics.

We now outline the content of the paper. The Riemann–Hilbert problem (RHP) representation of the orthogonal polynomials, introduced in [15], is briefly described in Section 2.1. In
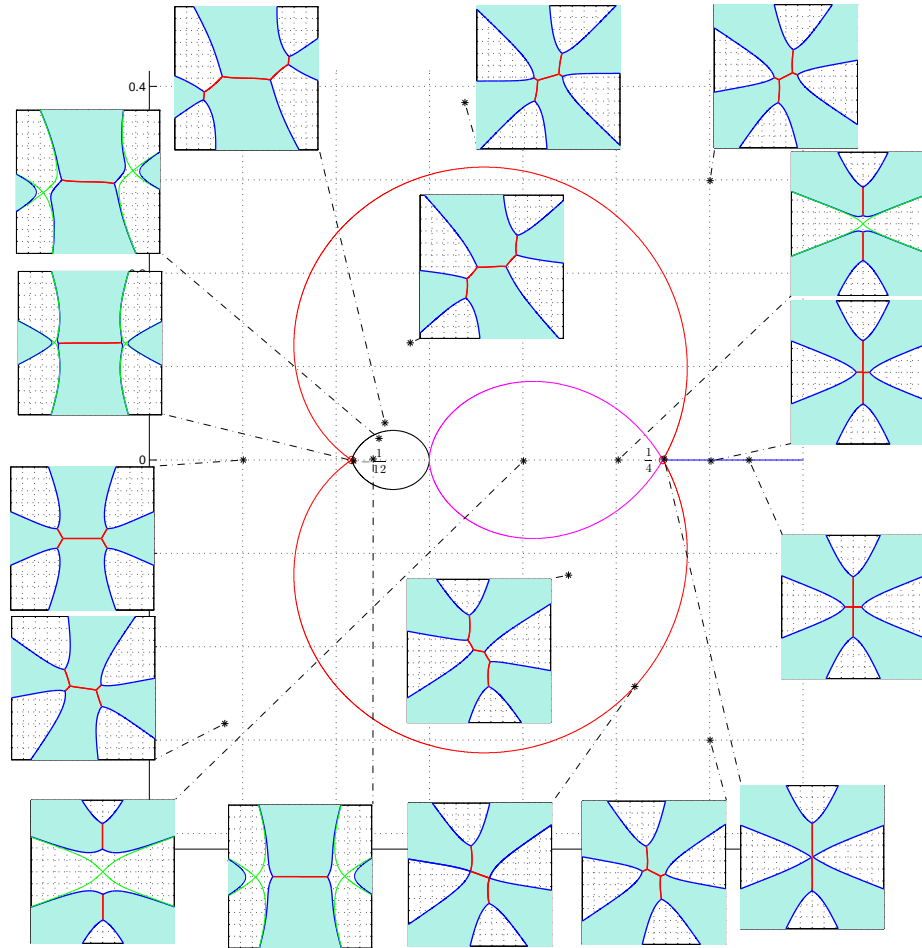
Figure 2. Asymptotic regions on the complex t plane in the case of generic traffic configuration. The smaller contour plots, attached to a particular value of t (shown by the $*$ and connected with the plot by a dashed line), show zero level curves $\Re h(z;t) = 0$ in the complex z-plane. On these plots, the $\Re h(z;t) > 0$ regions are shown by green (darker) colour whereas the $\Re h(z;t) < 0$ regions are shown by white (lighter) colour; the solid red (dark) lines with darker colour on both sides show the main arcs, whereas the solid red (dark) lines with different colors on each side indicate bounded complementary arcs (actual complementary arcs, not shown here, can be obtained by deforming off the corresponding red (dark) line into the lighter region while keeping their endpoints fixed). Four unbounded complementary arcs, also not shown on the panels, are located in the lighter regions. Each of them connects one of the branchpoints with $z = \infty$. The asymptotic directions between the neighboring unbounded complementary arcs differ by the angle $\pi/2$. Complementary arcs for some of the panels are shown on Fig. 3 below. The region inside the curve joining $\frac{1}{4}$ to 0 is of genus 1; inside the curve that joins $t = -\frac{1}{12}$ and $t = 0$ it is of genus 0. Everywhere else the genus is 2, except for the degeneration to genus 0 occurring on the curve that joins $t = -\frac{1}{12}$ and $t = \frac{1}{4}$, and to genus 1 on the ray $[\frac{1}{4}, \infty)$. Local behavior near $t = t_0 = -\frac{1}{12}$ is expressed through Painlevé I transcendents whereas local behavior near $t = t_2 = \frac{1}{4}$ is expressed through Painlevé II transcendents.

the rest of Section 2, we use the nonlinear steepest descent method of Deift–Zhou and the g-function mechanism to find the large N asymptotics of the corresponding RHP. Most of the statements/results of this section are valid for general polynomial potentials $f(z) = f(z; \vec{t})$. In Section 2.2 we define the g function and the associated h function (also known as the modified external field) as solutions to certain scalar RHPs, whose jump contours are yet to be defined.
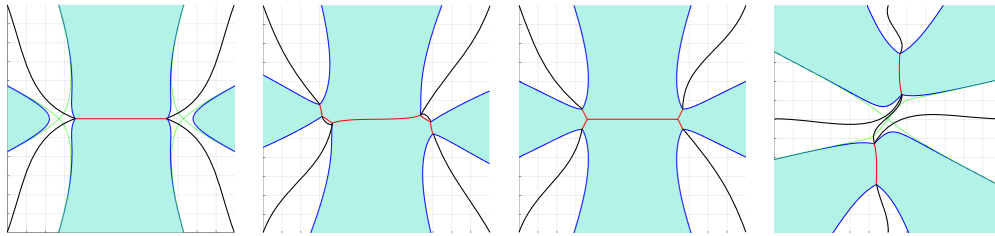
Figure 3. Four panels from Fig. 2 with complementary arcs shown as solid black curves and main arcs shown as solid red (dark) lines, surrounded by green (darker) regions. The panels correspond to the genus 0 (left), 1 (right), and 2 (center) regions, the latter is represented by two panels as there are two topologically different configurations of main and complementary arcs for genus two. Bounded complementary arcs connect branchpoints with each other whereas the unbounded complementary arcs connect some branchpoints with $z = \infty$ along the four asymptotic directions in the corresponding lighter regions. The actual position of the contours is irrelevant (within the lighter regions).

In addition, $\Re g$ must satisfy a set of inequalities, known as sign conditions (sign requirements). Defining the jump contours for g starts with defining their end points, which are also the branchpoints of the hyperelliptic Riemann surface $\mathcal{R} = \mathcal{R}(\vec{t})$, associated with g. Various forms of modulation equations that define these branchpoints λ_j are discussed in Section 2.3. As the genus of $\mathcal{R}(\vec{t})$ can change when we deform $\vec{t}$, we can talk about regions of different genera in the space of parameters $\vec{t}$. Schematic description of the steepest descent method and explicit formulae for the recurrence coefficients and the pseudonorms in the genus L region, $L \in \mathbb{N}$, can be found in Section 2.4. Some basic facts about hyperelliptic Riemann surfaces and Riemann theta functions are also provided there.

In Section 3 we calculate the g function in the regions of different genera. (Our approach of calculating g follows [18, 25], since the branchcuts of g are in $\mathbb{C}$ and the usual variational approach used for the equilibrium problem on the real line is not applicable.) This calculation is still formal, as we postpone the proof of the sign requirements until Section 4. In Section 3.1, following [4], we solve the modulation equations for the branchpoints and calculate the g and h functions in the genus zero region for our quartic potential $f(z, t)$. In Section 3.2, we construct the g and h functions for higher genera regions in the determinantal form and derive new forms of the modulation equations. In particular, we proved Theorem 3.4, stating that the modulation equations for a branchpoint λ is equivalent to $\frac{\partial g}{\partial \lambda} \equiv 0$ for all $z \in \mathbb{C}$ and is equivalent to the fact that real jump constants (that we will refer to as phases) in the RHP for g, see (2.8) and (2.9), are all independent of λ. This statement is valid for a larger class of RHPs, see, for example, [23], as applied to the semiclassical limit of the focusing Nonlinear Schrödinger equation (NLS).

In Section 4 we prove the existence of g function for any $t \in \mathbb{C}^* \setminus \{t_0, t_2\}$ for our quartic potential $f(z; t)$ in the Generic case, that is, we prove that there is a formal g function satisfying the sign conditions. We also derive equations for the breaking curves on complex t plane, separating regions of different genera, and define critical (breaking) points. Following [4], in Section 4.1 we remind the basic steps in proving the existence of g-function in the genus zero region, which contains, in particular, the interval $(-\frac{1}{12}, 0)$. That implies that the asymptotics of the recurrence coefficients is proven on the genus zero region. In Section 4.2 we derived the breaking curve equations for a general polynomial potential $f(z; \vec{t})$ and study properties of breaking curves.

One of the central points of this paper, discussed in Section 4.3, is a proof that once the asymptotics of the recurrence coefficients is established in some region of the t-plane (or even just at one regular point t), the *continuation principle for Boutroux deformations* would establish the existence of the asymptotics of recurrence coefficients for any $t \in \mathbb{C}^* \setminus \{t_0, t_2\}$. That is to say

that if there exists the nonlinear steepest descent solution to the RHP with some regular value of t, then the nonlinear steepest descent solution to the RHP exists for any regular value of $t \in \mathbb{C}^*$; moreover, it can be obtained by continuing the solution along any smooth curve (in the parameter space), connecting the original and the final values of t that avoids $t = 0$ as well as critical values of $t = t_0$ and $t = t_2$. Equivalently, one can say that if the g-function satisfying all the sign conditions (see Section 2.2) is established in some region of the t-plane, it can be continued to any regular value of $t \in \mathbb{C}^*$ while preserving all the above-mentioned sign conditions. In the process of such a continuation in the t-plane, the genus of the underlying Riemann surface $\mathcal{R}(t)$ will undergo the appropriate changes. The continuation principle for Boutroux deformations was (independently) introduced in [1] in the context of general polynomial potential $f(z, \vec{t})$ and in [24] in the context of the semiclassical NLS, where $f = f(z; x, t)$ is a Schwarz-symmetrical function in z that depends on real parameters x, t (space and time). Following [1, 24], we outline the proof of the continuation principle for general potentials $f(z, \vec{t})$. The essentially novel part in this proof, presented separately in Section 5, is the proof that the branchpoints λ_j of $\mathcal{R}(\vec{t})$ are bounded under Boutroux deformations on any compact subset of $\vec{t}$ that preserves the degree of the polynomial $f(z; \vec{t})$. This particular issue was not entirely addressed in [1]. The problem discussed in Section 4.3 is closely related to the problem of existence of a curve with S-property for the polynomial external field that, using the approach of [21], was recently solved in [20].

In Section 6 we present explicit formulae for all the breaking curves for our quartic potential $f(z, t)$ and prove that they separate regions of particular genera, as shown on Fig. 2. In short, we prove the topology of regions on the "asymptotic phase portrait" from Fig. 2. Explicit formulae for the leading order behavior of the recurrence coefficients $\alpha_n(t)$ and the pseudonorms $\mathbf{h}_n(t)$ in the regions of genera one and two are given in Section 7.

2 Steepest descent analysis of the RHP (2.1)

In this section we first (Section 2.1) express the recurrence coefficients $\alpha_n(t)$ and the pseudonorms $\mathbf{h}_n(t)$ through the solution of a matrix RHP Y, and then (Sections 2.2 and 2.3) describe the nonlinear steepest descent approach to find the asymptotics of Y as $n = N \to \infty$. The leading order asymptotics for recurrence coefficients and pseudonorms can be found in Section 2.4. The results of this section can be extended for general potentials $f(z; \vec{t})$ in an obvious way. Our exposition in Section 2.1 and in some other parts of this section follow [4].

2.1 The RHP for recurrence coefficients

It is well known [11] that the existence of the above-mentioned orthogonal polynomials $\pi_n(z)$ is equivalent to the existence of the solution to the following RHP (2.1). More precisely, relation between the RHP (2.1) and the orthogonal polynomials $\pi_n(z)$ is given by the following proposition [13], which has the standard proof (see [11]).

Proposition 2.1. *Define $\nu \colon \Omega \to \mathbb{C}$ (where $\Omega = \bigcup_{j=1}^{3} \Xi_j$) by $\nu(z) = \nu_j$ when $z \in \Xi_j$. Then the solution of the following RHP problem*

$$
\begin{aligned}
Y(z) \quad &\text{is analytic in } \ \mathbb{C} \setminus \Omega, \\
Y_+(z) = Y_-(z) &\begin{pmatrix} 1 & \nu(z)e^{-Nf(z,t)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Omega, \\
Y(z) = \left(\mathbf{1} + O\!\left(z^{-1}\right)\right) &\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad z \to \infty,
\end{aligned}
\tag{2.1}
$$

exists (and it is unique) if and only if there exist a monic polynomial $p(z)$ of degree n and a polynomial $q(z)$ of degree $\leq n - 1$ such that

$$\langle p(z), z^k \rangle_{\vec{\nu}} = 0, \qquad for\ all\quad k = 0, 1, 2, \ldots, n - 1,$$

$$\langle q(z), z^k \rangle_{\vec{\nu}} = 0, \qquad for\ all\quad k = 0, 1, 2, \ldots, n - 2, \qquad and \qquad \langle q(z), z^{n-1} \rangle_{\vec{\nu}} = -2\pi i,$$

where $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$. In that case the solution to the RHP (2.1) is given by

$$Y(z) = \begin{pmatrix} p(z) & C_\Omega\big[p(z)\nu(z)e^{-Nf(z,t)}\big] \\ q(z) & C_\Omega\big[q(z)\nu(z)e^{-Nf(z,t)}\big] \end{pmatrix}, \qquad z \in \mathbb{C} \setminus \Omega,$$

where

$$C_\Omega[\phi] = \frac{1}{2\pi i} \int_\Omega \frac{\phi(\zeta)d\zeta}{\zeta - z}$$

is the Cauchy transform of $\phi(z)$.

Remark 2.2. We remind that the contour Ω in (2.1) should coincide with the contour of integration in the bilinear form (1.3), so if any traffic $\nu_j = 0$ the corresponding contour Ξ_j should not be included in Ω.

If $\pi_{n+1}(z)$, $\pi_n(z)$, $\pi_{n-1}(z)$ are monic orthogonal polynomials then they satisfy the three term recurrence relation (1.7) for certain recurrence coefficients α_n, β_n [9, 22]. The following well known statements (see, for example, [11, 13, 15]) show the connection between the RHP (2.1), the orthogonal polynomials $\pi_n(z)$ and their recurrence coefficients.

Proposition 2.3. *Let $Y^{(n)}(z)$ denote the solution of the RHP (2.1). If we write*

$$Y^{(n)}(z) = \left(\mathbf{1} + \frac{Y_1^{(n)}}{z} + \frac{Y_2^{(n)}}{z^2} + O(z^{-3})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad z \to \infty, \tag{2.2}$$

then the pseudo-norms $\mathbf{h}_n$, appearing in (1.6), and the recurrence coefficients α_n, appearing in (1.7), are given by

$$\mathbf{h}_n = -2i\pi\big(Y_1^{(n)}\big)_{12}, \qquad \alpha_n = \big(Y_1^{(n)}\big)_{12} \cdot \big(Y_1^{(n)}\big)_{21}, \qquad \beta_n = \frac{\big(Y_2^{(n)}\big)_{12}}{\big(Y_1^{(n)}\big)_{12}} - \big(Y_1^{(n)}\big)_{22}. \tag{2.3}$$

Proposition 2.4. *Suppose the RHP (2.1) has solution $Y^{(n)}(z)$. Let (2.2) be its expansion at ∞ and let α_n, β_n be given by (2.3). If $\alpha_n \neq 0$, then the monic orthogonal polynomials $\pi_{n+1}(z)$, $\pi_n(z)$, $\pi_{n-1}(z)$ exist and satisfy the three term recurrence relation (1.7).*

The Deift–Zhou steepest descent analysis is a powerful tool in finding the asymptotic behavior of a matrix RHP with respect to a given asymptotic parameter. In the case of the RHP (2.1), the asymptotic parameter is the large parameter $N = n$. The remaining parameters of the RHP will be called external parameters. In the case of the RHP (2.1), the external parameters are t, $\vec{\nu}$, however, we will consider $\vec{\nu}$ to be fixed in the analysis below. The steepest descent analysis, as applied to the RHP (2.1), will be outlined here. As customary, the RHP undergoes a sequence of modifications into equivalent RHPs until it can be effectively solved (so-called model RHP) while keeping the error terms under control.

● One starts with the RHP (2.1) for Y and seeks an auxiliary scalar function $g = g(z; t)$, called the g-function, which is analytic in $\mathbb{C}$ except for a collection Σ of appropriate contours to be described in Section 2.2 and which behaves like $\ln z + \mathcal{O}(z^{-1})$ near $z = \infty$. The contour Ω

of the RHP (2.1) can be deformed because the RHP (2.1) has an analytic in z jump matrix. If two or more parts of Ω are deformed into a single oriented arc, the traffic on this arc is a signed (according to the orientation) sum of the traffics on the parts of Ω that formed the arc. The parts, if any, of the deformed Ω carrying zero traffic (no integration upon) should be removed from the contour Ω.

- Then we introduce a new matrix

$$T(z) := \mathrm{e}^{-N\ell\frac{\sigma_3}{2}}Y(z)\mathrm{e}^{-N(g(z,t)-\frac{\ell}{2})\sigma_3}, \tag{2.4}$$

where the constant $\ell \in \mathbb{C}$ is to be defined. Direct calculations show that $T(z)$ solves the RHP

$$T(z) \text{ is analytic in } \mathbb{C} \setminus \Omega,$$

$$T_+(z) = T_-(z) \begin{pmatrix} \mathrm{e}^{-\frac{N}{2}(h_+-h_-)} & \nu(z)\mathrm{e}^{\frac{N}{2}(h_++h_-)} \\ 0 & \mathrm{e}^{\frac{N}{2}(h_+-h_-)} \end{pmatrix}, z \in \Omega, \tag{2.5}$$

$$T(z) = \left(\mathbf{1} + O\!\left(z^{-1}\right)\right), \qquad z \to \infty,$$

where $h(z,t) := 2g(z,t) - f(z,t) - \ell$.

- At this point the Deift–Zhou method can proceed provided that the function g, the constant ℓ and the collection of contours Ω into which we have deformed the original contour of integration fulfill a rather long collection of equalities and – most importantly – *inequalities* (sign conditions) that we briefly describe in Section 2.2 (see, for example, [2, 25] for more details). If all these requirements are fulfilled, the leading order asymptotics for the problem (the solution to the model problem) can be obtained in terms of Riemann theta functions on a suitable hyperelliptic Riemann surface of a positive genus (with the case of zero genus not requiring the Riemann theta function).

We construct the g-function (or, more precisely, the corresponding h-function, see (2.5)), not by using the variational principle, as often done for orthogonal polynomials on $\mathbb{R}$, but by satisfying the collection of the above-mentioned equalities. These equalities can be considered as jump conditions in the corresponding scalar RHP for g (or for h). If the obtained g-function does not satisfy the inequality requirements, we need to change the genus of the scalar RHP for g. The continuation principle, see Section 4.3, enables us to trace the correct genus of g through the complex t-plane.

2.2 General requirements on the g-function

The final configuration of the contour Ω can be partitioned into two subsets of oriented arcs that we shall denote by $\mathfrak{M}$ and term *main arcs* (bands), and by $\mathfrak{C}$ and term *complementary arcs* (gaps), $\Omega = \mathfrak{M} \cup \mathfrak{C}$, where all the main arcs are bounded and have a nonzero traffic. The final configuration of Ω must contain all the contours where g is not analytic. This partitioning defines the hyperelliptic Riemann surface $\mathfrak{R}(t)$, whose branchcuts are the main arcs $\gamma_{m,j}$, where $\mathfrak{M} = \cup_{j=0}^{L}\gamma_{m,j}$. It is subordinated to the list of requirements for the function g (or for the corresponding function h), associated with the Riemann surface $\mathfrak{R}(t)$ of genus L, given in the following two subsections. For example, on Fig. 2, main arcs on each panel are the (red) arcs separating two blue (darker) regions. The genus of the problem is the number of the main arcs reduced by one. A finite complementary arc (these arcs are not usually shown on the panels) is a bounded curve within a white (lighter) region (or on its boundary), connecting the corresponding endpoints of the main arcs, see, for example, the panel in the low left corner. On the two panels in the middle of Fig. 2, finite complementary arcs are the (red) arcs, separating the blue (darker) and the white (lighter) regions. On some panels of positive genus (e.g., panels 2, 3 from the bottom on the left edge) there are no finite complementary arcs.

2.2.1   Equality requirements and formula for g

1) $g(z)$ (to shorten notations, we will often drop the t variable) is analytic in $\mathbb{C} \setminus \Sigma$, where $\Sigma = (\mathfrak{M} \cup \mathfrak{C})$, and has the asymptotic behaviour

$$g(z) = \ln z + \mathcal{O}\big(z^{-1}\big), \qquad z \to \infty; \tag{2.6}$$

2) $g(z)$ is analytic along all the *unbounded* complementary arcs except for exactly one *unbounded* complementary arc which we will denote by γ_0 (we assume γ_0 is oriented away from infinity), where

$$g_+(z) - g_-(z) = 2i\pi, \qquad z \in \gamma_0 \subset \mathfrak{C} \tag{2.7}$$

(note that the function $\mathrm{e}^{Ng(z)}$ from (2.4) is analytic across the unbounded complementary arc since $N = n \in \mathbb{N}$ by definition). In fact, γ_0 could be any simple contour connecting ∞ with a point on $\mathfrak{M}$ that has no other intersections with Ω;

3) on each *bounded* complementary arc $\gamma_{c,j}$, $j = 1, 2, \ldots, L$, the function $g(z)$ has a constant jump

$$g_+(z) - g_-(z) = 2\pi i \eta_j, \qquad \eta_j \in \mathbb{R}, \qquad z \in \gamma_{c,j} \subset \mathfrak{C} \tag{2.8}$$

(note that there are no bounded complementary arcs in the genus zero case $L = 0$);

4) across each main arc $\gamma_{m,j}$, $j = 0, 1, 2, \ldots, L$, the function $g(z)$ has a jump

$$g_+(z) + g_-(z) = f(z) + \ell + 2\pi i \varpi_j, \qquad \varpi_j \in \mathbb{R}, \qquad z \in \gamma_{m,j} \subset \mathfrak{M}, \tag{2.9}$$

where $\varpi_0 = 0$. We stress that the constant ℓ is the same for all the main arcs;

5) the limiting values $g_\pm(z)$ along all the contours belong to L^2_{loc}.

