ANALYSIS OF THE NON-STATIONARY MODEL OF COUPLED OSCILLATORS WITH INDUCTIVE COUPLING

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The model of coupled oscillators plays an important role in modern physics. It is used for description of various processes: from oscillations of atoms in solid states to electromagnetic oscillations in slow