Remark 2.5. The assumption that all the main and complementary arcs are simple smooth curves that can intersect each other only at the endpoints would have simplified the following discussion. In reality, though, main (complementary) arcs could intersect each other in a finite number of interior points forming, for example, crosses, "Y"-shaped contours (whiskers), etc., see, for example, various panels on Fig. 2. If an endpoint of a main arc $\gamma_{m,j-1}$ is in the interior of the neighboring main arc $\gamma_{m,j}$ for some $j = 1, \ldots, L$, then there is no complementary arc $\gamma_{c,j}$ that connects these two main arc. Thus, the number of the complementary arcs in (2.8) can be less than L. However, in this case, the constant ϖ_j from (2.9) on $\gamma_{m,j}$ will take different values on the different parts of this arc (separated by the intersection point). This jump condition will be equivalent to the situation where the constant ϖ_j is the same on all of the $\gamma_{m,j}$, but there is another jump (2.8) coming from the arc $\gamma_{c,j}$, which is the part of $\gamma_{m,j}$ that connects one of the endpoints with the point of intersection. Thus, we can assume that there are always L arcs $\gamma_{c,j}$ in (2.8), with the understanding that only proper complementary arcs have to satisfy the sign requirements from Section 2.2.2.

Remark 2.6. Assuming that the contours constituting $\mathfrak{M}$ and $\mathfrak{C}$, as well as all the constants $\ell, \eta_j, \varpi_j, j = 1, 2, \ldots, L$, are known, the conditions 1-5 form an RHP for the function $g(z)$. If a branch of $\ln z$ in (2.6) is fixed, that RHP would have a unique solution (constructed below), provided that the constants can be chosen so that the first requirement (2.6) is satisfied.

We start the construction of $g(z)$ by removing the jump on the unbounded contour γ_0. Let λ_{2j}, λ_{2j+1} be the beginning and the end points of the (oriented) main arc $\gamma_{m,j}$, $j = 0, \ldots, L$. We define the radical

$$R(z) = \sqrt{\prod_{j=0}^{2L+1} (z - \lambda_j)}, \tag{2.10}$$

where the branchcuts of $R(z)$ are $\gamma_{m,j}$, $j = 0, 1, \ldots, L$, and the branch of R is defined by $\lim_{z\to\infty} \frac{R(z)}{z^{L+1}} = 1$. The hyperelliptic Riemann surface $\mathfrak{R}(t)$ is the Riemann surface of $R(z)$. Let $\widetilde{\omega} = \frac{\widetilde{P}(\zeta)d\zeta}{R(\zeta)}$ be a normalized meromorphic differential of the third kind with two simple poles at $\infty_\pm$ with residues ± 1 respectively (i.e., $\widetilde{P}(z)$ is monic of degree L). It is well known [14] that the above conditions define $\tilde{P}(z)$ uniquely. The definition of "normalized" means that

$$\int_{\gamma_{m,j}} \frac{\widetilde{P}(\zeta)d\zeta}{R_+(\zeta)} = 0 \qquad \text{for all} \quad j = 1, \ldots, L. \tag{2.11}$$

Let

$$u(z) = \int_{\lambda_0}^z \frac{\widetilde{P}(\zeta)}{R(\zeta)} d\zeta, \tag{2.12}$$

where λ_0 is the beginning of the main arc $\gamma_{m,0}$. It will be also convenient to choose λ_0 as the (finite) endpoint of the contour γ_0. Then $u(z)$ is analytic in $\mathbb{C} \setminus (\mathfrak{M} \cup \gamma_0)$, has analytic limiting values on $\mathfrak{M} \cup \gamma_0$ (except, possibly, the branchpoints) and satisfies

$$u(\lambda_0) = 0, \qquad u_+(z) - u_-(z) = 2i\pi, \quad z \in \gamma_0, \qquad u_+(z) + u_-(z) = 0, \quad z \in \gamma_{m,0},$$

and $u(z) - \ln z$ is analytic at $z = \infty$. We will be looking for $g(z)$ in the form $g(z) = u(z) + v(z)$. Then $v(z)$ must be analytic in $\bar{\mathbb{C}}$ except for the main arcs $\mathfrak{M}$ and bounded complementary arcs $\cup_{j=1}^L \gamma_{c,j}$, where, according to (2.9), (2.7), it satisfies jump conditions

$$v_+(z) + v_-(z) = f(z) - [u_+(z) + u_-(z)] + \ell + 2\pi i \varpi_j, \qquad v_+(z) - v_-(z) = 2\pi i \eta_j$$

on the corresponding contours. By Sokhotski–Plemelj formula,

$$v(z) = \frac{R(z)}{2\pi i} \Bigg[\int_{\mathfrak{M}} \frac{f(\zeta) - [u_+(\zeta) + u_-(\zeta)] + \ell}{(\zeta - z)R(\zeta)_+} d\zeta$$
$$+ \sum_{j=1}^L \left(\int_{\gamma_{m,j}} \frac{2\pi i \varpi_j d\zeta}{(\zeta - z)R(\zeta)_+} + \int_{\gamma_{c,j}} \frac{2\pi i \eta_j d\zeta}{(\zeta - z)R(\zeta)} \right) \Bigg].$$

The solution $g(z)$ to the RHP from Remark 2.6 is then given by

$$g(z) = u(z) + \frac{R(z)}{2\pi i} \Bigg[\int_{\mathfrak{M}} \frac{f(\zeta) - [u_+(\zeta) + u_-(\zeta)] + \ell}{(\zeta - z)R(\zeta)_+} d\zeta$$
$$+ \sum_{j=1}^L \left(\int_{\gamma_{m,j}} \frac{2\pi i \varpi_j d\zeta}{(\zeta - z)R(\zeta)_+} + \int_{\gamma_{c,j}} \frac{2\pi i \eta_j d\zeta}{(\zeta - z)R(\zeta)} \right) \Bigg]$$
$$= u(z) + \frac{R(z)}{4\pi i} \Bigg[\oint_{\hat{\mathfrak{M}}} \frac{f(\zeta) - 2u(\zeta) + \ell}{(\zeta - z)R(\zeta)} d\zeta$$
$$+ \sum_{j=1}^L \left(\oint_{\hat{\gamma}_{m,j}} \frac{2\pi i \varpi_j d\zeta}{(\zeta - z)R(\zeta)} + \oint_{\hat{\gamma}_{c,j}} \frac{2\pi i \eta_j d\zeta}{(\zeta - z)R(\zeta)} \right) \Bigg], \tag{2.13}$$

where $\hat{\gamma}_{m,j}$, $\hat{\gamma}_{c,j}$ is a negatively oriented loop around $\gamma_{m,j}$, $\gamma_{c,j}$ respectively, with the latter traversing both sheets of the Riemann surface $\mathfrak{R}(t)$, $j = 1, \ldots, L$, and $\hat{\mathfrak{M}}$ denotes a negatively oriented loop around $\mathfrak{M}$ that (because of the $u(z)$ term) passes through λ_0. Here z is chosen outside all the loops.

Remark 2.7. Note that the jumps of g are analytic (in z) functions; thus, the interior of any main or complementary arc can be smoothly deformed without affecting the value of $g(z)$, as long as the deformed arc does not pass through z. Moreover, the limiting values $g_\pm(z)$ are analytic functions on the contour Ω with the exception of the branchpoints of the Riemann surface $\mathfrak{R}(t)$ (endpoints of the main and complementary arcs).

Remark 2.8. Similarly to g, the function

$$h = 2g - f - \ell \tag{2.14}$$

can be considered as the (unique) solution of the scalar RHP with the jumps

$$h_+(z) - h_-(z) = 4\pi i \eta_j, \quad z \in \gamma_{c,j}, \qquad h_+(z) + h_-(z) = 4\pi i \varpi_j, \quad z \in \gamma_{m,j},$$
$$h_+(z) - h_-(z) = 4i\pi, \quad z \in \gamma_0, \tag{2.15}$$

across all the main and all the bounded complementary arcs and with the asymptotic behavior

$$h(z) = -f(z) - \ell + 2\ln z + \mathcal{O}(z^{-1}), \qquad z \to \infty. \tag{2.16}$$

It follows immediately from (2.15) that $\Re h(z)$ is continuous across the complementary arcs $\mathfrak{C}$. Also, (2.13) implies that

$$h(z) = \frac{R(z)}{2\pi i} \left[\oint_{\hat{\mathfrak{M}}} \frac{f(\zeta) + \ell - 2u(\zeta)}{(\zeta - z)R(\zeta)} d\zeta \right.$$
$$\left. + \sum_{j=1}^{L} \left(\oint_{\hat{\gamma}_{m,j}} \frac{2\pi i \varpi_j d\zeta}{(\zeta - z)R(\zeta)} + \oint_{\hat{\gamma}_{c,j}} \frac{2\pi i \eta_j d\zeta}{(\zeta - z)R(\zeta)} \right) \right], \tag{2.17}$$

where z is inside the loop $\hat{\mathfrak{M}}$ but outside all other loops. Notice that ℓ can be removed from (2.17) since $\oint_{\hat{\mathfrak{M}}} \frac{d\zeta}{(\zeta-z)R(\zeta)} = 0$ when z is inside $\hat{\mathfrak{M}}$.

2.2.2 Inequality (sign) requirements (or sign distribution requirements) for h and the modulation equation

The following sign distribution requirements must be satisfied:

1) in the interior of every bounded complementary arc $\gamma_{c,j} \subset \mathfrak{C}$, $j = 1, \ldots, L$, as well as in the interior of every unbounded complementary arc, we have $\Re h(z) < 0$;

2) if z_0 is an interior point of a main arc $\gamma_{m,j} \subset \mathfrak{M}$, $j = 0, 1, \ldots, L$ then on both sides $\gamma_{m,j}$ in close proximity of z_0 we have $\Re h(z) > 0$.

Remark 2.9. The second requirement together with (2.15) imply that $\Re h(z) \equiv 0$ on $\mathfrak{M}$. Thus, the jump conditions (2.15) imply that $\Re h(z)$ *is continuous everywhere in* $\mathbb{C}$.

Remark 2.10. We shall call the above mentioned case *regular*, with the same connotation as in [10]. Violation of the above mentioned strict inequalities is allowed in exceptional (non-regular) cases at no more than a finite number of interior points of the corresponding main and/or complementary arcs $\Omega = \mathfrak{C} \cup \mathfrak{M}$. Let $z_* \in \Omega$ denote one of such points. Then $\Re h(z_*) = 0$. Indeed, this is true when $z_* \in \mathfrak{M}$, see Remark 2.9; otherwise, when $z_* \in \mathfrak{C}$, this is implied by the continuity of $\Re h(z)$. Moreover, the above requirements imply that there are at least four zero level curves of $\Re h(z) = 0$, emanating from $z = z_*$ (pinching).

2.3   Modulation equations

The sign distribution requirements of Section 2.2.2 imply that $\Re h(z)$ should attain both positive and negative values in a vicinity of any branchpoint λ, where a main and a complementary arcs meet. On the other hand, if an endpoint λ of a main arc is an interior point of another main arc, then there are at least three zero level curves of $\Re h(z)$ emanating from λ. Combined with the requirement $g_\pm \in L^2_{\mathrm{loc}}(\Omega)$, we see that

$$\Re h(z) = \mathcal{O}(z - \lambda_k)^{\frac{3}{2}}, \qquad z \to \lambda_k, \tag{2.18}$$

at every branchpoint λ_k, $k = 0, 1, \ldots, 2L + 1$. In other words, $\Re h(z) \to 0$ at least at the order $(z - \lambda_k)^{\frac{3}{2}}$ (or faster) as $z \to \lambda_k$.

  Consider equation (2.17) in the light of the latter requirement. Moving z to a vicinity of a branchpoint λ_k, $k = 0, \ldots, 2L + 1$, one has to cross some loops $\hat{\gamma}_{m,j}$, $\hat{\gamma}_{c,l}$, which yield terms $2\pi i \omega_j$ and $\pm 2\pi i \eta_l$ respectively, $j, l = 1, \ldots, L$. These terms do not affect $\Re h(z)$. It now follows from (2.17) that $\Re h(z) = \mathcal{O}(z - \lambda_k)^{\frac{1}{2}}$, $k = 0, 1, \ldots, 2L + 1$. Then, the conditions (2.18) can be equivalently restated as

$$\oint_{\hat{\mathfrak{M}}} \frac{f(\zeta) + \ell}{(\zeta - \lambda_k) R(\zeta)_+} d\zeta + \sum_{j=0}^{L} \oint_{\hat{\gamma}_{m,j}} \frac{2\pi i \varpi_j d\zeta}{(\zeta - \lambda_k) R(\zeta)_+}$$
$$+ \sum_{j=1}^{L} \oint_{\hat{\gamma}_{c,j}} \frac{2\pi i \eta_j d\zeta}{(\zeta - \lambda_k) R(\zeta)} + \int_{\widetilde{\gamma}_0} \frac{2\pi i d\zeta}{(\zeta - \lambda_k) R(\zeta)} = 0,$$

where $k = 0, 1, \ldots, 2L + 1$ and $\widetilde{\gamma}_0$ is a positively oriented loop around γ_0. Assuming that all the constants ℓ, η_j, ω_j, $j = 1, 2, \ldots, L$, are known (see Section 3.2), the system (2.3) is a system of $2L + 2$ equations (2.18) for the $2L + 2$ unknown branchpoints λ_k. It is known as the system of *modulation equations* that governs the location of the endpoints λ_k provided the genus L of the corresponding hyperelliptic Riemann surface $\mathfrak{R}(t)$ is given.

  The modulation equations (2.18) guarantee that there are *at least* three zero level curves of $\Re h$ emanating from each branch-point λ_k, $k = 0, 1, \ldots, 2L + 1$. They do not guarantee, however, that the sign requirements for h are satisfied. The latter requires a particular choice of the genus L of $\mathfrak{R}(t)$ depending on t.

  The modulation equations (2.18) imply that solutions of the RHPs for g and for h commute with differentiation, since the limiting values of $g_\pm$, $h_\pm$ and $g'_\pm$, $h'_\pm$ along Ω belong to $L^2_{\mathrm{loc}}(\Omega)$. Thus, the modulation equations (2.18) can be equivalently stated as

$$h'(z) = \mathcal{O}\big(\sqrt{z - \lambda_k}\big), \qquad z \to \lambda_k, \qquad k = 0, 1, \ldots, 2L + 1$$

(note that $h'(z)$ is not continuous on the main arcs). The function h' can be obtained as the solution of the RHP for $h'(z)$ with the following jump conditions and the asymptotics at infinity

$$h'_+(z) + h'_-(z) = 0, \qquad z \in \gamma_{m,j}, \qquad j = 0, 1, \ldots, L,$$
$$h'(z) = -f'(z) + \frac{2}{z} + \mathcal{O}\big(z^{-2}\big), \qquad z \to \infty. \tag{2.19}$$

In view of (2.14), the RHP for $g'(z)$ (the jump conditions and asymptotics at infinity) is given by

$$g'_+(z) + g'_-(z) = f'(z), \qquad z \in \gamma_{m,j}, \qquad j = 0, 1, \ldots, L,$$
$$g'(z) = \frac{1}{z} + \mathcal{O}\big(z^{-2}\big), \qquad z \to \infty.$$

Then solution to (2.3) is given by

$$g'(z) = \frac{R(z)}{4\pi i} \oint_{\hat{\mathfrak{M}}} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)} d\zeta, \tag{2.20}$$

where z is outside the contour of integration. Using the residue theorem, we see that

$$h'(z) = \frac{R(z)}{2\pi i} \oint_{\hat{\mathfrak{M}}} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)} d\zeta = R(z)M(z), \tag{2.21}$$

where z is now *inside* the contour $\hat{\mathfrak{M}}$. Equation (2.21) shows that $M(z)$ is a polynomial of degree $\deg f' - L - 1$. If $R(z)$ (that is, the Riemann surface $\mathfrak{R}(t)$) is given, then $M(z)$ is defined by the asymptotics in (2.19). This argument, in particular, shows that for a quartic polynomial f the genus L of $\mathfrak{R}(t)$ cannot exceed 2. In fact, our Riemann surface $\mathfrak{R}(t)$ is the nodal curve

$$y^2(z) = h'^2(z) = M^2(z) \prod_{j=0}^{2L+1} (z - \lambda_j) = P(z), \tag{2.22}$$

where $P(z)$ is a polynomial and $\deg P = 2\deg f - 2$. In case of a general potential $f(z; \vec{t})$, the genus L is bounded by

$$L \leq \deg f - 2. \tag{2.23}$$

The modulation equations now can be again recast as the requirements that: i) the meromorphic differential $h'(z)dz$ on the hyperelliptic surface $\mathfrak{R}(t)$ has the asymptotics as required in (2.19), and ii) the Riemann surface $\mathcal{R}(t)$ satisfies the *Boutroux condition* for all $t \in \mathbb{C}^*$, that is, all the periods of $h'(z)dz$ are purely imaginary (for all $t \in \mathbb{C}$). The corresponding equations can be written as

$$\frac{1}{4\pi i} \oint_{\hat{\mathfrak{M}}} \frac{\zeta^k f'(\zeta)d\zeta}{R(\zeta)} = -\delta_{k,L+1}, \qquad \Re \oint_{\hat{\gamma}_{m,j}} h'(\zeta)d\zeta = 0, \qquad \Re \oint_{\hat{\gamma}_{c,j}} h'(\zeta)d\zeta = 0, \tag{2.24}$$

where $k = 0, 1, \ldots, L+1$, $j = 1, \ldots, L$ and $\delta_{k,l}$ is the Kronecker delta. The first $L+2$ equations in (2.24), also known as moment conditions, express the fact that the large z asymptotics of $g'(z)$ is given by (2.3) (requirement i)). The remaining equations (2.24), also known as integral conditions [25], express the requirement ii).

Equivalence of modulation equations (2.18) and (2.24) follows from the fact that the solution of the RHP for g, h commutes with the differentiation and from the existence and uniqueness of the solutions of the RHPs for g, h, g', h'.

2.4 Schematic conclusion of the steepest descent analysis

In this subsection we briefly describe the construction of the leading order asymptotics in the case of a positive genus. The corresponding construction for the genus zero case was given in [4]. For convenience of the reader, we repeat here some common parts (for any genus).

Fix some $t \in \mathbb{C}^* \setminus \{t_0, t_2\}$. Let $\vec{\lambda} = (\lambda_0, \ldots, \lambda_{2L+1})$ be a solution to the modulation equations with some $L \in \mathbb{N} \cup \{0\}$. Deform the contour Ω in such a way that $\lambda_k \in \Omega$ for all $k = 0, \ldots, 2L+1$ and partition it into a collection of the main and complementary arcs as discussed in Section 2.2. If the g-function (2.13) satisfies all the requirements of Section 2.2 (equalities and inequalities), then we say that t belongs to the genus L region of $\mathbb{C}$ and proceed with the construction of the leading order asymptotics of the recurrence coefficients, as outlined below. The choice of the correct genus for a given $t \in \mathbb{C}^* \setminus \{t_0, t_2\}$ will be discussed in Section 4.3.

The remaining steps in the steepest descent analysis involve inserting additional contours, the *lenses*, which enclose each main arc and lie entirely within the $-\Re h < 0$ region, the so-called sea. (On the panels in Fig. 2, these regions (the sea) are shown in blue (darker) color.) One then re-defines $T(z)$ from the RHP (2.5) within the regions between the main arc and its corresponding lens by using the factorization

$$\begin{pmatrix} a & d \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}d^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ad^{-1} & 1 \end{pmatrix} \tag{2.25}$$

of the jump matrices of T so that

$$T_+(z) = T_-(z) \begin{bmatrix} 1 & 0 \\ \nu_m^{-1}e^{-Nh_-} & 1 \end{bmatrix} \begin{bmatrix} 0 & \nu_m \\ -\nu_m^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \nu_m^{-1}e^{-Nh_+} & 1 \end{bmatrix},$$

where ν_m is the (constant!) value of $\nu(z)$ on a given main arc γ_m under consideration. (For simplicity, we use here notation γ_m instead of $\gamma_{m,j}$ for a given particular main arc.) Therefore, defining $\widehat{T}(z)$ as T outside of the lenses and by

$$\widehat{T}(z) := T(z) \begin{bmatrix} 1 & 0 \\ \mp\nu_m^{-1}e^{-Nh} & 1 \end{bmatrix}$$

in the regions within the lenses and adjacent to the $\pm$ sides of γ_m one obtains a new RHP for $\widehat{T}(z)$. In this RHP, the jump matrices on the lenses and on the complementary arcs turn out to be exponentially close to the identity in any L^p, $(1 \leq p < +\infty)$ as $N \to \infty$ due to the inequality requirements of Section 2.2.2, see, for example, [2] for details. (Because of the asymptotics (2.16), where $f(z)$ is a polynomial, we can always assume that $\sup_{z\in\Omega} \Re h(z) < 0$ for all sufficiently large $z \in \Omega$; that establishes the above statement for any unbounded complementary arc.) The problem corresponding to the remaining jumps can be solved exactly. The fact that the neglected jump matrices do not approach the identity matrix $\mathbf{1}$ in L^∞ is addressed by the construction of appropriate local solutions of the RHP called parametrices [2, 10]. The type of local RHP depends on the behavior of $h(z)$ near the endpoints. For regular values of t (away from the breaking curves) we use the standard Airy parametrix at any branchpoint λ_j where a main arc meets a complementary arc. This Airy parametrix should be appropriately modified [2] if at least three main arcs come together at λ_j. Finally, if t is a regular point on the breaking curve and if λ is one of the interior points of $\mathfrak{M} \cup \mathfrak{C}$, where the sign requirement of Section 2.2.2 is violated, the local parametrix near λ_j is of the same type as in [3]. Consider, for simplicity, a regular point t. In the final steps of the approximation we fix sufficiently small disks $\mathbb{D}_j$, around the endpoints λ_j, $j = 0, \ldots, L+1$, and define a suitable approximate solution

$$\Phi(z) := \begin{cases} \Phi_{\text{ext}}(z) & \text{for } z \text{ outside } \cup_j \mathbb{D}_j, \\ \Phi_{\text{ext}}(z)\mathcal{P}_j(z) & \text{inside } \mathbb{D}_j, \end{cases}$$

such that the error matrix $\mathcal{E}(z) := \widehat{T}(z)\Phi^{-1}(z)$ solves a small-norm Riemann–Hilbert problem (as $N \to \infty$) and thus can be – in principle – completely solved in Neumann series. Here by $\mathcal{P}_j(z)$ we denote the parametrices near the endpoints λ_j respectively, $j = 0, \ldots, L+1$. The RHP for $\Phi_{\text{ext}} = \Psi_0$ ("model solution" or "exterior parametrix") will be discussed below, see (2.26).

Once we have achieved a suitable approximation for $\widehat{T}(z)$, the recurrence coefficients for the orthogonal polynomials can and will be recovered via Proposition 2.3:

$$\mathbf{h}_n = -2i\pi e^{N\ell} (T_1)_{12}, \qquad \alpha_n = (T_1)_{12}(T_1)_{21}, \qquad \beta_n = \frac{(T_2)_{12}}{(T_1)_{12}} - (T_1)_{22}.$$

Here we used (2.4) and the fact that $\widehat{T}(z)$ near ∞ equals $T(z)$ (since we are in the exterior region) and has expansion

$$\widehat{T}(z) = T(z) = \mathbf{1} + \frac{T_1}{z} + \frac{T_2}{z^2} + \cdots, \qquad z \to \infty.$$

The matrix entry $T_{1,2}$ can be obtained from the corresponding expansion of $\Phi_{\text{ext}}(z)$ near infinity, to within the error determined by the error matrix $\mathcal{E} = \widehat{T}\Phi^{-1}$. In the case of a regular t, the standard error analysis (which we do not report here) shows that $\mathcal{E}$ introduces an error of order $\mathcal{O}(N^{-1})$ that is uniform on compact subsets of regular t.

The higher genus case (i.e., more than one main arc) with real potentials and on the real line was first fully treated in [10] and was later extended to the complex plane and complex potentials in [2]. Consider first the case of interlacing main and complementary arcs, i.e., the region inside the two Schwarz-symmetrical curves connecting the points $t_0 = -\frac{1}{12}$ and $t_2 = \frac{1}{4}$ on Fig. 2 (we, obviously, exclude the genus zero region bounded by contours connecting the points $t_0 = -\frac{1}{12}$ and $t = 0$).

Remark 2.11. Let ρ_j denote the traffic on the main arc $\gamma_{m,j}$, $j = 0, 1, \ldots, L$. Without any loss of generality, see Remark 2.2, we assume that $\rho_0 = 1$. Alternatively, we can replace ρ_j by ρ_j/ρ_0 in the formulae below.

Let

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

denote the Pauli matrices. Following the standard arguments, we obtain the following RHP (model problem)

$$
\begin{aligned}
&\Psi_0(z) && \text{is analytic in } \mathbb{C} \setminus \widehat{\Omega}, \\
&\Psi_{0+}(z) = \Psi_{0-}(z)e^{(2\pi i \varpi_j N + \ln \rho_j)\sigma_3} i\sigma_2 && \text{on } \gamma_{m,j}, \quad j = 0, 1, \ldots, L, \\
&\Psi_{0+}(z) = \Psi_{0-}(z)e^{-2\pi i \eta_j N\sigma_3} && \text{on } \gamma_{c,j}, \quad j = 1, \ldots, L, \\
&\Psi_0(z) = \mathbf{1} + \mathcal{O}(z^{-1}) && \text{as } z \to \infty, \\
&\Psi_0(z) = \mathcal{O}\big((z - \lambda_k)^{-\frac{1}{4}}\big), && z \to \lambda_k, \quad k = 0, 1, \ldots, 2L+1
\end{aligned}
\tag{2.26}
$$

for the "model solution" or "exterior parametrix" matrix $\Psi_0(z)$. Here $\widehat{\Omega}$ is the collection of all main and all bounded complementary arcs, the real constants ϖ_j, η_j are given by (2.15) and the constants $\rho_j \in \mathbb{C}^*$ are equal to the traffic on the oriented main arc $\gamma_{m,j}$, obtained by deformation of the original contours into Ω. In the case with whiskers, that is, when three main arcs come together, some jumps $\gamma_{c,j}$ are created according to Remark 2.5. In this case, the diagonal jump matrix on $\gamma_{c,j}$ in (2.26) will have additional $\ln \rho$ terms in the exponential.

2.5 Solution of the model problem

To solve the model RHP (2.26) in terms of Riemann theta-functions we follow the approach of [12], see also [25] (for an alternative approach in the case of real main arcs see [19]). First, we use the transformation $\widetilde{\Psi}_0(z) = e^{-N\widetilde{g}(\infty)\sigma_3}\Psi_0(z)e^{N\widetilde{g}(z)\sigma_3}$ in order to remove the jumps $e^{-2\pi i \eta_j N\sigma_3}$ of $\Psi_0(z)$ on the complementary arcs. The function $\widetilde{g}$ is analytic in $\mathbb{C} \setminus \widehat{\Omega}$ and have constant jumps on $\widehat{\Omega}$. If V denotes the jump matrix for $\Psi_0(z)$, then the jump matrix $\widetilde{V}$ for $\widetilde{\Psi}_0(z)$ is

$$\widetilde{V} = e^{-N\widetilde{g}_-(z)\sigma_3}Ve^{N\widetilde{g}_+(z)\sigma_3}. \tag{2.27}$$

To remove the jumps on $\gamma_{c,j}$, we require that

$$\widetilde{g}_+ - \widetilde{g}_- = 2\pi i \eta_j \quad \text{on } \gamma_{c,j} \qquad \text{and} \qquad \widetilde{g}_+ + \widetilde{g}_- = 2\pi i \varkappa_j \quad \text{on } \gamma_{m,j}, \quad j = 1, \ldots, L,$$

where the complex constants $\varkappa_j$ are defined by the requirement that $\widetilde{g}$ is analytic at $z = \infty$. Then $\widetilde{\Psi}_0(z)$ solves the RHP

$$
\begin{aligned}
&\widetilde{\Psi}_0(z) && \text{is analytic in } \mathbb{C} \setminus \widehat{\Omega}, \\
&\widetilde{\Psi}_{0+}(z) = \widetilde{\Psi}_{0-}(z)e^{(2\pi i N \widetilde{\varpi}_j + \ln \rho_j)\sigma_3} i\sigma_2 && \text{on } \gamma_{m,j}, \quad j = 0, 1, \ldots, L, \\
&\widetilde{\Psi}_0(z) = \mathbf{1} + O(z^{-1}) && \text{as } z \to \infty, \\
&\widetilde{\Psi}_0(z) = \mathcal{O}(z - \lambda_k)^{-\frac{1}{4}}, && z \to \lambda_k, \quad k = 0, 1, \ldots, 2L + 1,
\end{aligned}
\tag{2.28}
$$

where $\widetilde{\varpi}_j = \varpi_j + \varkappa_j$, $j = 1, \ldots, L$, and $\widetilde{\varpi}_0 = 0$.

The solution to the RHP (2.28) is known [12, 25]. To present it, we need to introduce the Riemann theta function associated with the hyperelliptic Riemann surface $\mathcal{R}(t)$. Let $A_j = \hat{\gamma}_{m,j}$, $j = 1, \ldots, L$ be A-cycles of $\mathcal{R}$, whereas the (standard) set of B-cycles B_j consists of the properly oriented loops passing through the branchcuts $\gamma_{m,0}$ and $\gamma_{m,j}$. The basis $\vec{\omega} = (\omega_1, \ldots, \omega_L)$ of normalized holomorphic differentials, dual to the A-cycles, is defined by

$$
\int_{\hat{\gamma}_{m,j}} \omega_k = \delta_{jk}, \qquad j, k = 1, \ldots, L,
\tag{2.29}
$$

where δ_{jk} denotes the Kronecker symbol. Each of these differentials ω_j has the form $\omega_j = \frac{P_j(z)}{R(z)} dz$, where $P_j(z)$ is a polynomial of degree less than L. The Riemann period matrix is defined

$$
\tau = (\tau_{kj}) = \left(\int_{B_j} \omega_k \right), \qquad k, j = 1, \ldots, L.
\tag{2.30}
$$

τ is known to be symmetric and have positive definite imaginary part. The Riemann theta function is defined as

$$
\Theta(s) = \sum_{l \in \mathbb{Z}^{2N}} e^{2\pi i(l,s) + \pi i(l, \tau l)}, \qquad s \in \mathbb{C}^{2N}.
$$

It satisfies

$$
\begin{aligned}
&\text{(a)} \quad \Theta(s) = \Theta(-s), && \text{(b)} \quad \Theta(s + e_j) = \Theta(s), \\
&\text{(c)} \quad \Theta(s + \tau_j) = e^{-2\pi i s_j - \pi i \tau_{jj}} \Theta(s),
\end{aligned}
\tag{2.31}
$$

where e_j is the j-th column of $\mathbf{1}_{L \times L}$ and $\tau_j = \tau e_j$. The set $\Lambda = \mathbb{Z}^L + \tau \mathbb{Z}^L$ is called *the period lattice* and $\mathbb{J} = \mathbb{C}^L \bmod \Lambda$ the *Jacobian*. Define the Abel map by

$$
\mathfrak{u}(z) = \int_{\lambda_0}^{z} \vec{\omega},
$$

and the vector of functions

$$
\mathcal{M}(z, d) \equiv (\mathcal{M}_1, \mathcal{M}_2) = \left(\frac{\Theta(\mathfrak{u}(z) - \widehat{W} + d)}{\Theta(\mathfrak{u}(z) + d)}, \frac{\Theta(-\mathfrak{u}(z) - \widehat{W} + d)}{\Theta(-\mathfrak{u}(z) + d)} \right),
\tag{2.32}
$$

where

$$
\widehat{W} = (\widehat{W}_1, \ldots, \widehat{W}_L) = \left(N\widetilde{\varpi}_1 + \frac{\ln \rho_1}{2\pi i}, \ldots, N\widetilde{\varpi}_L + \frac{\ln \rho_L}{2\pi i} \right)^t
\tag{2.33}
$$

and $d \in \mathbb{C}^L$ is a vector that will be chosen appropriately later on. Note that

$$\mathfrak{u}_+ + \mathfrak{u}_- = -\tau_j \quad \text{on } \gamma_{m,j}, \quad j = 0, 1, \ldots, L, \quad \text{where } \tau_0 = 0, \qquad \text{and}$$

$$\mathfrak{u}_+ = \mathfrak{u}_- - \sum_{k=j}^{L} e_k \quad \text{on } \gamma_{c,j}, \quad j = 1, \ldots, L, \tag{2.34}$$

and $\mathfrak{u}(\lambda_0) = 0$,

$$\mathfrak{u}(\lambda_{2j+1}) = -\frac{1}{2}\left(\tau_j + \sum_{k=j+1}^{L} e_k\right), \qquad j = 0, 1, \ldots, L,$$

$$\mathfrak{u}(\lambda_{2j}) = -\frac{1}{2}\left(\tau_j + \sum_{k=j}^{L} e_k\right), \qquad j = 1, \ldots, L. \tag{2.35}$$

Although $\mathfrak{u}(z)$ is multivalued, $\mathcal{M}(z, d)$ is single-valued and meromorphic on $\mathbb{C} \setminus \mathfrak{M}$. Moreover, according to (2.34) and the third equation in (2.31), $\mathcal{M}$ satisfies

$$\mathcal{M}_+ = \mathcal{M}_- \begin{pmatrix} 0 & e^{2\pi i \widehat{W}_j} \\ e^{-2\pi i \widehat{W}_j} & 0 \end{pmatrix}$$

on $\gamma_{m,j}$, $j = 0, 1, \ldots, L$, where $\widehat{W}_0 = 0$.

Theorem 2.12 ([14, p. 308]). *Let $\mathbf{f} \in \mathbb{C}^L$ be arbitrary, and denote by $\mathfrak{u}(p)$ the Abel map (extended to the whole Riemann surface $\mathcal{R}$). The (multi-valued) function $\Theta(\mathfrak{u}(z) - \mathbf{f})$ on the Riemann surface either vanishes identically or vanishes at L points $p_1, \ldots, p_L$ (counted with multiplicity). In the latter case we have*

$$\mathbf{f} = \sum_{j=1}^{L} \mathfrak{u}(p_j) + \mathcal{K}, \tag{2.36}$$

where $\mathcal{K} = \sum_{j=1}^{L} \mathfrak{u}(\lambda_{2j})$.

The sign of $\mathcal{K}$ in (2.36) is irrelevant since $2\mathcal{K} \in \Lambda$. An immediate consequence of Theorem 2.12 is the following statement.

Corollary 2.13. *The function Θ vanishes at $\mathbf{e} \in \Lambda$ if and only if there exist $L - 1$ points $q_1, \ldots, q_{L-1}$ on $\mathcal{R}$ such that*

$$\mathbf{e} = \sum_{j=1}^{L-1} \mathfrak{u}(q_j) + \mathcal{K}.$$

Definition 2.14. The *Theta divisor* is the locus $\mathbf{e} \in \mathbb{J}$ such that $\Theta(\mathbf{e}) = 0$. It will be denoted by the symbol (Θ).

Let

$$r(z) = \left(\prod_{j=0}^{L} \frac{z - \lambda_{2j+1}}{z - \lambda_{2j}}\right)^{\frac{1}{4}}, \tag{2.37}$$

where the branch cuts of $r(z)$ are along the main arcs $\mathfrak{M}$ and $r(z)$ is normalized by $\lim_{z\to\infty} r(z) = 1$. One can also construct the solution to the RHP (2.28) in the form of

$$\mathcal{L}(z) = \frac{1}{2} \begin{pmatrix} (r(z) + r^{-1}(z))\mathcal{M}_1(z,d) & -i(r(z) - r^{-1}(z))\mathcal{M}_2(z,d) \\ i(r(z) - r^{-1}(z))\mathcal{M}_1(z,-d) & (r(z) + r^{-1}(z))\mathcal{M}_2(z,-d) \end{pmatrix}, \tag{2.38}$$

where the vector $d \in \mathbb{C}^L$ is to be defined and $\mathcal{M}(z,d)$, $r(z)$ were given by (2.32), (2.37) respectively. Indeed, using the facts that $(r \pm r^{-1})_+ = i(r \mp r^{-1})_-$, it is straightforward to check that $\mathcal{L}(z)$ satisfies the jump conditions listed in (2.28).

The vector $d \in \mathbb{C}^L$ is found by requiring the analyticity of $\mathcal{L}(z)$ in $\mathbb{C} \setminus \mathfrak{M}$; to this end, consider the equation

$$r^4(z) - 1 = 0 \equiv \frac{\prod_{j=0}^{L}(z - \lambda_{2j+1}) - \prod_{j=0}^{L}(z - \lambda_{2j})}{R(z)} = 0.$$

Note that $r^4(z) - 1$ is a meromorphic function on the Riemann surface $\mathcal{R}$ with a zero $z_0 = \infty_+$ (infinity on the main sheet) and $2L$ (counted with multiplicities) zeroes $z_1, \ldots, z_L$ counted twice (for each of the two sheets of $\mathcal{R}$). Zeroes $z_1, \ldots, z_L$ are finite provided

$$\sum_{j=0}^{L} \lambda_{2j} \neq \sum_{j=0}^{L} \lambda_{2j+1}.$$

With a slight abuse of notation, we denote by $z_1, \ldots, z_L$ zeroes of $r^2(z) - 1 = 0$ other than z_0 that can be located on either sheet of $\mathcal{R}$. It is straightforward to check that if z is a zero of $r^2(z) - 1 = 0$ then $\widehat{z}$ is a zero of $r^2(z) + 1 = 0$, where $\widehat{z}$ denotes the projection of z on the other sheet of the hyperelliptic surface $\mathcal{R}$. If we now choose

$$d = -\sum_{j=1}^{L} \int_{\lambda_{2j}}^{\hat{z}_j} \vec{\omega},$$

then, according to Theorem 2.12, the set of all zeroes of $\Theta(\mathfrak{u}(z) + d)$ in $\mathcal{R}$ consists of $\hat{z}_j$, $j = 1, \ldots, L$. Indeed,

$$\Theta(\mathfrak{u}(z) + d) = \Theta\left(\mathfrak{u}(z) + \mathcal{K} - \sum_{j=1}^{L} \mathfrak{u}(\lambda_{2j}) - \sum_{j=1}^{L} \int_{\lambda_{2j}}^{\hat{z}_j} \vec{\omega}\right) = \Theta\left(\mathfrak{u}(z) + \mathcal{K} - \sum_{j=1}^{L} \mathfrak{u}(\hat{z}_j)\right).$$

Since

$$\Theta(\mathfrak{u}(z_j) - d) = \Theta(-\mathfrak{u}(\widehat{z}_j) - d) = \Theta(\mathfrak{u}(\widehat{z}_j) + d) = 0$$

we obtain that the set of all zeroes of $\Theta(-\mathfrak{u}(z) + d)$ in $\mathcal{R}$ consists of z_j, $j = 1, \ldots, L$.

Since all the zeroes z_j, $j = 1, \ldots, L$, of $\Theta(\mathfrak{u}(z) - d)$ are also zeroes of $r(z) - \frac{1}{r(z)}$ and all the zeroes $\widehat{z}_j$, $j = 1, \ldots, L$ of $\Theta(\mathfrak{u}(z) + d)$ are also zeroes of $r(z) + \frac{1}{r(z)}$, we conclude that $\mathcal{L}(z)$ does not have poles on $\mathcal{R}$. In particular, $\mathcal{L}(z)$ is analytic in $\mathbb{C} \setminus \mathfrak{M}$. It obviously satisfies the boundary requirements of (2.28) at the branchpoints.

Lemma 2.15.

$$d = -\mathfrak{u}(\infty_+) \bmod \Lambda, \tag{2.39}$$

where Λ is the period lattice.

Proof. By construction, $r^2(z) - 1$ is a meromorphic function on $\mathcal{R}$ with simple poles at λ_{2j}, $j = 0, 1, \ldots, L$, and simple zeroes at z_j, $j = 1, \ldots, L$, and at infinity on the main sheet $\infty = \infty_+$. Let $\mathcal{D}$ denote the divisor of $r^2(z) - 1$. Then according to Abel's theorem (see [14]), $U(\mathcal{D}) = 0$ mod Λ. Thus,

$$\mathfrak{u}(\infty) + \sum_{j=1}^{L} \mathfrak{u}(z_j) = \sum_{j=0}^{L} \mathfrak{u}(\lambda_{2j}) \bmod \Lambda. \tag{2.40}$$

Then (2.39) follows from (2.40),

$$\sum_{j=1}^{L} \mathfrak{u}(z_j) = \sum_{j=0}^{L} \mathfrak{u}(\lambda_{2j}) + d$$

and $\mathfrak{u}(\lambda_0) = 0$. ∎

With Lemma 2.15, we obtain the following lemma.

Lemma 2.16. *The matrix* $\mathcal{L}(z)$ *with* $d = \mathfrak{u}(\infty_+)$ *and* $\widehat{W}$ *as in* (2.33) *is analytic in* $\mathbb{C} \setminus \mathfrak{M}$ *and satisfies the same jump conditions as* $\widetilde{\Psi}_0(z)$ *from the RHP* (2.28).

Since $\mathcal{L}(z)$ satisfies the same jump conditions as $\widetilde{\Psi}_0(z)$, the determinant $\det \mathcal{L}(z)$ must be analytic in $\mathbb{C}$. On the other hand, according to (2.38) and (2.37)

$$\begin{aligned}
\det \mathcal{L}(\infty) &= \mathcal{M}_1(\infty, d)\mathcal{M}_2(\infty, -d) \\
&= \frac{\Theta\big(\mathfrak{u}(\infty) - \widehat{W} + d\big)\Theta\big(\mathfrak{u}(\infty) + \widehat{W} + d\big)}{\Theta^2(\mathfrak{u}(\infty) + d)} = \frac{\Theta^2\big(\widehat{W}\big)}{\Theta^2(0)}.
\end{aligned} \tag{2.41}$$

$\mathcal{L}(z)$ is bounded at $z = \infty$, so that

$$\det \mathcal{L}(z) \equiv \frac{\Theta^2\big(\widehat{W}\big)}{\Theta^2(0)}.$$

Thus, we obtain

$$\widetilde{\Psi}_0(z) = \mathcal{L}^{-1}(\infty)\mathcal{L}(z).$$

Consequently we have proven the following analog of Theorem 2.17.

Theorem 2.17. *The RHP* (2.28) *admits a solution* $\widetilde{\Psi}_0$ *if and only if* $\Theta\big(\widehat{W}\big) \neq 0$.

Corollary 2.18. *If* $\widehat{W} = \mathfrak{u}(\lambda_{2j}) \bmod \Lambda$ *for some* $j = 1, \ldots, L$, *that is*

$$N\vec{\widetilde{\varpi}} + \frac{\vec{\ln \rho}}{2\pi i} = \frac{1}{2}\left(\tau_j + \sum_{k=j}^{L} e_k\right) \bmod \Lambda, \tag{2.42}$$

for some $j = 1, \ldots, L$, *where* $\vec{\widetilde{\varpi}} = (\widetilde{\varpi}_1, \ldots, \widetilde{\varpi}_L)$, $\vec{\ln \rho} = (\ln \rho_1, \ldots, \ln \rho_L)$, *then the RHP* (2.28) *does not have a solution. Condition* (2.42) *is necessary and sufficient in the case* $L = 1$.

Proof. The corollary follows from Corollary 2.13 and (2.41), (2.35) and (2.33). ∎

Theorem 2.19. *Assume that the h function $h(z;t)$, defined by the RHP (2.15), (2.16) with $L+1$ main arcs, satisfies the sign requirements (strict inequalities) in some compact region $\mathcal{T} \subset \mathbb{C}^*$. Then the recurrence coefficients $\alpha_n = \alpha_n(t)$ and the pseudonorms $\mathbf{h}_n = \mathbf{h}_n(t)$ satisfy*

$$\alpha_n = \frac{1}{16} \left(\sum_{j=0}^{L} (\lambda_{2j} - \lambda_{2j+1}) \right)^2 \frac{\Theta(2\mathfrak{u}_\infty - \widehat{W})\Theta(2\mathfrak{u}_\infty + \widehat{W})\Theta^2(0)}{\Theta^2(\widehat{W})\Theta^2(2\mathfrak{u}_\infty)} + O(N^{-1}), \qquad (2.43)$$

$$\mathbf{h}_n = -\frac{\pi}{2} \left(\sum_{j=0}^{L} (\lambda_{2j} - \lambda_{2j+1}) \right) \frac{\Theta(2\mathfrak{u}_\infty + \widehat{W})\Theta(0)}{\Theta(\widehat{W})\Theta(2\mathfrak{u}_\infty)} e^{N\ell} + O(N^{-1}), \qquad (2.44)$$

in the limit $N \to \infty$ uniformly in $t \in \mathcal{T}$ provided that $\Theta(\widehat{W})$ is separated from zero. Here $\mathfrak{u}_\infty = \mathfrak{u}(\infty_+)$ and ℓ, $\widehat{W}$ are given by (3.19) and (2.33), (3.18) respectively.

Proof. Note that

$$r(z) - r^{-1}(z) = \frac{1}{2z} \sum_{j=0}^{L} (\lambda_{2j} - \lambda_{2j+1}) + O(z^{-2}) \qquad (2.45)$$

as $z \to \infty$. According to (2.45) and (2.41), the $O(z^{-1})$ (residue) term in the expansion of $\widetilde{\Psi}_0$ at infinity has $(1,2)$ and $(2,1)$ terms given by

$$-i \sum_{j=0}^{L} (\lambda_{2j} - \lambda_{2j+1}) \cdot \frac{\mathcal{M}_2(\infty, d)}{4\mathcal{M}_1(\infty, d)} \qquad \text{and} \qquad i \sum_{j=0}^{L} (\lambda_{2j} - \lambda_{2j+1}) \cdot \frac{\mathcal{M}_1(\infty, -d)}{4\mathcal{M}_2(\infty, -d)}. \qquad (2.46)$$

Then, combining (2.3), (2.4), (2.27), we obtain

$$\alpha_n = \frac{1}{16} \left(\sum_{j=0}^{L} (\lambda_{2j} - \lambda_{2j+1}) \right)^2 \frac{\Theta(\mathfrak{u}_\infty - \widehat{W} - d)\Theta(\mathfrak{u}_\infty + \widehat{W} - d)\Theta^2(\mathfrak{u}_\infty + d)}{\Theta(\mathfrak{u}_\infty - \widehat{W} + d)\Theta(\mathfrak{u}_\infty + \widehat{W} + d)\Theta^2(\mathfrak{u}_\infty - d)}.$$

We now use (2.3), Lemma 2.16 and properties (2.31) of theta functions to obtain (2.43), (2.44). The error estimates follow from assumptions of the theorem. ∎

Remark 2.20. Theorem 2.19 is valid for general polynomial potentials $f(z; \vec{t})$.

3 Formal calculation of the *g*-function and *h*-function

In this section, following [4], we calculate explicitly $g(z;t)$ and $h(z;t)$ in the genus zero case. We will also derive the determinantal expressions for g and h in the higher genera regions for general polynomial potentials $f = f(z; \vec{t})$.

3.1 Explicit form of *g* and *h* functions in the genus zero region

In the genus zero case, $\mathfrak{M}$ consists of a single main arc γ_m with the endpoints $\lambda_{0,1} = \lambda_{0,1}(t)$. Applying the Sokhotski–Plemelj formula for the RHP (2.3) and using the analyticity of f', we obtain

$$g'(z) = \frac{R(z)}{4\pi i} \oint_{\hat{\gamma}_m} \frac{f'(\zeta)}{(\zeta - z)R(\zeta)_+} d\zeta, \qquad R(z) = \sqrt{(z - \lambda_1)(z - \lambda_0)}, \qquad (3.1)$$

where the branchcut of $R(z)$ is γ_m and the branch of R is defined by $\lim_{z\to\infty} \frac{R(z)}{z} = 1$; the contour $\hat\gamma_m$ encircles the contour γ_m and has counterclockwise orientation, and; z is outside $\hat\gamma_m$. Note that if z is inside $\hat\gamma_m$, the right hand side of (3.1) is equal to $g'(z) - \frac{1}{2}f'(z) = \frac{1}{2}h'(z)$. The modulation equations (2.24) in the case $L = 0$ reduce to the following moment conditions:

$$\oint_{\hat\gamma_m} \frac{f'(\zeta)}{R(\zeta)_+}d\zeta = 0 \qquad \text{and} \qquad \oint_{\hat\gamma_m} \frac{\zeta f'(\zeta)}{R(\zeta)_+}d\zeta = -4i\pi. \tag{3.2}$$

We use the moment conditions (3.2) *to define the location of the endpoints* $\lambda_{0,1}$.

Since f is a polynomial, the equations (3.2) can be solved using the residue theorem on equations (3.2), setting $\lambda_0 = a - b$, $\lambda_1 = a + b$. We obtain the system

$$a + ta\left(a^2 + \frac{3}{2}b^2\right) = 0 \qquad \text{and} \qquad a^2 + \frac{1}{2}b^2 + t\left(a^4 + 3a^2b^2 + \frac{3}{8}b^4\right) = 2. \tag{3.3}$$

There are two possibilities: $a = 0$ or $a \neq 0$. The Generic configuration of the traffic, see Section 1, considered in the paper, forces the first (*symmetric*) case, see [4]. In this case we obtain solutions to the system (3.3) as

$$a = 0, \qquad b^2 = -\frac{2}{3t}\left(1 \mp \sqrt{1 + 12t}\right), \tag{3.4}$$

$$\lambda_{0,1} = \mp b = \mp\sqrt{-\frac{2}{3t}\left(1 - \sqrt{1 + 12t}\right)}. \tag{3.5}$$

The choice of the negative sign in (3.4) comes from the requirement that b is bounded as $t \to 0$. Observe that for $t > t_0 = -\frac{1}{12}$ the values $\pm b$ coincide with the branch-points, derived in [13]. The second pair of roots $b = \pm\sqrt{-\frac{2}{3t}(1 + \sqrt{1 + 12t})}$ are sliding along the real axis from $\pm\infty$ to $\pm b_0 = \pm\sqrt{8}$ as real t varies from 0^- to t_0, and are sliding along the imaginary axis from $\pm i\infty$ to 0 as real t varies from 0^+ to $+\infty$. At the point

$$t = t_0 = -\frac{1}{12}$$

the two pairs of roots (3.4) coincide, creating five zero level curves of $\Re h(z)$ emanating from the endpoints $\pm b_0$ (see [4, 13]). Once the values of branch-points (endpoints) $\lambda_{0,1} = \lambda_{0,1}(t)$ are determined, one can calculate explicitly $g(z) = g(z;t)$ and $h(z) = h(z;t)$, where $h(z) = 2g(z) - f(z) - \ell$. We are first looking for $h'(z) = -(k + tz^2)\sqrt{z^2 - b^2}$, where the endpoints $\pm b$ of γ_m are given by (3.4). The asymptotics in (2.19) implies $k = 1 + \frac{1}{2}tb^2$, so that

$$h'(z) = -\left[tz^2 + 1 + \frac{tb^2}{2}\right](z^2 - b^2)^{\frac{1}{2}} = -\left[tz^2 + \frac{\sqrt{1 + 12t}}{3} + \frac{2}{3}\right](z^2 - b^2)^{\frac{1}{2}}. \tag{3.6}$$

Since the branch-cut of the radical is $[-b, b]$ we conclude that $h'(z)$ is an odd function. Direct calculations yield

$$h(z) = 2\ln\frac{z + \sqrt{z^2 - b^2}}{b} - \frac{z}{8}\left(2tz^2 + tb^2 + 4\right)\left(z^2 - b^2\right)^{\frac{1}{2}}. \tag{3.7}$$

It is clear that $h(b) = 0$. Combined with (3.6), that implies

$$h(z) = O(z - b)^{\frac{3}{2}} \tag{3.8}$$

(or a higher power of $(z - b)$). It is easy to check that at $t_0 = -\frac{1}{12}$ the order $O(z - b)^{\frac{3}{2}}$ in (3.8) can be replaced $O(z - b)^{\frac{5}{2}}$. There is the oriented branch-cut of $h(z)$ along the ray $(-\infty, -b)$, where $h_+(z) - h_-(z) = 4\pi i$. Because of this jump, the function $h(z)$ does not have $O(z + b)^{\frac{3}{2}}$ behavior near $z = -b$; however, $\Re h(z)$ does have $O(z + b)^{\frac{3}{2}}$ behavior near $z = -b$.

Remark 3.1. One can verify directly that $h(z)$ satisfies the following RHP:

1) $h(z)$ is analytic (in z) in $\mathbb{C} \setminus \{\gamma_m \cup (-\infty, \lambda_0)\}$ and

$$h(z) = -\frac{1}{4}tz^4 - \frac{1}{2}z^2 + 2\ln z - \ell + O(z^{-1}) \quad \text{as} \ \ z \to \infty,$$

where

$$\ell = \ln \frac{b^2}{4} - \frac{b^2}{8} - \frac{1}{2};$$

2) $h(z)$ satisfies the jump condition

$$h_+ + h_- = 0 \quad \text{on} \ \ \gamma_m \qquad \text{and} \qquad h_+ - h_- = 4\pi i \quad \text{on} \ \ (-\infty, \lambda_0).$$

This RHP is the genus zero case ($L = 0$) of the general RHP for h from Remark 2.8.

Remark 3.2. As it was mentioned above, the solution to the scalar RHP for h commutes with differentiation in z; on the same basis, it commutes with differentiation in t as well. Thus, we obtain the following RHP for h_t:

1) $h_t(z)$ is analytic (in z) in $\bar{\mathbb{C}} \setminus \gamma_m$ and

$$h_t(z) = -\frac{1}{4}z^4 - \ell_t + O(z^{-1}) \quad \text{as} \ \ z \to \infty, \tag{3.9}$$

where

$$\ell_t = \frac{3}{32}b^4 = \frac{2 + 12t - 2\sqrt{1 + 12t}}{24t^2};$$

2) $h_t(z)$ satisfies the jump condition

$$h_{t+} + h_{t-} = 0 \quad \text{on} \ \ \gamma_m.$$

This RHP has the unique solution

$$h_t(z) = -\frac{z}{8}\left(2z^2 + b^2\right)\sqrt{z^2 - b^2} \tag{3.10}$$

that can be verified directly. The explicit formula of h_t is useful in controlling the sign of $\Re h(z; t)$ when t is varying.

3.2 The g-function and h-function and their properties in the genus L case

For the rest of this section we will be considering the case of a general polynomial potential

$$f(z) = f(z, \vec{t}) = \sum_{j=1}^{n} \frac{t_j}{j}z^j \tag{3.11}$$

with $\vec{t} = (t_1, \ldots, t_n)$ being a vector of (complex) coefficients of f. Then the genus L of the corresponding Riemann surface $\mathcal{R} = \mathcal{R}(\vec{t})$ satisfies (2.23).

In the case of genus $L > 0$, we start construction of g, h by defining the constants $\eta_j, \varpi_j \in \mathbb{R}$, $j = 1, 2, \ldots, L$, and $\ell \in \mathbb{C}$. Using (2.6), (2.13) (the second expression) and the expansion $\frac{1}{\zeta - z} = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{\zeta^k}{z^k}$, we obtain

$$
\frac{1}{2\pi i} \oint_{\hat{\mathfrak{M}}} \frac{\zeta^k [f(\zeta) - 2u(\zeta)]}{R(\zeta)} d\zeta + \frac{\ell}{2\pi i} \oint_{\hat{\mathfrak{M}}} \frac{\zeta^k d\zeta}{R(\zeta)}
$$
$$
+ \sum_{j=1}^{L} \varpi_j \oint_{\hat{\gamma}_{m,j}} \frac{\zeta^k d\zeta}{R(\zeta)} + \sum_{j=1}^{L} \eta_j \oint_{\hat{\gamma}_{c,j}} \frac{\zeta^k d\zeta}{R(\zeta)} = -2\delta_{k,L} u_*, \tag{3.12}
$$

$k = 0, 1, \ldots, L$, where $u_* = \lim_{z \to \infty} u(z) - \ln z$ and $\delta_{k,L}$ is the Kronecker delta. It is clear that the second integral in (3.12) is zero for all $k < L$, so that the first L equations in (3.12) is a system of linear equations for $2L$ real unknowns η_j, ϖ_j. Then from the last equation we obtain

$$
\ell = 2u_* + \frac{1}{2\pi i} \oint_{\hat{\mathfrak{M}}} \frac{\zeta^L [f(\zeta) - 2u(\zeta)]}{R(\zeta)} d\zeta + \sum_{j=1}^{L} \varpi_j \oint_{\hat{\gamma}_{m,j}} \frac{\zeta^L d\zeta}{R(\zeta)} + \sum_{j=1}^{L} \eta_j \oint_{\hat{\gamma}_{c,j}} \frac{\zeta^L d\zeta}{R(\zeta)}. \tag{3.13}
$$

Taking complex conjugates of the first L equations (3.12), we obtain

$$
-\frac{1}{2\pi i} \overline{\oint_{\hat{\mathfrak{M}}} \frac{\zeta^k (f(\zeta) - 2u(z))}{R(\zeta)} d\zeta} + \sum_{j=1}^{L} \varpi_j \overline{\oint_{\hat{\gamma}_{m,j}} \frac{\zeta^k d\zeta}{R(\zeta)}} + \sum_{j=1}^{L} \eta_j \overline{\oint_{\hat{\gamma}_{c,j}} \frac{\zeta^k d\zeta}{R(\zeta)}} = 0, \tag{3.14}
$$

$k = 0, 1, \ldots, L - 1$.

The first L equations from (3.12) combined with (3.14) form a system of linear equations

$$
D \begin{pmatrix} \vec{\varpi} \\ \vec{\eta} \end{pmatrix} = \frac{1}{2\pi i} \left(\oint_{\hat{\mathfrak{M}}} \frac{(f(\zeta) - 2u(z))}{R(\zeta)} \vec{\varsigma}(\zeta) d\zeta, \quad -\overline{\oint_{\hat{\mathfrak{M}}} \frac{(f(\zeta) - 2u(z))}{R(\zeta)} \vec{\varsigma}(\zeta) d\zeta} \right)^t, \tag{3.15}
$$

where $\vec{\varpi}$, $\vec{\eta}$ are vectors of constants $\varpi_1, \ldots, \varpi_L$ and $\eta_1, \ldots, \eta_L$ respectively, $\vec{\varsigma}(\zeta) = (1, \zeta, \ldots, \zeta^{L-1})^t$ and

$$
D^t = \begin{pmatrix}
\oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat{\gamma}_{m,L}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,L}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{m,L}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{m,L}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)}} \\
\oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat{\gamma}_{c,L}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,L}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{c,L}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{c,L}} \frac{\zeta^{L-1} d\zeta}{R(\zeta)}}
\end{pmatrix}. \tag{3.16}
$$

Here $*^t$ denotes the transposition. Note that the contours $\hat{\gamma}_{m,j}$ can be taken as α-cycles of the Riemann surface $\mathcal{R} = \mathcal{R}(\vec{t})$. Then the contours $\hat{\gamma}_{c,1}$, $\hat{\gamma}_{c,1} \cup \hat{\gamma}_{c,2}$, $\ldots$, $\cup_{j=1}^{L} \hat{\gamma}_{c,j}$ with the opposite orientation form the β-cycles. Let us denote

$$
D = \begin{pmatrix} A & B \\ \bar{A} & \bar{B} \end{pmatrix},
$$

where A and B are the corresponding $L \times L$ blocks of D. Then, taking proper linear combinations of the rows of A we can replace the integrands $\frac{\zeta^j}{R(\zeta)}$ with the normalized holomorphic differentials $\frac{p_j(\zeta)}{R(\zeta)}$ of the Riemann surface $\mathcal{R}$, where $p_j(\zeta)$ are polynomials of degree not exceeding $L-1$. Thus the matrix A is nonsingular and

$$D = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \begin{pmatrix} \mathbf{1} & A^{-1}B \\ \mathbf{1} & \bar{A}^{-1}\bar{B} \end{pmatrix},$$

(3.17)

so that

$$D^t = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \theta & \bar{\theta} \end{pmatrix} \begin{pmatrix} A^t & 0 \\ 0 & \bar{A}^t \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -H \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \tau & \bar{\tau} \end{pmatrix} \begin{pmatrix} A^t & 0 \\ 0 & \bar{A}^t \end{pmatrix},$$

where $\theta^t = A^{-1}B$, H is a lower triangular matrix with all entries on and below the main diagonal being one. Then $\theta = -H\tau$, where τ is the Riemann matrix of periods. As it is well known, $\Im\tau$ is a positive definite matrix. Thus

$$|D| = |D^t| = (2i)^L |A|^2 |\Im\tau|,$$

where $|D|$ denotes $\det D$, and we have proved the following lemma.

Lemma 3.3 ([24]). *If all the branchpoints $\lambda_0, \lambda_1, \ldots, \lambda_{2L+1}$ are distinct then $(-i)^L|D| > 0$.*

Lemma 3.3 shows that there exists a unique set of real constants $\vec{\varpi}, \vec{\eta}$ satisfying the first L equations (3.12). Of course, this system of equations could be solved with all $\vec{\eta} = 0$ if we allow $\vec{\varpi}$ to have complex components. Denoting the corresponding complex vector by $\vec{\tilde{\varpi}}$, we obtain

$$\begin{aligned} \vec{\tilde{\varpi}} &= -\frac{A^{-1}}{2\pi i} \oint_{\hat{\mathfrak{M}}} \frac{[f(\zeta) - 2u(\zeta)]}{R(\zeta)} \vec{\zeta}(\zeta) d\zeta \\ &= -\frac{1}{2\pi i} \oint_{\hat{\mathfrak{M}}} [f(\zeta) - 2u(\zeta)]\vec{\omega} = -\frac{1}{2\pi i} \oint_{\hat{\mathfrak{M}}} f(\zeta)\vec{\omega} + 2\mathfrak{u}_\infty. \end{aligned}$$

(3.18)

According to (2.11), we can also reduce (3.13) to

$$\ell = 2u_* + \frac{1}{2\pi i} \oint_{\hat{\mathfrak{M}}} \frac{[f(\zeta) - 2u(\zeta)]\tilde{P}(\zeta)}{R(\zeta)} d\zeta = 2u_* + \frac{1}{2\pi i} \oint_{\hat{\mathfrak{M}}} \frac{f(\zeta)\tilde{P}(\zeta)}{R(\zeta)} d\zeta - 2\pi^2,$$

(3.19)

since, according to (2.12),

$$\oint_{\hat{\mathfrak{M}}} \frac{u(\zeta)\tilde{P}(\zeta)}{R(\zeta)} d\zeta = \oint_{\hat{\mathfrak{M}}} u(\zeta)u'(\zeta)d\zeta = \frac{1}{2}u^2(z)\Big|_{z=\lambda_0}^{z=\lambda_0 e^{-2\pi i}} = -2\pi^2.$$

In view of Lemma 3.3, the system (3.15) has a unique solution given by

$$\begin{pmatrix} \vec{\varpi} \\ \vec{\eta} \end{pmatrix} = D^{-1} \begin{pmatrix} \vec{r} \\ \vec{\bar{r}} \end{pmatrix},$$

where the vector $\vec{r}$ denotes the first L entries in the right hand side of (3.15). Moreover, this solution is real since the system (3.15) can be written as

$$\begin{pmatrix} \Re A & \Re B \\ \Im A & \Im B \end{pmatrix} \begin{pmatrix} \vec{\varpi} \\ \vec{\eta} \end{pmatrix} = \begin{pmatrix} \Re\vec{r} \\ \Im\vec{r} \end{pmatrix}.$$

Using (3.15) and (3.17), we calculate

$$\vec{\eta} = \left(\Im(A^{-1}B)\right)^{-1}\Im(A^{-1}\vec{r}), \qquad \vec{\varpi} = \Re(A^{-1}\vec{r}) - \Re(A^{-1}B)\left(\Im(A^{-1}B)\right)^{-1}\Im(A^{-1}\vec{r}). \quad (3.20)$$

Alternatively, one can say that the vector $-(\vec{\varpi}^t, \vec{\eta}^t)$ coincides with the last row of the matrix $K(z) \operatorname{diag}((D^t)^{-1}, 1)$

Now, according to (3.12), the expression (2.17) for $h(z)$ can be represented as

$$h(z) = \frac{R(z)}{|D|} K(z), \tag{3.21}$$

where

$$K(z) = \frac{1}{2\pi i} \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)}} & \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{m,L}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,L}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{m,L}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{m,L}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)}} & \oint_{\hat{\gamma}_{m,L}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{c,L}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,L}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,L}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,L}} \frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,L}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\mathfrak{M}}} \frac{\tilde{f}(\zeta)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\mathfrak{M}}} \frac{\zeta^{L-1}\tilde{f}(\zeta))d\zeta}{R(\zeta)} & \oint_{\hat{\mathfrak{M}}} \frac{\tilde{f}(\zeta)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\mathfrak{M}}} \frac{\zeta^{L-1}\tilde{f}(\zeta))d\zeta}{R(\zeta)} & \oint_{\hat{\mathfrak{M}}} \frac{(\tilde{f}(\zeta)+\ell)d\zeta}{(\zeta-z)R(\zeta)} \end{vmatrix} \tag{3.22}$$

with $\tilde{f}(\zeta) = f(\zeta) - 2u(z)$, and where z is inside the loop $\hat{\mathfrak{M}}$ but outside all other loops. Indeed, multiplying the first L rows of $K(z)$ by $2\pi i\varpi_1, 2\pi i\varpi_2, \ldots, 2\pi i\varpi_L$ respectively, the next L rows of $K(z)$ by $2\pi i\eta_1, 2\pi i\eta_2, \ldots, 2\pi i\eta_L$, and adding them to the last row, we obtain (3.21). Equation (3.21) gives $2g(z)$ when z is outside the loop $\hat{\mathfrak{M}}$.

It is clear that moving z inside the loops $\hat{\gamma}_{m,l}$, $\hat{\gamma}_{c,k}$ would generate residue terms $2\pi i\varpi_l$ and $\pm 2\pi i\eta_k$ (depending on the direction z crosses the oriented loop $\hat{\gamma}_{c,k}$) in the right hand side of (3.21). Combining this fact with (2.3), we obtain a new form of modulation equations

$$K(\lambda_j) = 0, \qquad j = 0, 1, \ldots, 2L+1. \tag{3.23}$$

Theorem 3.4. *Let λ denote an arbitrary branchpoint λ_j in the set of $2L+2$ distinct branchpoints $j = 0, 1, \ldots, 2L+1$. Then the following statements are equivalent: 1) $K(\lambda) = 0$; 2) $\frac{\partial(\vec{\varpi}, \vec{\eta})^t}{\partial\lambda} = 0$; 3) $\frac{\partial h(z)}{\partial\lambda} \equiv 0$ for all $z \in \mathbb{C}$; 4) $\frac{\partial g(z)}{\partial\lambda} \equiv 0$ for all $z \in \mathbb{C}$.*

Proof. Combining (2.17) with the identity

$$\frac{\partial}{\partial\lambda}\left[\frac{R(z)}{(\zeta-z)R(\zeta)}\right] = -\frac{R(z)}{2(z-\lambda)(\zeta-\lambda)R(\zeta)},$$

we obtain

$$\frac{\partial h(z)}{\partial\lambda} = -\frac{R(z)K(\lambda)}{2(z-\lambda)|D|} + \frac{R(z)}{4\pi i}\left(\oint_{\hat{\mathfrak{M}}} \frac{\left(\frac{\partial\ell}{\partial\lambda} - 2\frac{\partial u(\zeta)}{\partial\lambda}\right)d\zeta}{(\zeta-z)R(\zeta)}\right.$$
$$\left. + \sum_{j=1}^{L}\left[\frac{\partial\varpi_j}{\partial\lambda}\oint_{\hat{\gamma}_{m,j}} \frac{2\pi i d\zeta}{(\zeta-z)R(\zeta)_+} + \frac{\partial\eta_j}{\partial\lambda}\oint_{\hat{\gamma}_{c,j}} \frac{2\pi i d\zeta}{(\zeta-z)R(\zeta)}\right]\right), \tag{3.24}$$

where z is inside $\hat{\mathfrak{M}}$.

According to (2.12),

$$\frac{\partial u(z)}{\partial\lambda} = \frac{1}{2}\int_{\lambda_0}^{z} \frac{\tilde{P}(\zeta)d\zeta}{(\zeta-\lambda)R(\zeta)} + \int_{\lambda_0}^{z} \frac{\tilde{P}'(\zeta)d\zeta}{R(\zeta)} \tag{3.25}$$

provided that $\lambda \neq \lambda_0$. If $\lambda = \lambda_0$, the integral in (2.12) seems to have singularity at $z = \lambda_0$, but it is compensated by the extra term $-\left.\frac{\tilde{P}(\zeta)}{R(\zeta)}\right|_{\zeta \to \lambda_0}$ obtained when differentiating by λ_0. In any case, $\frac{\partial u(\zeta)}{\partial \lambda}$ is analytic outside the closed contour $\hat{\mathfrak{M}}$ and at $\zeta = \infty$. Thus, the first integral in (3.24) vanishes for every λ. According to (2.16),

$$\frac{\partial h(z)}{\partial \lambda} = O(1). \tag{3.26}$$

Then, $K(\lambda) = 0$ implies the system of equations

$$\sum_{j=1}^{L} \left[\frac{\partial \varpi_j}{\partial \lambda} \oint_{\hat{\gamma}_{m,j}} \frac{2\pi i \zeta^k d\zeta}{R(\zeta)_+} + \frac{\partial \eta_j}{\partial \lambda} \oint_{\hat{\gamma}_{c,j}} \frac{2\pi i \zeta^k d\zeta}{R(\zeta)} \right] = 0, \qquad k = 0, 1, \ldots, L-1. \tag{3.27}$$

Considering (3.27) together with L complex conjugated equations, we obtain the system (3.15) with zero right hand side. Then, according to Lemma 3.3, $\frac{(\vec{\varpi}, \vec{\eta})^t}{\partial \lambda} = 0$. Hence, we proved that 1) implies 2). Similarly, (3.26) combined with $\frac{(\vec{\varpi}, \vec{\eta})^t}{\partial \lambda} = 0$ imply 3), that is, 2) implies 3). Let us assume 3). Then, according to (2.14), (2.6), $\frac{\partial \ell}{\partial \lambda} = 0$ and so $\frac{\partial g(z)}{\partial \lambda} \equiv 0$ for all $z \in \mathbb{C}$. Thus, 3) implies 4).

Let us now assume 4). Then, differentiating in λ the jump conditions (2.7)–(2.9) for $g(z)$, we obtain $\frac{\partial \ell}{\partial \lambda} = 0$ (ℓ is the jump on $\gamma_{m,0}$) and $\frac{\partial(\vec{\varpi}, \vec{\eta})^t}{\partial \lambda} = 0$ (the jumps on the remaining main and complementary arcs). Now 1) follows from (3.24), (3.26). ∎

Remark 3.5. According to (3.10), in the case of the quartic potential $f(z; t)$, the function $h(z; t)$ is analytic in t in the genus zero region when z is on the Riemann surface $\mathcal{R}(t)$ away from the branchpoints. But, according to (3.21), (3.22), in general, in the higher genera regions the function is only smooth in t since the determinant K depends on both t and $\bar{t}$.

Fix some $j \in \{0, 1, \ldots, 2L+1\}$. The equation $K(\lambda_j) = 0$ defines a co-dimension one manifold Λ_j in the $\mathbb{C}^{2L+2}$ space of all branchpoints λ_j. Then Theorem 3.4 implies the following corollary.

Corollary 3.6. *Let potential $f = f(z; \vec{t})$ be given by (3.11). If $\vec{\lambda} \in \Lambda$, where $\Lambda = \cap_{j=0}^{2L+1} \Lambda_j$, then*

$$\frac{d}{dt_j} h(z; \vec{t}) \equiv \frac{\partial}{\partial t_j} h(z; \vec{t}), \qquad \frac{d}{dt_j} g(z; \vec{t}) \equiv \frac{\partial}{\partial t_j} g(z; \vec{t}),$$
$$\frac{d}{dt_j} \varpi_k(\vec{t}) = \frac{\partial}{\partial t_j} \varpi_k(\vec{t}), \qquad \frac{d}{dt_j} \eta_k(\vec{t}) = \frac{\partial}{\partial t_j} \eta_k(\vec{t}).$$

Similar formulae are valid for the derivatives in $\bar{t}_j$.

According to Corollary 3.6 and (3.11), modulation equation from (3.23) can be written in the so-called *hodograph form* as

$$\sum_{m=1}^{n} t_m K_{t_m}(\lambda_j) + K_0(\lambda_j) = 0, \qquad j = 0, 1, \ldots, 2L+1,$$

where $K_{t_m}(z)$, $K_0(z)$ can be obtained from $K(z)$ by replacing $\tilde{f}$ in (3.22) by $\frac{\zeta^m}{m}$ and by $-2u(\zeta)$ respectively.

Let us now rewrite the equations (3.20) in terms of the data of the Riemann surface $\mathcal{R} = \mathcal{R}(\vec{t})$. If $\vec{\omega}$ denotes the vector of normalized holomorphic (first kind) differentials and τ of $\mathcal{R}$, then

$A^{-1}\vec{r} = \frac{1}{2\pi i}\oint_{\mathfrak{M}}(f-2u)\vec{\omega}$, $A^{-1}B = \tau^t(-H^t)$, so that (3.20) become

$$\vec{\eta} = J(\Im\tau^t)^{-1}\Im\left[\frac{1}{2\pi i}\oint_{\mathfrak{M}}(f-2u)\vec{\omega}\right],$$

$$\vec{\varpi} = \Re\left[\frac{1}{2\pi i}\oint_{\mathfrak{M}}(f-2u)\vec{\omega}\right] - \Re(\tau^t(-H^t))\vec{\eta}, \tag{3.28}$$

where $J = -(H^t)^{-1}$ is the $L \times L$ Jordan block with the eigenvalue -1. Since f is linear in the entries t_m of $\vec{t}$, Corollary 3.6 implies another set of hodograph equations

$$\sum_{m=1}^{n} t_m \partial_m \vec{\eta} + \vec{\eta}_0 = 0, \qquad \sum_{m=1}^{n} t_m \partial_m \vec{\varpi} + \vec{\varpi}_0 = 0, \tag{3.29}$$

where $\partial_m = \frac{\partial}{\partial t_m}$,

$$\partial_m\vec{\eta} = J(\Im\tau^t)^{-1}\Im\operatorname*{res}_{\zeta=\infty}\frac{\zeta^m\vec{\omega}}{m}, \qquad \partial_m\vec{\varpi} = \Re\operatorname*{res}_{\zeta=\infty}\frac{\zeta^m\vec{\omega}}{m} + \Re\tau^t H^t \partial_m\vec{\eta}, \qquad m = 1, \ldots, L,$$

and $\vec{\eta}_0, \vec{\varpi}_0$ can be obtain from (3.28) by setting $f = 0$, $\vec{\eta} = \vec{\eta}_0$, $\vec{\varpi} = \vec{\varpi}_0$ there. The following set of conservation equations

$$\Im\operatorname*{res}_{\zeta=\infty}\left[\partial_l\frac{\zeta^m}{m} - \partial_m\frac{\zeta^l}{l}\right]\vec{\omega} = \Im(\partial_m\tau^t)H^t\partial_l\vec{\eta} - \Im(\partial_l\tau^t)H^t\partial_m\vec{\eta},$$

$$\Re\operatorname*{res}_{\zeta=\infty}\left[\partial_l\frac{\zeta^m}{m} - \partial_m\frac{\zeta^l}{l}\right]\vec{\omega} = \Re(\partial_m\tau^t)H^t\partial_l\vec{\eta} - \Re(\partial_l\tau^t)H^t\partial_m\vec{\eta}$$

or

$$\operatorname*{res}_{\zeta=\infty}\left[\partial_l\frac{\zeta^m}{m} - \partial_m\frac{\zeta^l}{l}\right]\vec{\omega} = \partial_l\tau^t(\Im\tau^t)^{-1}\Im\operatorname*{res}_{\zeta=\infty}\frac{\zeta^m\vec{\omega}}{m} - \partial_m\tau^t(\Im\tau^t)^{-1}\Im\operatorname*{res}_{\zeta=\infty}\frac{\zeta^l\vec{\omega}}{l},$$

$$m, l = 1, \ldots, L,$$

follows from (3.29).

Lemma 3.7. *If $\lambda = \lambda_j$, $j = 0, 1, \ldots, 2L+1$ and the modulation equations (3.23) hold, then*

$$\frac{\partial K(z)}{\partial\lambda} = \left[\frac{\partial\ln|D|}{\partial\lambda} + \frac{1}{2(z-\lambda)}\right]K(z). \tag{3.30}$$

In particular, for any $j, m = 0, 1, \ldots, 2L+1$

$$\frac{\partial}{\partial\lambda_m}K(\lambda_j) = 0 \quad \text{if} \ \ j \neq m \qquad \text{and} \qquad \frac{\partial}{\partial\lambda_j}K(\lambda_j) = \frac{1}{2}K'(z)|_{z=\lambda_j}. \tag{3.31}$$

Proof. Formula (3.30) is a direct consequence of Theorem 3.4 and (3.21), whereas (3.31) follows from the analyticity of $K(z)$ at $z = \lambda_j$ (3.22) and (3.23). The analyticity of $K(z)$ at $z = \lambda_0$ follows from the fact that $\frac{u(z)}{R(z)}$ does not have a jump across $\gamma_{m,0}$ and, thus,

$$\frac{R(z)}{2\pi i}\oint_{\hat{\mathfrak{M}}}\frac{u(\zeta)d\zeta}{(\zeta-z)R(\zeta)} = u(z) + \frac{R(z)}{2\pi i}\oint_{\widetilde{\mathfrak{M}}}\frac{u(\zeta)d\zeta}{(\zeta-z)R(\zeta)}.$$

Here $\widetilde{\mathfrak{M}}$ is a small deformation of $\hat{\mathfrak{M}}$ near λ_0, such that $\widetilde{\mathfrak{M}}$ crosses $\gamma_{m,0}$ and does not contain a neighborhood of λ_0. ∎

Corollary 3.8. *For any $j = 0, 1, \ldots, 2L+1$ and any $m = 1, \ldots, L$ the motion of λ_j with respect to t_m is given by*

$$\partial_m(\lambda_j) = -\frac{2\partial_m K(\lambda_j)}{K'(\lambda_j)}.$$

Moreover, the system of branchpoints satisfies the so-called Whitham equations

$$\partial_m K(\lambda_j)\partial_l \lambda_j = \partial_l K(\lambda_j)\partial_m \lambda_j, \tag{3.32}$$

for any $l = 1, \ldots, L$.

The proof follows directly from Lemma 3.7. Equations (3.32) are, in fact, an instance of the well-known Whitham equations for the evolution of branchpoints of the Riemann surface $\mathcal{R}(\vec{t})$, see, for example, [23].

4   Existence of genus zero g and h functions, breaking curves and continuation principle

The $L = 0$ g-function constructed in Section 3 for our quartic potential $f(z; t)$ satisfies the equality requirements (jump conditions) of Section 2.2.1, but so far there is no information about the sign requirements of Section 2.2.2. We first discuss the sign requirements in the genus zero region (they were established in [13] for $t \in (-1/12, 0)$ and later in [4] for any t) Then we introduce the notion of the breaking curve, derive equations for the breaking curve and find their exact locations. Finally, we show that the sign requirements of Section 2.2.2 are satisfied in the region of genera one and two, that is, at any regular point of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

4.1   Sign requirements in the genus zero region

The explicit expression for h, given by (3.7), satisfies the RHP in Remark 3.1 and, thus, $g = \frac{1}{2}(h + f + \ell)$ satisfies the equality requirements of Section 2.2.1. It will be convenient to make some deformations, see Remark 2.7, of the contour Ω that are similar to that of [13]. Consider first $t \in (-\frac{1}{12}, 0)$, that is, $\arg t = -\pi$. The contour Ω can first be deformed into the union of the segment $[-b, b]$ and four rays connecting the endpoints $\pm b$ with infinity, where $b = b(t)$ is given by (3.5), see Fig. 3, left. The two rays emanating from b have directions $\pm \frac{\pi}{4}$ and the two rays emanating from $-b$ have directions $\pm \frac{3\pi}{4}$. The rays in the left half-plane are oriented towards $z = -b$ whereas the rays in the right half-plane are oriented towards infinity; the segment $[-b, b]$ has the standard (left to right) orientation. As we consider Generic traffic configuration, see (1.4), (1.5), the traffic on $[-b, b]$ equals 1, while all the traffics ϱ_j on the four rays are different from zero. The following Lemma 4.1 [4] shows that the segment $[-b, b]$ represents the main arc, whereas the rays can be deformed into unbounded complementary arcs.

Lemma 4.1. *If $t \in (-\frac{1}{12}, 0)$, the four rays of the contour Ω can be deformed so that the function $\Re h(z)$, where $h(z)$ is given by (3.7), satisfies the sign requirements of Section 2.2.2 along the contour Ω.*

Lemma 4.1 implies the existence of the genus zero h function (and, thus, the existence of the genus zero g-function) for all $t \in (-\frac{1}{12}, 0)$. The following arguments shows that the genus zero region extends from $(-\frac{1}{12}, 0)$ into a region in $\mathbb{C}^*$.

First observe that if a point $t \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a regular point of genus L, see Remark 2.10, that is, the branchpoints in $\vec{\lambda}(t)$ are distinct and the sign requirements of Section 2.2.2 for $h(z; t)$ are satisfied in every interior point of $\mathfrak{M}$, $\mathfrak{C}$, then there is a neighborhood U of t consisting of regular point of genus L. The proof of this well known fact is given in the lemma below (see also [1, 25] etc.).

Lemma 4.2. *The set of regular points is open. This statement is true for a general polynomial potential $f = f(z; \vec{t})$, see (3.11).*

Proof. Writing modulation equations (3.23) as $F(\vec{\lambda}; \vec{t}) = 0$, we obtain from Lemma 3.7 that the Jacobian matrix

$$\frac{\partial F}{\partial \vec{\lambda}} = \frac{1}{2} \operatorname{diag}\left(K'(\lambda_0), \dots, K'(\lambda_{2L+1})\right). \tag{4.1}$$

If $\vec{t}_0$ is a regular point then all the branchpoints are finite and distinct. In this case the determinant $K(z)$ has simple zeroes at every branchpoint λ_j, see (2.18) and (3.21), so that the Jacobian matrix (4.1) is invertible. By the implicit function theorem, we obtain a unique continuation of the branchpoints $\vec{\lambda} = \vec{\lambda}(\vec{t})$ into a neighborhood of the original parameter vector $\vec{t}_0$. Since $\vec{\lambda}(\vec{t})$ is a continuous function of $\vec{t}$ in this neighborhood, then, according to (3.21), so is $h(z; \vec{t})$. Thus, if this neighborhood is sufficiently small, then the sign requirements of Section 2.2.2 for h will be preserved, and we complete the proof. ∎

4.2 Breaking curves. Regular and critical breaking points. Symmetries

Let the h-function $h = h(z; t)$ satisfy the sign requirements of Section 2.2.2 except, possibly, at finitely many points. Define z_0 as a saddle point of $h(z; t)$ if $h'(z; t)$ has zero at z_0 of order at least one, that is, $h'(z; t) = O(z - z_0)$ as $z \to z_0$. A point $t \in \mathbb{C}^*$ is called a breaking (non-regular) point, if there is a saddle point z_0 of $h(z; t)$ that is either on a main arc or on a complementary arc that cannot be deformed away from z_0. In the latter case, $\Re h(z_0, t) = 0$ and we say that the complementary arc is pinched at z_0. So, if t_b is a breaking point, then there exists a saddle point $z_0 \in \Omega$ satisfying the equations

$$h'(z_0; t_b) = 0 \qquad \text{and} \qquad \Re h(z_0; t_b) = 0. \tag{4.2}$$

(In the case z_0 is a branchpoint, the first equation (4.2) should be replaced by the condition: $h'(z; t)$ is of order at least $O(z - z_0)^{\frac{3}{2}}$.)

A breaking point $t_b \in \mathbb{C}^*$ is called *critical* breaking point if one of the following applies:

1) a saddle point z_0 of $h(z; t)$ coincides with a branchpoint; in this case we have $h'(z; t_b) = O(z - z_0)^m$ with $m \geq \frac{3}{2}$;

2) there are at least two (counted with multiplicity) saddle points of $h(z; t)$ on Ω that, in the case when a saddle point is on $\mathfrak{C}$, cannot be deformed away.

For the case $\Omega = \mathbb{R}$ considered in [10], critical points correspond to special cases of *irregular* potentials. A breaking point t_b that is not a critical point is called a regular breaking point. In this case the saddle point z_0, corresponding to t_b, is simple and does not coincide with any branchpoint. It is called a *double point*. This name reflects the fact that a double point can be considered as a double branchpoint, that can move apart and form an extra main or complementary arc of Ω in a higher genus region, that is, an extra branchcut of $\mathcal{R}$.

Considering (4.2) as a system $G(u, v, t_j) = 0$, $j = 1, 2$ of three real equations with four real variables $z = u + iv$ and $t = t_1 + it_2$, we calculate the Jacobian

$$\det\left(\frac{\partial G}{\partial(u, v, t_j)}\right) = i^{j-1} |h''(z; t)| \Re h_{t_j}(z; t)\Big|_{(z,t)=(z_0,t_b)}, \tag{4.3}$$

where we can choose j to be either 1 or 2. If t_b is a regular breaking point and if $h_t(z_0; t_b) \neq 0$, then, by the implicit function theorem, there is a smooth curve passing through $t_b \in \mathbb{C}$ that consists of breaking points. Such curves are called breaking curves. Thus, we have proved the following lemma.

Lemma 4.3. *Any regular breaking point t_b belongs to a locally smooth branch of the breaking curve provided that at least one of the derivatives $h_{t_j}(z_0; t_b)$, $j = 1, 2$, is not zero.*

Lemma 4.3 can be extended for a critical breaking point t_b corresponding to a situation where there are several double points not coinciding with the branchpoints. In this case each double point, according to Lemma 4.3, will define its own local piece of a breaking curve, passing through t_b. Thus, generically, there would be several breaking curves intersecting each other at t_b. However, under certain symmetries, all these breaking curves could coincide and form just one breaking curve that is smooth at t_b. Examples of such symmetries include the case of the semiclassical focusing NLS [25], where the Schwarz symmetry of $h(z)$ implies that the saddle points of $h(z)$ on Ω appears in complex conjugated pairs. As it will be shown below, they also include the case of Generic traffic configuration.

In particular, we will prove that $h'(z; t)$ is an odd function (in z) and, thus, the saddle points $\pm z_0 \neq 0$ appear simultaneously on the contour Ω at a critical point t_b. We will also show that at least one of the derivatives $h_{t_j}(z_0; t_b) \neq 0$ if t_b is a breaking point and z_0 is a corresponding double point, so that the implicit function theorem guarantees existence of a smooth local breaking curve for each double point $\pm z_0$. Because of the symmetry, these (local) breaking curves coincide with each other. Therefore, in the case of symmetric h' *we extend the notion of a regular breaking point t_b to the cases when there is a pair of symmetric double points on Ω that generate the same smooth breaking curve passing through t_b.*

Lemma 4.4. *Under the assumption of Generic traffic configuration, the branchpoints $\vec{\lambda}$ are symmetrical for any $t \in \mathbb{C}^*$, that is, λ is a branchpoint if and only if $-\lambda$ is a branchpoint.*

Proof. Let $\varrho_0 = \varrho_2$ and $\varrho_1 = \varrho_3$ in (1.1). Then, since the weight is even, the orthogonal polynomials $\pi_n(z)$ are also symmetrical, and so, the curves of accumulation of zeroes will also be even. It is well known that these curves coincide with the main arcs (see for example [2]). But, within the Generic traffic configuration, the RHP contour Ω does not depend on the traffics $\vec{\varrho}$. Thus, in the Generic case with any admissible traffics, the branchpoints are symmetrical. ■

Corollary 4.5. *If the branchpoints $\vec{\lambda} = \vec{\lambda}(t)$ are symmetrical, the function $h'(z; t)$ is odd and the functions $h_{t_j}(z; t)$ are even, $j = 1, 2$, $t = t_1 + it_2$.*

Proof. If the branchpoints are symmetrical, the corresponding radical $R(z)$, see (2.10), is even for odd L and odd for even L. It follows then that the solution $g'(z; t)$ to the scalar RHP (2.3), given by (2.20), is odd, and so, $h'(z; t)$ is odd, regardless of the genus. Because of the modulation equations, solution of the RHP (2.15), (2.16) for h commutes with the operator $\frac{d}{dt_j}$. Writing solution to the differentiated RHP for h_{t_j} in the same form as (2.17), we can show, in a similar way to h', that h_t is even. ■

Now, according to (4.2), a breaking point t_b is critical if either

$$\left. \frac{h'(z; t_b)}{R(z)} \right|_{z=\pm z_0} = 0 \qquad \text{or} \qquad h'(z_0; t_b) = h''(z_0; t_b) = 0, \tag{4.4}$$

where z_0 is a branchpoint in the former case and an interior point on a main or a complementary arc in the latter case. In the genus zero case, according to (3.5), the branchpoints are $\pm b$. Using equation (3.6) for h', we easily calculate that the first equation (4.4) yields $t = t_0 = -\frac{1}{12}$, whereas the second set of equations yields $z_0 = 0$, $\frac{tb^2}{2} = -1$ and, thus, $t = t_2 = \frac{1}{4}$. As we will show below, there are no other critical points for Generic traffic configurations (the critical point $t_1 = \frac{1}{15}$ occurs for different traffic configurations, see [4]).

It follows then that the arcs of breaking curves begin and end at a critical point ($t_0 = -\frac{1}{12}$, $t_2 = \frac{1}{4}$) or at $t = 0$ or at infinity. They separate either regions of different genera or regions

of the same genus but with different topology of main arcs. In Section 6 we shall exclude the possibility of a breaking curve forming a smooth closed curve (loop) in the complex t-plane while not passing through any critical point or through $t = 0$.

4.3   Continuation principle for Boutroux deformations

In Section 3.2 we have derived the explicit formula (3.21), (3.22) for the h function in the genus L region, where $L = 1, 2, \ldots$. For quartic exponential weight considered here ($L \leq 2$) there is no guarantee yet that the function h, given by (3.21), (3.22), satisfies the sign requirements of Section 2.2.2. In Section 4.1, we showed that the sign requirements are satisfied in the genus zero region. To prove that the correct signs of $\Re h$ persist beyond the genus zero region, we use the continuation principle for Boutroux deformations. Let ϑ be a smooth bounded curve in the external parameters space (complex t-plane in our case). According to (2.24), the deformation of the vector of (all the) branchpoints $\vec{\lambda}(t)$ along ϑ, governed by the modulation equations, preserves the Boutroux condition (Boutroux deformation). The idea of the continuation principle is that moving along ϑ, we can show that the sign conditions for $\Re h(z; t)$ will be satisfied for all t along the curve, provided that:

- the sign conditions for $\Re h(z; t)$ were satisfied at the beginning point of ϑ;

- the curve ϑ does not contain any critical point or any singular point ($t = 0$), and;

- every time a breaking curve is crossed (at a regular breaking point), the genus is properly adjusted.

Let t be a regular point in a genus L region, that is, all the branchpoints in $\vec{\lambda}(t)$ are distinct and $\Re h(z; t)$ satisfies the strict inequalities from Section 2.2.2 at every point of the contour $\Omega = \Omega(t)$ except the branchpoints. Take another regular point t^* in the same open connected component $\mathfrak{G}$ of the genus L region, that is, $t, t^* \in \mathfrak{G}$, and let $\nu \subset \mathfrak{G}$ be a simple smooth contour, connecting t and t^*. Using Lemma 4.2, we can continue $\vec{\lambda}(t)$ from t to t^* along ν preserving the condition $\lambda_j(t) \neq \lambda_k(t)$, $j \neq k$. Therefore, we can also continue $h(z; t)$ from t to t^* along ν. The only possibility for the inequalities for $h(z; t)$ to fail at some point $t \in \nu$ is if t is a breaking point, which is excluded by construction of ν. Thus, we proved the existence of h-function satisfying the strict inequalities at any point of $\mathfrak{G}$.

In view of the above results, it remains only to prove that the strict inequalities can be preserved after a transversal crossing (along ϑ) of the breaking curve at a regular breaking point t_b. Since $L \leq 2$, at any regular breaking point t_b, there can be either one or two symmetrical double points $z_0 = 0$ or $\pm z_0$ respectively. Because of the symmetry of $h_z(z; t)$, it is sufficient to consider only one double point z_0 in any case.

At a regular breaking point t_b the topology of zero level curves of $\Re h(z; t)$ undergoes a change according to one of the following four scenarios:

- two endpoints of a main arc collide and the arc becomes a double point z_0;

- two endpoints of neighboring main arcs collide and the corresponding bounded complementary arc turns into a double point z_0;

- a complementary arc is pinched by zero level curves of $\Re h(z; t)$ and a new double point z_0 appears;

- a main arc collides with (another) zero level curve of $\Re h(z; t)$ and a new double point z_0 appears.

The first two scenarios lead to decrease of the genus whereas the last two lead to its increase.

Let $h^{(n)}(z; t)$ denote the h function (that means that all the strict inequalities of Section 2.2.2 are satisfied) in the genus n region.

Theorem 4.6. *Let t_b be a regular breaking point on a breaking curve, separating regions of genus n and m, where $n, m \in \mathbb{N} \cup \{0\}$. Then $h_z^{(m)}(z; t_b) \equiv h_z^{(n)}(z; t_b)$ and $h_{zt}^{(m)}(z; t_b) \equiv h_{zt}^{(n)}(z; t_b)$ for all $z \in \mathbb{C}$.*

For the RHP, describing the NLS evolution, and with $|m - n| = 2$, the first identity of the theorem was proved in [25, Theorem 3.1], and the second identity – in [24, Theorem 3.1]. In the context of orthogonal polynomials, this theorem was proved in [1]. Finally, in the most general case which include both regular breaking points and the critical points, the theorem follows from the continuity of Boutroux-normalized meromorphic differentials, proven in [5].

Corollary 4.7. *In the conditions of Theorem 4.6, $h^{(m)}(z; t_b) \equiv h^{(n)}(z; t_b)$ and $h_t^{(m)}(z; t_b) \equiv h_t^{(n)}(z; t_b)$ for all $z \in \mathbb{C}$.*

We can now return to the continuation principle for Boutroux deformations. Let $t(s)$, $s \in \mathbb{R}$, be a parametrization of the smooth curve ϑ that is transversal to the breaking curve at a regular breaking point $t_b = t(0)$. Then, by Lemma 6.1, $h_t(z_0, t_b) \neq 0$, where $z(s)$ is the corresponding saddle point and $z_0 = z(0)$. Since $\Re[h_t(z_0; t_b)\Delta t] = 0$ when Δt is tangential to the breaking curve at t_b, we conclude that $\Re \frac{dh}{ds} = \Re[h_t \frac{dt}{ds}] \neq 0$ at z_0, t_b.

Let us assume, for example, that $\Re \frac{dh}{ds}(z_0; t_b) < 0$, and that the sign conditions for $h(z, t) = h^{(L)}(z, t)$ hold when $t = t(s)$ with $s < 0$. Then, as $s \to 0^-$, either a main arc γ_{mj} collapsed into the double point $z = z_0$, or a main arc collides with another zero level curve of $\Re h(z; t)$ at z_0. We now need to prove the sign requirements on the other side of the breaking curve in a small vicinity of z_0, that is, when $t = t(s)$ with a small $s > 0$.

Consider first the case of a collapsing simple (without whiskers) main arc γ_{mj}. This case can be illustrated by Fig. 3 as transition from the second to the first (counting from the left) panel, caused by collapse of the two symmetrical small main arcs (on the right and on the left). We will now consider only a vicinity of z_0 and disregard the symmetrical to z_0 double point. If $s < 0$ is sufficiently close to zero then γ_{mj} is the only main arc in a vicinity of z_0. Moreover, z_0 is a saddle point of $\Re h^{(L)}(z, t_b)$ with $\Re h^{(L)}(z_0, t_b) = 0$. Let us desingularize the hyperelliptic Riemann surface $\mathfrak{R}(t_b) = \mathfrak{R}^{(L)}(t_b)$ of the genus L by removing a pair of branchpoints that collided into $z = z_0$. The genus of the desingularized Riemann surface $\mathfrak{R}^{(L-1)}(t_b)$ is $L - 1$. The corresponding $h^{(L-1)}(z; t_b) \equiv h^{(L)}(z; t_b)$ satisfies the strict sign inequalities everywhere except $z = z_0$. However, since $h_t^{(L-1)}(z; t_b) \equiv h_t^{(L)}(z; t_b)$ and $\Re \frac{dh^{(L)}}{ds}(z_0; t_b) < 0$, we conclude that for any small $s > 0$ the function $\Re h^{(L-1)}(z; t(s)) < 0$ in some deleted neighborhood of z_0. Then the strict sign inequalities for $h^{(L-1)}(z; t(s))$, where $s > 0$ and small, will be valid everywhere in this neighborhood. So, we proved that $h(z, t(s)) = h^{(L)}(z, t(s))$, $s < 0$, can be continued by $h(z, t(s)) = h^{(L-1)}(z, t(s))$, $s > 0$, across the breaking curve while preserving the sign inequalities.

Consider the case when γ_{mj} is not a simple main arc, that is, γ_{mj} is a disappearing "whisker", see the center top panel in Fig. 2 and the other panel right below it (see also transition from the third to the second panel (counting from the left) panel on Fig. 3). In this case we do not change the genus after crossing the breaking curve, so that $h(z, t(s)) = h^{(L)}(z, t(s))$ for all s close to zero. By repeating the previous arguments we see that a new complementary arc is formed near $z = z_0$, so that strict inequalities are valid everywhere for small $s > 0$. Here we would like to mention one more issue: at the breaking point t_b solution of the modulation equations $F(\vec{\lambda}, t) = 0$ does not produce a set of distinct branchpoints; in fact, two of them collided at $z = z_0$. Under this circumstances we cannot use Theorem 3.3 from [25] to prove existence and continuity of the branchpoints $\vec{\lambda}(t(s))$ for $s > 0$, since the Jacobian $\frac{\partial F}{\partial \vec{\lambda}} = 0$ at $t = t_b$. However, after a proper rescaling of $t - t_b$, it was shown that $\vec{\lambda}(t(s))$ can be continuously extended for $s > 0$ in [25, Theorem 6.4]. This fact also follows from Remark 5.9.

The remaining case of a main arc colliding with another zero level curve can be considered similarly. Finally, the remaining two cases of different topological change at the breaking curve correspond to $\Re\frac{dh}{ds}(z_0; t_b) > 0$. They can be considered in the similar way by just reversing the orientation of the curve $t(s)$, i.e., by replacing $s \mapsto -s$.

Thus, we have essentially proved the following continuation principle: if $t(s)$, $0 \le s \le 1$, is a smooth curve in the complex t-plane that does not contain critical points, and if $h(z, t(0))$ satisfies all the conditions (equations and inequalities) for h-functions, then $h(z, t(0))$ can be continuously (in s) deformed into $h(z, t(1))$, so that for any $s \in [0, 1]$, the function $h(z, t(s))$ satisfies all the conditions for h-functions. Since the curve $t(s)$ and the breaking curve are smooth and their intersections are transversal, there can be no more than a finite values of $s \in [0, 1]$ for which the inequalities are not strict; for these values of s, $t(s)$ are regular breaking points.

The next step in the proof of the sign requirements for $h(z; t)$ for any $t \in \mathbb{C}^* \setminus \{t_0, t_2\}$ is to show that for all $s \in [0, 1]$ the set of branchpoints $\lambda(t(s))$ is bounded. (Note that the dimension of the vector $\lambda(t(s))$ may vary with s). This will be done in Section 5.

5   Boundedness of the branchpoints

Continuation principle allows us to prove that if the g-function (or h-function) exists for some $t = \hat{t}_b$, then it can be continued to a given nonsingular $t = \hat{t}_1$ along any simple, compact, piece-wise smooth contour $t(s)$ in $\mathbb{C}$, where $s \in [0, 1]$ and $t(0) = \hat{t}_b$, $t_1 = \hat{t}(1)$, provided that $t(s)$ does not contain any critical points. However, the arguments of Section 4.3, as well as general theorem about continuation principle from [1], does not exclude the possibility that some of the branchpoints $\lambda_j(t(s))$ may approach infinity as $s \to s_*$ for some $s_* \in (0, 1)$. In this section we prove the boundedness of the branchpoints and nodes of a nodal hyperelliptic curve $\mathcal{R}$ under the Boutroux evolution, which implies boundedness of the branchpoints for general polynomial potentials.

We start our analysis by considering polynomials $P(z)$ of the form

$$P(z) = M^2(z)S(z), \qquad M(z) := t^2 \prod_{j=1}^{m}(z - \mu_j), \qquad S(z) := R^2(z) = \prod_{j=0}^{2L+1}(z - \lambda_j), \quad (5.1)$$

where all λ_j are distinct. We are motivated by the fact that (5.1) coincides with (2.22), where $y(z) = \sqrt{P(z)}$. Let us assume that

$$y(z) = f'(z) + \frac{T}{z} + \mathcal{O}(z^{-2}), \tag{5.2}$$

where $T \in \mathbb{R}$ is a residue of $y(z)$ at infinity and the polynomial potential f has the form

$$f(z) = f(z, \vec{t}) = \sum_{j=1}^{m+L+2} \frac{t_j}{j} z^j \tag{5.3}$$

with $\vec{t} = (t_1, \ldots, t_{m+L+2})$ being a vector of (complex) coefficients of f and $t = t_{m+L+2}$. (The constant term in $f(z)$ does not play any significant role and hence is conventionally set to zero.)

Note that $y^2(z) = P(z)$ defines a nodal hyperelliptic curve $\mathcal{R}$ of the (geometric) genus L. We further assume that the meromorphic differential $y(z)dz$ satisfies the Boutroux condition:

$$\Re \oint_\gamma y(z)dz = 0 \tag{5.4}$$

along any loop γ on $\mathcal{R}$.

Remark 5.1. A simple counting of the parameters shows that the $m+L+2$ complex coefficients from $\vec{t}$, the $T \in \mathbb{R}$ and the $2L+1$ real constraints (5.4) account for all the $m+2L+3$ complex parameters ($\vec{\lambda}, \vec{\mu}$ and t) defining a polynomial $P(z)$ of the form (5.1).

Remark 5.2. It is clear that $y(z)$ satisfies the jump conditions (2.19), where $\gamma_{m,j}$ are the branchcuts of $\mathcal{R}$, and the asymptotics of y at infinity is given by (5.2). Then the Boutroux condition implies that $h(z) = \int_{\lambda_0}^{z} y(u)du$ satisfies the jump conditions (2.15) with some *real* constants η_j, ω_j. Note that the jump across γ_0 in (2.15) has to be replaced by $2\pi i T$. Here $\gamma_{c,j}$ is a collection of complementary loops on $\mathcal{R}$ and γ_0 is a contour connecting λ_0 and ∞. The asymptotics of $h(z)$ at infinity is given by

$$h(z) = f(z) + \ell + T \ln z + \mathcal{O}(z^{-1}), \qquad z \to \infty,$$

where $\ell \in \mathbb{C}$ and f is given by (5.3). Then various form of modulation equations, obtained in Section 2, as well as determinantal formulae for h, obtained in Section 4, are valid for $h(z) = \int_{\lambda_0}^{z} y(u)du$. In particular, condition (5.4) together with $T \in \mathbb{R}$ is, of course, equivalent to the integral conditions of (2.24). Then, as was shown in Section 2.3, conditions (5.4), (5.2) define the branchpoints $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{m+L+2})$, the nodes $\vec{\mu} = (\mu_1, \ldots, \mu_m)$ and $T \in \mathbb{R}$.

Our goal is to study finite deformations of the parameters $\vec{t}$ (we consider $T = const$) that preserve the Boutroux condition. Such deformations will be called Boutroux deformations. Let $\vec{t}(s)$ be a smooth finite curve in $\mathbb{C}^{m+L+2}$, where $s \in [0,1]$ is the arclength parameter. We assume the residue T to be constant. Consider a Boutroux deformation $P(z,s) := P(z, \vec{t}(s))$ of the polynomial $P(z) = P(z; \vec{t})$. The simple roots $\vec{\lambda}(s)$ and the double roots $\vec{\mu}(s)$ now depend continuously on s (as long as they are distinct). As we move along $\vec{t}$ the genus of the nodal curve $\mathcal{R} = \mathcal{R}(s)$ can change as some simple roots may become double roots and vice versa. We want to prove that the roots $\vec{\lambda}(s)$, $\vec{\mu}(s)$ are bounded provided that $\vec{t}(s)$ is separated from the hyperplane $t = t_{m+L+2} = 0$.

The smooth deformation contour $\vec{t}(s)$ can be then interpreted as a parametrized integral curve of the corresponding vector field $\vec{v}(\vec{t})$, that is, $\dot{\vec{t}}(s) = \vec{v}(\vec{t}(s))$ and $\|\vec{v}(\vec{t})\| = 1$. Thus, we need to show how an arbitrary unitary vector field on the $\vec{t}$ induces an infinitesimal deformation of the coefficients of $P(z)$, namely, a vector field on the space of polynomials $P(z)$ of the form (5.1) that is tangent to the manifold of constraints (5.4).

5.1 A general theorem on ODE's for polynomials

Let

$$P(z,s) = p_0(s)z^n + p_1(s)z^{n-1} + \cdots + p_{n-1}(s)z + p_n(s), \tag{5.5}$$

where the coefficients $\vec{p}(s) = (p_1(s), \ldots, p_n(s))$ of P satisfy an *autonomous* system of differential equations of the form

$$\dot{\vec{p}}(s) = \vec{g}(\vec{p}(s)), \qquad s \in [0,1], \tag{5.6}$$

where $\vec{g}(\vec{p}) = (g_1(\vec{p}), \ldots, g_n(\vec{p}))$. To prove that the roots of the polynomial P are bounded (in s), it would be sufficient to show that $p_0(s)$ is separated from zero and all the coefficients $\vec{p}(s)$ are bounded on $[0,1]$.

Theorem 5.3. *Let $\vec{g}(\vec{p})$ in (5.6) be continuous and suppose that there exists $K > 0$ such that the functions $g_j(\vec{p})$ satisfy the bound*

$$|g_j(\vec{p})| \le K\rho^j, \qquad j = 1, \ldots, n, \tag{5.7}$$

where

$$\rho = \rho(\vec{p}) := \max_{\ell \geq 1} |p_\ell|^{\frac{1}{\ell}}. \tag{5.8}$$

Then for any solution of the system (5.6) *the function* $\rho(s) := \rho(\vec{p}(s))$ *remains bounded for all* $s \in [0, 1]$.

To prove Theorem 5.3, we need to establish the following statements from nonsmooth analysis.

Lemma 5.4. *Let* $\vec{p}(s) \in C^1[0, 1]$. *Then* $\rho(s)$, *defined by* (5.8), *admits Dini's derivative*

$$D^+\rho(s) := \limsup_{\widetilde{s} \to s_+} \frac{\rho(\widetilde{s}) - \rho(s)}{\widetilde{s} - s}, \qquad s \in [0, 1]. \tag{5.9}$$

Moreover,

$$D^+\rho(s) = \max_{1 \leq j \leq n} \left\{ \frac{d}{ds} |p_j(s)|^{\frac{1}{j}} : \rho(s) = |p_j(s)|^{\frac{1}{j}} \right\}. \tag{5.10}$$

In particular, the derivative is finite at all points where $\rho(s) > 0$.

Proof. Denote $f_j(s) := |p_j(s)|^{\frac{1}{j}}$. Let s be such that $\rho(s) > 0$. Fix a sequence $s_k \to s$ from above and let j_k be such that $f_{j_k}(s_k) = \rho(s_k)$. Since the set of indices is finite, there is a subsequence of j_k that eventually becomes constant, say j_0. Then, along any such subsequence $\{j_{k_\ell}\}$,

$$f_{j_0}(s) = \lim_\ell f_{j_{k_\ell}}(s) = \lim_\ell f_{j_{k_\ell}}(s_{k_\ell}) = \lim_\ell \rho(s_{k_\ell}) = \rho(s).$$

Then

$$D^+\rho(s) \geq \lim_\ell \frac{\rho(s_{k_\ell}) - \rho(s)}{s_{k_\ell} - s} = \lim_\ell \frac{f_{j_{k_\ell}}(s_{k_\ell}) - \rho(s)}{s_{k_\ell} - s} = \lim_\ell \frac{f_{j_0}(s_{k_\ell}) - f_{j_0}(s)}{s_{k_\ell} - s} = f'_{j_0}(s), \quad (5.11)$$

where we used the fact that the subsequence j_{k_ℓ} is eventually constant with limiting value j_0. Since $f'_{j_0}(s)$ belongs to the set in (5.10) we prove the inequality.

Now let us prove ($\leq$). Let the limsup in (5.9) is attained along the sequence $\{s_k\}$, introduced above. We have

$$D^+\rho(s) = \lim_k \frac{\rho(s_k) - \rho(s)}{s_k - s} = \lim_k \frac{f_{j_k}(s_k) - \rho(s)}{s_k - s} \leq \lim_k \frac{f_{j_k}(s_k) - f_{j_k}(s)}{s_k - s}.$$

The limit set of the sequence j_k necessarily belongs to the set $\{\widetilde{j} : f_{\widetilde{j}}(s) = \rho(s)\}$ and hence

$$\lim_k \frac{f_{j_k}(s_k) - f_{j_k}(s)}{s_k - s} \in \left\{ \frac{d}{ds} f_j(s) : f_j(s) = \rho(s) \right\}.$$

Thus the inequality $\leq$ holds. The latter equation and (5.11) prove (5.10). ■

Next we state the following well known fact that will be used in the proof of Theorem 5.3.

Proposition 5.5 ([17]). *If* $D^+M(s)$ *exists everywhere (finite) in the interval* $[a, b]$ *then*

$$M(b) - M(a) \leq \overline{\int_a^b} D^+M(t)\delta t$$

where $\overline{\int_a^b}$ *denotes the upper Riemann integral.*

Proof of Theorem 5.3. Since $(\ell \geq 1)$, we have

$$\frac{d}{ds}|p_\ell(s)|^{\frac{1}{\ell}} = \frac{1}{\ell}|p_\ell(s)|^{\frac{1}{\ell}-1}\frac{d}{ds}|p_\ell(s)| \leq |p_\ell(s)|^{\frac{1}{\ell}-1}\frac{d}{ds}|p_\ell(s)| \leq |p_\ell(s)|^{\frac{1}{\ell}-1}|\dot{p}_\ell(s)|$$

$$= |p_\ell(s)|^{\frac{1}{\ell}-1}|g_\ell(\vec{p}(s))| \leq K\rho^\ell|p_\ell(s)|^{\frac{1}{\ell}-1}, \tag{5.12}$$

where the assumption of the theorem was used in the latter inequality. By formula (5.10) we have that $D^+\rho(s)$ is the maximum of all derivatives amongst the functions $f_\ell = |p_\ell(s)|^{\frac{1}{\ell}}$ that realize the maximum at s. Thus in (5.12)

$$\frac{d}{ds}|p_\ell(s)|^{\frac{1}{\ell}} \leq K\rho^\ell\rho^{1-\ell} = K\rho$$

and, taking into the account (5.10), we obtain

$$D^+\rho(s) \leq K\rho(s).$$

We would like to use Gronwall's theorem to complete the proof. In order to use this theorem, we introduce $F(s) = \max\{1, \rho(s)\}$. Since $\rho(s)$ is the maximum of locally Lipshitz functions f_j, $j = 1, \ldots, n$, (when $\rho(s) > 1$) then $F(s)$ is locally Lipshitz as well. It is straightforward to verify that

$$D^+F(s) = \begin{cases} 0, & \rho(s) < 1, \\ D^+\rho(s), & \rho(s) > 1, \\ \max\{0, D^+\rho(s)\}, & \rho(s) = 1. \end{cases}$$

Then, if $\rho(s) > 1$ then $D^+F(s) = D^+\rho(s) \leq K\rho(s) \leq KF(s)$; if $\rho(s) = 1$ then also $D^+F(s) = \max\{0, D^+\rho(s)\} \leq \max\{0, K\} = KF(s)$. If $\rho(s) < 1$ then $D^+F(s) = 0 \leq K\rho(s) \leq KF(s)$. Thus in all cases

$$D^+F(s) \leq KF(s).$$

Integrating this inequality from 0 to s and using (5.5), we obtain

$$F(s) - F(0) \leq \overline{\int_0^s} D^+F(t)dt \leq \overline{\int_0^s} KF(t)dt = \int_0^s KF(t)dt,$$

where in the last equality we have replaced the upper Riemann integral by a regular Riemann integral because $M(t)$ is continuous. Now Gronwall's theorem yields $F(s) \leq F(0)e^{Ks}$ for any $s \in [0, 1]$. The theorem is proved. ∎

Corollary 5.6. *If $p_0(s)$ remains separated from 0 and the remaining coefficients $p_j(s)$, $j = 1, \ldots, n$, of (5.5) satisfy the assumptions of Theorem 5.3, on $[0, 1]$ then the roots of $P(z; s)$ are uniformly bounded on $[0, 1]$.*

Proof. According to Rouché's theorem, the roots of the polynomial (5.5) are within a circle of the radius

$$N\max_{n \geq \ell \geq 1}\left|\frac{p_\ell(s)}{p_0(s)}\right|^{\frac{1}{\ell}} \leq \frac{N\rho(s)}{\min\{|p_0(s)|, 1\}}, \tag{5.13}$$

where the constant N depends only on the order of the polynomial $P(z; s)$. Now the corollary follows from Theorem 5.3. ∎

5.2   Properties of the polynomial (5.1)

In this section we prove that the ODEs (5.6) for the coefficients $p_j(s)$, $j = 1, \ldots, n$ of the polynomial

$$P(z, s) = t^2 \prod_{j=1}^{m} (z - \mu_j(s))^2 \prod_{j=0}^{2L+1} (z - \lambda_j(s)), \tag{5.14}$$

see (5.1), where $n = 2m + 2L + 2$, satisfy the conditions (5.7). Under the additional assumption that $p_0(s) = t^2(s)$ is separated from zero on $[0, 1]$, it would then follow from Corollary 5.6 that the roots $\mu_j(s)$, $\lambda_j(s)$ of (5.14) are uniformly bounded on $[0, 1]$.

Let $\partial_j = \frac{\partial}{\partial t_j}$, $j = 0, 1, \ldots, n$. Then $\dot{P}(z, s) = \nabla_{\vec{t}} P \cdot \vec{v}(\vec{t})$ and, taking into account $\dot{\vec{t}}(s) = \vec{v}(\vec{t})$ and (5.6), we obtain

$$g_\ell(\vec{p}) = \dot{p}_\ell(s) = \nabla_{\vec{t}} p_\ell \cdot \vec{v}(\vec{t}).$$

Since $\|\vec{v}\| = 1$, it is clear that if there exist some $K_{j,\ell} > 0$ such that

$$|\partial_j p_\ell| < K_{j,\ell} \rho^\ell \qquad \text{for all} \;\; j = 1, \ldots, m + L + 2, \quad \ell = 1, \ldots, n, \tag{5.15}$$

then the condition (5.7) in Theorem 5.3 will hold for all ℓ. Therefore, we need to prove (5.15) for some fixed j and all $\ell = 1, \ldots, n$. According to (5.1), we have

$$y_j = \frac{P_j}{2\sqrt{P}} = \frac{Q}{R}, \tag{5.16}$$

where the subindex j is used to denote $\frac{\partial}{\partial t_j}$, so that

$$P_j = 2MQ. \tag{5.17}$$

Expressions (5.16), (5.17) together with (5.3), (5.2) imply that Q is a polynomial of degree $L + j$, and the condition (5.4) in differentiated form implies that the differential $y_t \mathrm{d}z = \frac{Q(z)\mathrm{d}z}{R(z)}$ (5.16) is a second kind differential on the Riemann surface of $R(z)$ uniquely determined by the condition that $\frac{Q}{R} = z^j + \mathcal{O}(z^{-2})$ for $|z| \to \infty$ and by the requirement that it is an *imaginary normalized*, namely, all integrals on closed paths are purely imaginary. Then the results of [5] apply and we have the following theorem.

Theorem 5.7 ([5]). *The polynomial $Q(z; \vec{\lambda})$ is continuous in $\vec{\lambda}$.*

Lemma 5.8. *The polynomial $Q(z) = Q(z; \vec{\lambda})$ satisfy the homogeneity*

$$Q(z; \vec{\lambda}) = \nu^{L+j} Q(\nu^{-1} z; \nu^{-1} \vec{\lambda}), \qquad \nu \in \mathbb{R}_+.$$

Proof. The Boutroux condition (5.4) guarantees the validity of Corollary 3.6. Using this corollary, (3.21), (3.22) and the fact that $y(z) = h'(z)$, see (2.22), we obtain

$$y_j(z) = h'_j(z) = \frac{1}{R(z)} \left[\frac{\left(R^2(z)\right)' K_j(z) + R^2(z) K'_j(z)}{2D} \right],$$

so that, according to (5.16)

$$Q(z; \vec{\lambda}) = \frac{\left(R^2(z; \vec{\lambda})\right)' K_j(z; \vec{\lambda}) + R^2(z; \vec{\lambda}) K'_j(z; \vec{\lambda})}{2D(\vec{\lambda})}, \tag{5.18}$$

where $()'$ denotes $\frac{d}{dz}$. Taking into account (3.16), a straightforward calculation shows that

$$D\left(\frac{\vec{\lambda}}{\nu}\right) = \nu^{(L+1)L}D(\vec{\lambda}), \qquad R\left(\frac{z}{\nu};\frac{\vec{\lambda}}{\nu}\right) = \nu^{-(L+1)}R(z;\vec{\lambda}),$$

$$\int \frac{\zeta^k d\zeta}{R(\zeta,\frac{\vec{\lambda}}{\nu})} = \nu^{L-k}\int \frac{\zeta^k d\zeta}{R(\zeta,\vec{\lambda})}. \tag{5.19}$$

Note that, according to Corollary 3.6,

$$K_j(z) = \frac{1}{2\pi i j}
\begin{vmatrix}
\oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{m,1}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)}} & \oint_{\hat{\gamma}_{m,1}}\frac{d\zeta}{(\zeta-z)R(\zeta)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat{\gamma}_{m,L}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,L}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{m,L}}\frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{m,L}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)}} & \oint_{\hat{\gamma}_{m,L}}\frac{d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{c,1}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)}} & \oint_{\hat{\gamma}_{c,1}}\frac{d\zeta}{(\zeta-z)R(\zeta)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat{\gamma}_{c,L}}\frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,L}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)} & \overline{\oint_{\hat{\gamma}_{c,L}}\frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\hat{\gamma}_{c,L}}\frac{\zeta^{L-1}d\zeta}{R(\zeta)}} & \oint_{\hat{\gamma}_{c,L}}\frac{d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat{\mathfrak{m}}}\frac{\zeta^j d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\mathfrak{m}}}\frac{\zeta^{j+L-1}d\zeta}{R(\zeta)} & \oint_{\hat{\mathfrak{m}}}\frac{\zeta^j d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\mathfrak{m}}}\frac{\zeta^{j+L-1}d\zeta}{R(\zeta)} & \oint_{\hat{\mathfrak{m}}}\frac{\zeta^j d\zeta}{(\zeta-z)R(\zeta)}
\end{vmatrix}. \tag{5.20}$$

In light of (5.19) and (5.20), we obtain

$$K_j\left(\frac{z}{\nu};\frac{\vec{\lambda}}{\nu}\right) = \nu^{(L+1)^2-j}K_j(z,\vec{\lambda}), \qquad K_j'\left(\frac{z}{\nu};\frac{\vec{\alpha}}{\nu}\right) = \nu^{(L+1)^2-j+1}K_j'(z,\vec{\lambda}). \tag{5.21}$$

Now the statement of the lemma follows from (5.18), (5.19) and (5.21). ∎

Let $\vec{\alpha} = (\vec{\lambda},\vec{\mu})$. According to (5.17), (see (5.1) for the definition of M)

$$\partial_j P(z;\vec{\alpha}) = 2M(z;\vec{\mu})Q(z;\vec{\lambda}) = G(z;\vec{\alpha}). \tag{5.22}$$

Then, as an immediate consequence of Lemma 5.8, we have

$$G(z;\vec{\alpha}) = \nu^{m+L+j}G\left(\nu^{-1}z;\nu^{-1}\vec{\alpha}\right), \qquad \nu \in \mathbb{R}_+.$$

Equation (5.22) implies $(p_\ell)_t = G_\ell$, where

$$G\left(z;\vec{\alpha}\right) = \sum_{k=0}^{L+m+j} G_k(\vec{\alpha})z^{L+m+j-k}. \tag{5.23}$$

Thus,

$$G_\ell\left(\nu^{-1}\vec{\alpha}\right) = \nu^{-\ell}G_\ell\left(\vec{\alpha}\right). \tag{5.24}$$

Remark 5.9. According to Theorem 5.7, the polynomial $Q(z;\vec{\lambda})$ is continuous even when any number of branchpoints in $\vec{\lambda}$ coincide. Therefore $G_\ell(\vec{\alpha})$ defined in (5.22) is continuous with respect to $\vec{\alpha}$. That also mean that $G_\ell(\vec{\alpha})$ is continuous in the symmetric polynomials p_ℓ of the entries of vector $\vec{\alpha}$. Thus, the right hand side of equation (5.22), considered as a nonlinear autonomous system of ODEs, is continuous in the p_ℓ's. This right hand side is Lipshitz when the branchpoints in $\vec{\lambda}$ are distinct, so the t-evolution of $\vec{\alpha}$ is defined uniquely. On the discriminant locus (where two or more α_j's coincide) the right hand side of equation (5.22) is still continuous but not Lipshitz-continuous; in fact, the uniqueness of solutions of (5.22) fails there (but the existence still holds). This non-uniqueness of the solutions for initial data with coinciding α's (i.e., on the breaking curves) is important because solutions of the continuation equations (5.22) of different genera coincide at those points and hence it allows us to select the appropriate solution that also satisfies the sign inequality requirements.

Lemma 5.10. *Conditions* (5.15) *are satisfied.*

Proof. We prove this statement for an arbitrary fixed $j = 1, \ldots, m + L + 2$. According to (5.22), (5.23), $\partial_j p_\ell = G_\ell(\vec{\alpha})$ for any $\ell = 0, 1, \ldots, n$. Let $\nu = \rho$, where ρ is defined by (5.8). Then, all the roots of the polynomial P, that is, all the components of $\vec{\alpha} = (\vec{\lambda}, \vec{\mu})$, are within the a circle $\widehat{K}\rho$, where $\widehat{K} = \frac{N}{\min\{|p_0|, 1\}}$, see (5.13). Since $G_\ell(\vec{\alpha})$ is a continuous function of $\vec{\alpha}$, it is bounded on the set $\max_k |\alpha_k| \le \widehat{K}$ by some $K_{j\ell} > 0$. Then, according to (5.24),

$$|\partial_j p_\ell| = |G_\ell(\vec{\alpha})| \le K_{j,\ell}\rho^\ell,$$

so that conditions (5.15) are satisfied. ∎

Thus, we have proved the following theorem.

Theorem 5.11. *Suppose that the polynomial $P(z)P = P(z; \vec{t})$, given by* (5.1), (5.2), *is such that the Boutroux condition* (5.4) *is satisfied and $t_{M+L+2} \neq 0$. Let $\vec{t}(s)$, $s \in [0, 1]$, be a smooth deformation of $\vec{t}$, with $\vec{t} = \vec{t}(0)$ preserving the Boutroux condition* (5.4) *and such that $t_{M+L+2}(s)$ remains separated from zero. Then the roots $\vec{\lambda}(s)$, $\vec{\mu}(s)$ are uniformly bounded for $s \in [0, 1]$.*

As an immediate consequence of Theorem 5.11 we obtain the following two corollaries.

Corollary 5.12. *Let $t(s)$, $s \in [0, 1]$, be simple, compact, separated from zero piece-wise smooth contour, connecting points $t(0) = \hat{t}_0$ and $t_1 = \hat{t}(1)$ of $\mathbb{C}$. If $f(z, t)$ is given by* (1.1) *and $T = 2$, then the roots of the polynomial $P(z, t(s))$, given by* (5.1), (5.2), *are uniformly bounded on $s \in [0, 1]$, provided that the Boutroux condition* (5.4) *holds for all $s \in [0, 1]$.*

Corollary 5.13. *For any $t \in \mathbb{C}^*$ there exists the g function $g(z; t)$, satisfying all the sign requirements of Section* 2.2.2*. The corresponding hyperelliptic surface $\mathcal{R}(t)$ has genus $L = 0, 1, 2$. The function h is given by* (3.7) *for $L = 0$ and by* (3.21) *$L > 0$.*

The explicit expression for h with $L = 1$ is given in (6.11).

6 Breaking curves for quartic potential $f(z; t)$

In this section, we prove existence of the breaking curve and confirm their topology, shown on Fig. 2. We start with the following lemma.

Lemma 6.1. *If t_b is a regular breaking point and $\pm z_0$ are the corresponding saddle points on Ω then $h_t(\pm z_0; t_b) \neq 0$.*

Proof. Since the maximal genus for a quartic exponential potential cannot exceed two, see equation (2.23), the genus can be only zero or one along any breaking curve. Comparing (3.6) with (3.10), we see that in the case of genus zero, the conditions

$$h_t(z; t) = 0 \qquad \text{and} \qquad h_z(z; t) = 0 \qquad \text{while} \quad z \neq \pm b \tag{6.1}$$

imply that $t = \frac{1}{4}$, which is the critical breaking point t_2.

In the remaining case of genus one with symmetrical branchpoints, the functions h_z, h_t are given by

$$h_z(z; t) = -tz\sqrt{(z^2 - \lambda_1^2)(z^2 - \lambda_2^2)}, \qquad h_t(z; t) = -\frac{1}{4}(z^2 + K)\sqrt{(z^2 - \lambda_1^2)(z^2 - \lambda_2^2)}, \tag{6.2}$$

where $\pm\lambda_1$, $\pm\lambda_2$ are the branchpoints and K is a constant in z. The h_z expression follows from (2.22). To derive the second expression, we observe that, according to Corollary 3.6, $h_t(z;t) = \frac{R(z)}{|D|}K_t(z)$. In the case of genus one with symmetrical branchpoints, $R(z)$ is an even function. Since the residue at $\zeta = \infty$ of any even function is zero, the last row of the determinant K_t (see (5.20)) contains all zero entries except of the diagonal (the last) entry. Thus, $h_t(z) = -\frac{R(z)}{8\pi i}\oint_{\hat{\mathfrak{M}}}\frac{\zeta^4 d\zeta}{(\zeta-z)R(\zeta)}$, which implies (6.2). Using the large z asymptotics (2.19) and (3.9) for h_z and h_t respectively, we obtain $\lambda_1^2 + \lambda_2^2 = -\frac{2}{t}$ and $K = -\frac{1}{t}$. Then, conditions (6.1), where $z \neq \pm b$ is replaced by $z \neq \pm\lambda_j$, $j = 1, 2$, are not compatible. Thus, we have shown that for the Generic traffic configuration zeroes of h_t cannot coincide with any regular breaking point. ∎

Let t_b be an arbitrary regular breaking point. Then, according to Lemmas 4.5 and 6.1, the breaking curve passing through t_b is a locally smooth curve. Thus, we obtain the following corollary.

Corollary 6.2. *Breaking curves are smooth simple curves consisting of regular breaking points (except, possibly, the endpoints). They do not intersect each other except, possibly, at the critical points $t_0 = -\frac{1}{12}$, $t_2 = \frac{1}{4}$, at the singularity $t = 0$ and at infinity. They can originate and end only at the above mentioned points.*

6.1   Existence of breaking curves

The continuation principle shows that the existence of the g-function (or h-function) on the interval $(-\frac{1}{12}, 0)$ implies the existence of the g function for any $t \in \mathbb{C}^*$. However, it does not describe the breaking curves or even prove their existence. The collection of the breaking for the Generic traffic configuration, the so-called "asymptotic phase portrait", shown on Fig. 2, was calculated numerically in [4].

Let us briefly characterize the breaking curves on Fig. 2. The curves connecting $t_0 = -\frac{1}{12}$ and $t = 0$ are the boundaries of the genus zero region. They are described by (4.2) with h given by (3.7), where $b = b(t)$ is given by (3.4) with the choice of the negative sign. The next breaking curves, connecting t_0 with $t_2 = \frac{1}{4}$ are given by the same equations, except that now $b = b(t)$ is given by (3.4) with the choice of the positive sign. Two different values of $b(t)$ given by (3.4) coincide at the critical point t_0. Moreover, with the positive choice of sign in (3.4) both equations (4.2) with h given by (3.7) are satisfied at $t = \frac{1}{4}$ and $z = 0$.

To derive equations of the genus zero breaking curves we note that, according to (3.6), the two saddle points $\pm z_0$ are defined by

$$z_0^2 = -\frac{1}{t} - \frac{b^2}{2}. \tag{6.3}$$

Thus, the values of h, h_t, see (3.7), (3.10) respectively, at either of the two saddle points $\pm z_0 = \pm z_0(t)$ are

$$h(z_0;t) = \ln\frac{2z_0^2 - b^2 + 2z_0\sqrt{z_0^2 - b^2}}{b^2} - \frac{z_0}{4}\sqrt{z_0^2 - b^2}, \qquad h_t(z_0;t) = \frac{z_0}{4t}\sqrt{z_0^2 - b^2}. \tag{6.4}$$

Introducing $u = \sqrt{1 + 12t} = 2\sqrt{3}\sqrt{t - t_0}$, where $\arg u = 0$ when $t \in (-\frac{1}{12}, 0)$, and using (3.4) and (6.3), we obtain

$$h(z_0(t);t) = \ln\frac{1 \pm 2u + \sqrt{3}\sqrt{u^2 \pm 2u}}{1 \mp u} - \frac{\sqrt{u^2 \pm 2u}}{4\sqrt{3}t}, \qquad h_t(z_0(t);t) = \frac{\sqrt{u^2 \pm 2u}}{4\sqrt{3}t^2}. \tag{6.5}$$

Thus, the equations of genus zero breaking curves are

$$\ln\left|\frac{1 \pm 2\sqrt{1+12t} + \sqrt{3}\sqrt{1+12t \pm 2\sqrt{1+12t}}}{1 \mp \sqrt{1+12t}}\right| = \Re\frac{\sqrt{1+12t \pm 2\sqrt{1+12t}}}{4\sqrt{3}t}. \tag{6.6}$$

Let us now consider the local structure of the breaking curves near $t = t_0$. It would be convenient to introduce a new "hyperbolic" variable p by

$$\cosh p = \frac{z_0}{b}.$$

Then

$$\sinh p = \sqrt{\frac{z_0^2}{b^2} - 1},$$

and, according to (6.4), equation $\Re h(z_0; t) = 0$ for genus zero breaking curves becomes

$$\Re\left[2p - \frac{b^2 \sinh 2p}{8}\right] = 0.$$

Direct calculations yields

$$\sinh^2 p = \frac{\pm 3u}{2(1 \mp u)}, \qquad b^2 = \frac{8}{1 \pm u} = 8\frac{2\cosh^2 p + 1}{4\cosh^2 p - 1} = 8\frac{\cosh 2p + 2}{2\cosh 2p + 1}, \tag{6.7}$$

which transforms (6.1) into

$$\Re[2p] = \Re\frac{\sinh 2p(\cosh 2p + 2)}{2\cosh 2p + 1}. \tag{6.8}$$

The right hand side of (6.8) has Taylor expansion $2p - \frac{8}{45}p^5 + \cdots$ at $p = 0$. Thus, for small p, equation (6.8) becomes

$$\Re p^5 \approx 0.$$

Taking into account (6.7), we obtain that the breaking curves (6.6) with upper signs approach $t_0 = -\frac{1}{12}$ at angles $\pm\frac{2\pi}{5}$, and the breaking curves (6.6) with lower signs approach $t_0 = -\frac{1}{12}$ at angles $\pm\frac{4\pi}{5}$.

In order to prove the transition from genus 0 to genus 2 along the breaking curve (6.6) with the upper sign, it is sufficient to prove that

$$\Re\left[h_t(z_0(t); t)\frac{dt}{ds}\right] > 0, \tag{6.9}$$

where $\frac{dt}{ds}$ is the direction of the crossing from the genus 0 to genus 2 region at a regular breaking point $t = t_b$. Indeed, the positive sign of the inequality reflects the fact that a complementary arc is pinched at a double point and, thus, a new main arc is born.

Let us, for simplicity, take $t = t_b$ sufficiently close to $t_0 = -\frac{1}{12}$. Then $|\arg(t_b - t_0)| \le \pi$ and we know $t_b \notin \mathbb{R}$. So, u is small and $\Re\sqrt{u} > 0, \Im\sqrt{u} \ne 0$. Then the logarithmic term in (6.6), and, thus, the right hand side of (6.6) are positive. But, according to (6.5), $\Re h_t(z_0(t); t) = \Re\left[t^{-1}\frac{\sqrt{u^2 \pm 2u}}{4\sqrt{3}t}\right] < 0$ since $\arg t \approx \pi$. We cross the breaking curve moving along the line $\Im t = \Im t_b$ in the negative direction, so $\frac{dt}{ds} < 0$. Thus, we have proved inequality (6.9) for t close to $t_0 = -\frac{1}{12}$. Since, according to Lemma 6.1, $h_t(z_0(t); t) \ne 0$ on the breaking curve, the inequality (6.9) will hold on the breaking curve until we reach the singularity $t = 0$. Hence we have proved the

transition from genus zero to genus two across the breaking curve (6.6) with the upper choice of the sign.

The second pair of breaking curves, emanating from $t_0 = -\frac{1}{12}$, is given by (6.6) with the lower sign. They end at $t_2 = \frac{1}{4}$. They separate two regions of genus 2 with different topology of the main arcs, see Fig. 2, two middle panels at the bottom and the panel in the middle (lower half plane). As we see, bounded complementary arcs shrink to points as we cross this breaking curve, so that the inequality (6.9) should hold in this case as well. The proof is identical to the previous arguments once we notice that $\Re\sqrt{-u} > 0$. We have finally proved the following lemma.

Lemma 6.3. *Four breaking curves, emanating from $t_0 = -\frac{1}{12}$, separate regions of genera zero and two, or two regions of genus two with different topology of main arcs, as shown on Fig. 2. They are given by equation (6.6).*

The genus one breaking curves consist of the ray $\Re z > \frac{1}{4}$ of $\mathbb{R}$ as well as two bounded curves, symmetrical with respect to $\mathbb{R}$, that connect $t_2 = \frac{1}{4}$ with $t = 0$. Let us first consider this case. Expanding the expression (6.2) for h' at $z = \infty$ and taking into the account the asymptotics (2.19) of h', we obtain

$$h'(z) = -tz^3 - z + \frac{2}{z} + O(z_2) = -tz^3 \left[1 - \frac{\lambda_1^2 + \lambda_2^2}{2z^2} - \frac{\left(\lambda_1^2 - \lambda_2^2\right)^2}{8z^4} + O\left(z^{-6}\right) \right],$$

which yields

$$\lambda_1^2 + \lambda_2^2 = -\frac{2}{t}, \qquad \lambda_1^2 - \lambda_2^2 = \frac{4}{\sqrt{t}} \qquad \text{or} \qquad \lambda_1^2 = \frac{2}{\sqrt{t}} - \frac{1}{t}, \qquad \lambda_2^2 = -\frac{2}{\sqrt{t}} - \frac{1}{t}. \tag{6.10}$$

It follows that the branchpoints are situated on the imaginary axis when $t \in (0, \frac{1}{4})$ and on the cross $(\pm\lambda_1 \in \mathbb{R}, \ \pm\lambda_2 \in i\mathbb{R})$ when $t > \frac{1}{4}$, see Fig. 2, the panel in lower left corner and the panels in the middle on the right side).

To obtain the equations of the genus one breaking curve we need to compute h. Substituting (6.10) into (6.2), we obtain

$$h'(z; t) = -tz\sqrt{\left(z^2 + \frac{1}{t}\right)^2 - \frac{4}{t}},$$

so that

$$h(z; t) = \ln \frac{tz^2 + 1 + \sqrt{(tz^2 + 1)^2 - 4t}}{-2\sqrt{t}} - \frac{tz^2 + 1}{4t} \sqrt{\left(tz^2 + 1\right)^2 - 4t}, \tag{6.11}$$

where we integrated from the branchpoint $-\lambda_2$ (because of the Boutroux condition, $\Re h$ does not depend on the choice of a particular branchpoint). Then

$$h(0; t) = \ln \frac{1 + \sqrt{1 - 4t}}{-2\sqrt{t}} - \frac{\sqrt{1 - 4t}}{4t}, \tag{6.12}$$

so that the equation of the breaking curves, connecting $t_2 = \frac{1}{4}$ with $t = 0$ is

$$\ln \left| \frac{1 + \sqrt{1 - 4t}}{-2\sqrt{t}} \right| = \Re \frac{\sqrt{1 - 4t}}{4t}. \tag{6.13}$$

Moreover, it is clear that

$$\Re h(0; t) < 0 \qquad \text{when} \quad t \in \left(0, \frac{1}{4}\right) \tag{6.14}$$

and $\Re h(0;t) = 0$ when $t > \frac{1}{4}$. Thus, the ray $t > \frac{1}{4}$ is a breaking curve whereas the interval $(0, \frac{1}{4})$ is not.

As one crosses the breaking curve (6.13) from the genus two region (Fig. 2, the upper panel in the center), to the genus one region (Fig. 2, the lower left panel), the central main arc in the first panel shrinks into a double point (on the breaking curve), which disappear as one gets into the region zero region. To prove this scenario, it is sufficient to prove the inequality along the breaking curve (6.13),

$$\Re\left[h_t(0;t)\frac{dt}{ds}\right] < 0. \tag{6.15}$$

As it was observed above, it is sufficient to prove (6.15) at one point of the breaking curve. Let us take this point, $t_b = \rho e^{i\phi}$, sufficiently close to $t = 0$. Then equation (6.13) implies

$$\cos\phi \approx -\rho\ln\rho \qquad \text{or} \qquad \phi \to \frac{\pi}{2} - 0 \tag{6.16}$$

as $\rho \to 0$. So, according to the second equation (6.2) and (6.16), we have

$$\Re[h_t(0;t_b)] = \Re\frac{\sqrt{1-4t_b}}{4t_b^2} \approx \frac{\cos 2\phi}{4\rho^2} < 0.$$

Finally, $\frac{dt}{ds} > 0$ when we cross the breaking curve (6.13) along the line $\Im t = \Im t_b$ with a sufficiently small $\Im t_b > 0$. Thus, we have proved the following lemma.

Lemma 6.4. *The breaking curves connecting $t_2 = \frac{1}{4}$ with $t = 0$, separate regions of genera one (inside the curves) and two (outside the curves), see Fig. 2. They are given by equation (6.13). The breaking curve $\Im t = 0$, $\Re t > \frac{1}{4}$ is surrounded by the genus two region.*

All the breaking curves, shown on Fig. 2, emanate from either critical point $t_0 = -\frac{1}{12}$ or $t_2 = \frac{1}{4}$. The former are genus zero breaking curves whereas the latter are genus one breaking curves (except those, that connect t_0 and t_2, which are genus zero). These breaking curves were described in Lemmas 6.3 and 6.4. To complete the proof of the asymptotic phase portrait of Fig. 2, we still need to show that there are no more breaking curves, that is, there are no breaking curves that do not pass through a critical point, or $t = 0$ or ∞. According to Corollary 6.2, any such curve must be a simple smooth closed loop, ν, that does not intersect any other breaking curve. We rule out existence of such ν by observing that, according to (4.2), each breaking curve is zero level curve of $\Re h(z_0(t),t) = 0$, where $z_0(t)$ is the double point. Then, it remains to show that $\Re h(z_0(t),t)$ is a harmonic function of t.

We start with a genus one level curve, where $z_0(t) \equiv 0$ and so $h(z_0(t),t) = h(0,t)$ is given by (6.12). This is a harmonic function with a singularity at $t = 0$ and the branchcut $\mathbb{R}^+$. Thus, if $\Re h(0,t) = 0$ along bounded ν, then $\nu \cap \mathbb{R}^+ \neq \varnothing$. But $\nu \cap (\frac{1}{4}, +\infty) = \varnothing$ since $(\frac{1}{4}, +\infty)$ is a breaking curve. Thus, $\nu \cap (0, \frac{1}{4}) \neq \varnothing$, which contradicts (6.14). Finally, if ν is an unbounded loop that does not intersect $\mathbb{R}^+$, then ν originates and ends at infinity and $\Re h(0,t)$ is harmonic inside ν. Since $\Re h(0,t) = O(t^{-1})$ as $t \to \infty$, we conclude that such loop cannot exist.

Consider now a genus zero level curve given by the first equation (6.5). The function $\psi(t) = \Re h^0(z_0(t),t)$, given by (6.4) (see also (6.5)) in the complex t plane, has singularities at $\{t_0, t_2, 0, \infty\}$, with branchcut $\mathbb{R} \setminus (-\frac{1}{12}, 0)$. Thus, if ν is bounded, it must intersect $(-\infty, -\frac{1}{12})$ or $(0, \frac{1}{4})$. In order to prove that this is not the case, we calculate

$$\psi'(t) = \Re h_t(z_0(t),t) + \Re[h'(z_0(t),t)z_0'(t)] = \Re h_t(z_0(t),t),$$

since $h'(z_0(t),t) = 0$ for every t and $z_0'(t)$ is bounded for $t \notin \{t_0, t_2, 0, \infty\}$.

Let us first consider the interval $(-\infty, -\frac{1}{12})$. According to (3.7), $\psi(t_0) = 0$. Thus, it is sufficient to prove that $\psi'(t) \neq 0$ on $(-\infty, -\frac{1}{12})$. Direct calculations (for $t \in \mathbb{R}$) yield

$$\psi'(t) = \frac{1}{8t} \Re \sqrt{z_0^2(t)\left[z_0^2(t) - b^2\right]} = \frac{\Re\sqrt{(1 + 12t) \pm 2\sqrt{1 + 12t}}}{8\sqrt{3}t^2}, \tag{6.17}$$

where we used

$$z_0^2(t) = \frac{-2 \mp \sqrt{1 + 12t}}{3t}, \qquad z_0^2(t) - b^2 = \frac{\mp\sqrt{1 + 12t}}{t}.$$

When $t \in (-\infty, -\frac{1}{12})$, we have $\sqrt{1 + 12t} \in i\mathbb{R}$, so that the radical is in the right half plane. Thus, $\psi'(t) > 0$, so that $\psi(t) < 0$ on this interval. Therefore, ν cannot cross $t \in (-\infty, -\frac{1}{12})$. This estimate also shows that genus zero breaking curves are bounded.

Consider the interval $t \in (0, \frac{1}{4})$, we distinguish the functions $\psi_l(t)$ and $\psi_u(t)$, corresponding to the lower/upper choice of signs in (6.4), (6.5) (and also (6.17)). It follows directly from (6.5) that $h^0(z_0(t), t)$ with the lower choice of the signs does not have branching at $t = 0$ (corresponds to $u = 1$ on the main sheet of t). Thus, we have to study only $\psi = \psi_u$. A straightforward computation yields

$$\psi_u\left(\frac{1}{4}\right) = \ln\left(2\sqrt{\frac{2}{3}} + \frac{1}{3}\right) + 2\sqrt{\frac{2}{3}} > 0 \qquad \text{and} \qquad \psi_u'(t) < 0,$$

since both $z_0^2(t)$, $z_0^2(t) - b^2$ have argument π on $(0, \frac{1}{4})$. Thus, ν cannot cross $(0, \frac{1}{4})$, and we completed the proof of the global asymptotic portrait from Fig. 2.

7 Asymptotics of recurrence coefficients in the higher genera regions

The leading order behavior of the recurrence coefficients and of the pseudonorms in the genus zero region and in full vicinities of the critical points t_0, t_2 was obtained in [4, 13] and [7] respectively. In this section we use Theorem 2.18 to find the leading order behaviour of $\alpha_n(t)$ and $\mathbf{h}_n(t)$ in regions of genus one and two.

7.1 Genus 1 region

Let us start with the genus one region $\mathfrak{D}_1$ that contains the segment $(0, \frac{1}{4})$. Explicit formulae for the corresponding branchpoints are given by (6.10), where, thanks to the symmetry, we use notations $-\lambda_2, -\lambda_1, \lambda_1, \lambda_2$ for consecutive branchpoints along the Riemann–Hilbert contour instead of $\lambda_0, \lambda_1, \lambda_2, \lambda_3$.

The negatively oriented loop $\hat{\gamma}_{m,1}$ around the branchcut λ_1, λ_2 is the A cycle, whereas the loop $-\hat{\gamma}_{c,1}$ is the B cycle. The period τ and the normalized holomorphic differential ω are elliptic integrals

$$\tau = \frac{-\int_{-\lambda_1}^{\lambda_1} \frac{d\zeta}{R(\zeta)}}{2\int_{\lambda_1}^{\lambda_2} \frac{d\zeta}{R(\zeta)}}, \qquad \omega(z) = \frac{dz}{2\int_{\lambda_1}^{\lambda_2} \frac{d\zeta}{R(\zeta)}R(z)},$$

where

$$R(\zeta) = \sqrt{\left(\zeta^2 - \lambda_1^2\right)\left(\zeta^2 - \lambda_2^2\right)}.$$

Direct calculations yield

$$2 \int_{\lambda_1}^{\lambda_2} \frac{d\zeta}{R(\zeta)} = -\frac{2\sqrt{t}}{\sqrt{1 + 2\sqrt{t}}} \mathbf{K} \left(\frac{2t^{\frac{1}{4}}}{\sqrt{1 + 2\sqrt{t}}} \right), \qquad \tau = i \frac{\mathbf{K} \left(\sqrt{\frac{1 - 2\sqrt{t}}{1 + 2\sqrt{t}}} \right)}{\mathbf{K} \left(\frac{2t^{\frac{1}{4}}}{\sqrt{1 + 2\sqrt{t}}} \right)}, \tag{7.1}$$

where $\mathbf{K}$ denotes complete elliptic integral. Using the fact that $R(z)$ is an even function and (3.18), we calculate

$$2\mathfrak{u}(\infty) = -\frac{\tau}{2}, \qquad \widetilde{\varpi} = 2\mathfrak{u}(\infty) = -\frac{\tau}{2}. \tag{7.2}$$

Let us normalize the traffics ϱ_j, see (1.1), so that the traffic on $\gamma_{m,0} = (-\lambda_2, -\lambda_1)$ is 1. We then denote the traffic on $\gamma_{m,0} = (\lambda_1, \lambda_2)$ by ρ_1. Note that in the Generic case, considered in the paper, $\rho_1 \neq 0$.

Lemma 7.1. *For any $t \in \mathfrak{D}_1$ the model RHP* (2.26) *has a solution* (*for all sufficiently large $N \in \mathbb{N}$*) *if and only if*

$$\frac{\ln \rho_1 - i\pi}{2\pi i} \neq 0 \mod \frac{\tau}{2}\mathbb{Z} \oplus \mathbb{Z}, \tag{7.3}$$

where τ is given by (7.1).

Lemma 7.1 follow from Corollary 2.18 and (7.2). Combining (7.2) with Theorem 2.19 and using the properties (2.31) of the Riemann theta function, we obtain the theorem below. The constant

$$\ell = -\frac{1}{2} + \frac{1}{4t} - \frac{1}{2}\ln(t) + i\pi$$

can be easily obtained from (6.11).

Theorem 7.2. *If $t \in \mathfrak{D}_1$ is such that* (7.3) *is satisfied then*

$$\alpha_n(t) = \frac{\sqrt{1 - 4t} - 1}{2t}$$

$$\times \begin{cases} \rho_1 e^{-2\pi i \frac{\tau}{2}} \dfrac{\Theta^2\left(\frac{\ln \rho_1}{2\pi i}\right)\Theta^2(0)}{\Theta^2\left(\frac{\ln \rho_1}{2\pi i} - \frac{\tau}{2}\right)\Theta^2\left(\frac{\tau}{2}\right)} + O\left(n^{-1}\right) & \text{if } n \text{ is odd}, \\[2em] \rho_1^{-1} \dfrac{\Theta^2\left(\frac{\ln \rho_1}{2\pi i} - \frac{\tau}{2}\right)\Theta^2(0)}{\Theta^2\left(\frac{\ln \rho_1}{2\pi i}\right)\Theta^2\left(\frac{\tau}{2}\right)} + O\left(n^{-1}\right) & \text{if } n \text{ is even}, \end{cases} \tag{7.4}$$

$$\mathbf{h}_n(t) = \pi i t^{-\frac{n}{2}} \frac{\sqrt{1 + 2\sqrt{t}} - \sqrt{1 - 2\sqrt{t}}}{(-1)^{(n+1)} e^{\frac{n}{2}\left(1 - \frac{1}{2t} + i\pi\tau\right)}\sqrt{t}}$$

$$\times \begin{cases} e^{-\frac{\pi i \tau}{2}} \dfrac{\Theta\left(\frac{\ln \rho_1}{2\pi i}\right)\Theta(0)}{\Theta\left(\frac{\ln \rho_1}{2\pi i} + \frac{\tau}{2}\right)\Theta\left(-\frac{\tau}{2}\right)} + O\left(n^{-1}\right) & \text{if } n \text{ is odd}, \\[2em] \dfrac{\Theta\left(\frac{\ln \rho_1}{2\pi i} - \frac{\tau}{2}\right)\Theta(0)}{\Theta\left(\frac{\ln \rho_1}{2\pi i}\right)\Theta\left(-\frac{\tau}{2}\right)} + O\left(n^{-1}\right) & \text{if } n \text{ is even}. \end{cases}$$

According to Theorem 7.2, the even and odd subsequences of $\alpha_n(t)$ attain, in general, different limits. The special values of ρ_1, when these limits are equal, are provided in the following corollary.

Corollary 7.3. *Under the same conditions as Theorem* 7.2, *the requirement*

$$\frac{\ln \rho_1}{2\pi i} = \left(\frac{1}{4} + k\right)\tau + m, \qquad where \quad k \in \mathbb{Z}, \quad m \in \frac{1}{2}\mathbb{Z}, \tag{7.5}$$

implies that the even and odd subsequences of $\alpha_n(t)$ *attain a single limit*

$$\alpha_n(t) = \frac{\sqrt{1-4t}-1}{2t}\frac{e^{2\pi i\left(m-\frac{\tau}{4}\right)}\Theta^2(0)}{\Theta^2\left(\frac{\tau}{2}\right)} + O\left(n^{-1}\right). \tag{7.6}$$

Proof. Denote $x = \frac{\ln \rho_1}{2\pi i}$. According to (7.4), the limits of the even and odd subsequences of $\alpha_n(t)$ coincide when

$$e^{2\pi i\left(2x-\frac{\tau}{2}\right)}\Theta^4(x) = \Theta^4\left(x - \frac{\tau}{2}\right). \tag{7.7}$$

Substituting (7.5) into (7.7) and using properties (2.31), we obtain the first statement. Direct calculations also lead to (7.6). ∎

7.2 Genus 2 solutions

In this section we adjust the results of Theorem 2.19 to the case of $L = 2$ and symmetric branchpoints with respect to the origin. Because of the symmetry, we denote the main arc containing zero as $\gamma_{m,0}$, its endpoints (branchpoints) as $\pm\lambda_0$, the remaining main arcs as $\gamma_{m,\pm 1}$ and their branchpoints as $\pm\lambda_1, \pm\lambda_2$ respectively. In the genus two region we distinguish two different configurations of the main arcs: the standard configuration shown on two panels in the middle of Fig. 2 and configuration with whiskers shown on the central and right upper panels, the second and third (from below) left panels, etc. These two regions are separated by two symmetrical genus zero breaking curves, connecting the critical points $t_0 = -\frac{1}{12}$ and $t_2 = \frac{1}{4}$; the whisker region is outside and the standard genus two region (together with genera zero and one regions) is inside these breaking curves.

We start with calculating the Riemann period matrix τ. We take negatively oriented loops around $\gamma_{m,1}, \gamma_{m,-1}$ as A cycles A_1, A_2 respectively, and oriented loops connecting λ_1 and λ_0 and $-\lambda_1$ and $-\lambda_0$ (on the main sheet and going back on the second sheet) as B cycles B_1, B_2 respectively.

The normalized holomorphic differentials $\omega_j = \frac{P_j(z)dz}{R(z)}$, defined by (2.29), in the case of genus two and symmetric branchpoints have the form

$$\omega_1 = \frac{\varkappa_1 z + \varkappa_2}{R(z)}dz, \qquad \omega_2 = \frac{-\varkappa_1 z + \varkappa_2}{R(z)}dz, \tag{7.8}$$

where

$$\varkappa_1 = \left(2\int_{\gamma_{m,1}}\frac{zdz}{R(z)}\right)^{-1}, \qquad \varkappa_2 = \left(2\int_{\gamma_{m,1}}\frac{dz}{R(z)}\right)^{-1},$$

with

$$R(z) = \sqrt{\prod_{j=0}^{2}\left(z^2 - \lambda_j^2\right)}.$$

Here we used the fact that $R(z)$ is an odd function in $\mathbb{C}\setminus\mathfrak{M}$. Further, the symmetry of B cycles implies that $\tau_{11} + \tau_{21} = \tau_{12} + \tau_{22}$, where $\tau_{j,k}$ are the entries of the Riemann period matrix τ,

see (2.30). Since τ is symmetric, we obtain $\tau_{11} = \tau_{22}$, i.e., the two by two matrix τ has equal diagonal entries.

Let us now consider the vector $\vec{\tilde{\omega}}$ from the RHP (2.28). Using (3.18) and Cauchy's residue theorem, we obtain

$$\vec{\tilde{\omega}} = 2\mathfrak{u}_\infty - \frac{1}{2\pi i} \int_{\hat{\mathfrak{M}}} \frac{\left(\frac{tz^4}{4} + \frac{z^2}{2}\right)(\pm \varkappa_1 z + \varkappa_2)dz}{R(z)} = 2\mathfrak{u}_\infty + \frac{1 + \frac{t}{4} \sum_{j=0}^{2} \lambda_j^2}{\int_{\gamma_{m,1}} \frac{dz}{R(z)}}(1,1)^t, \tag{7.9}$$

where, in view of (7.8), the plus sign corresponds to $\tilde{\omega}_1$ and the minus sign to $\tilde{\omega}_2$. According to (2.19), (2.22), in the genus two region

$$h'(z) = -t\sqrt{\prod_{j=0}^{2} (z^2 - \lambda_j^2)} = -tz^3 \sqrt{\prod_{j=0}^{2}\left(1 - \frac{\lambda_j^2}{z^2}\right)} = -tz^3 - z + \frac{2}{z} + O(z^{-2})$$

so that, similarly to (6.10), we obtain

$$\sum_{j=0}^{2} \lambda_j^2 = -\frac{2}{t}, \qquad \sum_{j<k} \lambda_j^2 \lambda_k^2 = \frac{1}{t^2} - \frac{4}{t}. \tag{7.10}$$

The remaining third condition for the branchpoints $\pm\lambda_j$, $j = 0,1,2$, is the Boutroux condition

$$\int_{\lambda_0}^{\lambda_1} h'(z)dz = -t\int_{\lambda_0}^{\lambda_1} \sqrt{\prod_{j=0}^{2} (z^2 - \lambda_j^2)} \in \mathbb{R}.$$

Equations (7.9) and (7.10) yield

$$\vec{\tilde{\omega}} = 2\mathfrak{u}_\infty + \frac{1}{2\int_{\gamma_{m,1}} \frac{dz}{R(z)}}(1,1)^t = 2\mathfrak{u}_\infty + \vec{\varkappa}_2, \tag{7.11}$$

where $\vec{\varkappa}_2 = \varkappa_2(1,1)^t$. According to (1.5), the traffic on $\gamma_{m,0}$ is 1. Let $\vec{\rho} = \frac{1}{2\pi i}(\ln \rho_1, \ln \rho_2)^t$, where $\ln \rho_j$ is the logarithmic term in the exponential of the jump matrix on $\gamma_{m,j}$ in the RHP (2.26).

Substituting (7.8), (7.11) into (2.43), (2.44) we obtain the following theorem.

Theorem 7.4. *Let $\mathcal{T}$ be a compact subset in the genus two region. The recurrence coefficients and the pseudonorms satisfy*

$$\alpha_n = (\lambda_0 + \lambda_2 - \lambda_1)^2 \frac{\Theta(2(N-1)\mathfrak{u}_\infty + N\vec{\varkappa}_2 + \vec{\rho})\Theta(2(N+1)\mathfrak{u}_\infty + N\vec{\varkappa}_2 + \vec{\rho})\Theta^2(0)}{4\Theta^2(2N\mathfrak{u}_\infty + N\vec{\varkappa}_2 + \vec{\rho})\Theta^2(2\mathfrak{u}_\infty)}$$

$$+ O(N^{-1}),$$

$$\mathbf{h}_n = \pi(\lambda_0 + \lambda_2 - \lambda_1)\frac{\Theta(2(N+1)\mathfrak{u}_\infty + N\vec{\varkappa}_2 + \vec{\rho})\Theta(0)}{\Theta(2N\mathfrak{u}_\infty + N\vec{\varkappa}_2 + \vec{\rho})\Theta(2\mathfrak{u}_\infty)}e^{N\ell} + O(N^{-1}), \tag{7.12}$$

as $N \to \infty$ uniformly in $t \in \mathcal{T}$ provided that

$$\Theta(2N\mathfrak{u}_\infty + N\vec{\varkappa}_2 + \vec{\rho}) \qquad \text{is separated from } 0.$$

Remark 7.5. If we change the orientation of the contour $\gamma_{m,2}$ in the model RHP (2.26), then, according to (2.43), (2.44), we will have to replace in (7.12) the prefactors $(\lambda_0 + \lambda_2 - \lambda_1)$ by λ_0, the vector $\vec{\rho}$ by $\vec{\rho} + (0,1)^t$ and the real part $\Re\tau = \frac{1}{2}\mathbf{1}$ of the period matrix for all the theta functions in (7.12) by $\Re\tau = \frac{1}{2}\sigma_3$. For justification and more details see [6, Section 3].

Acknowledgements

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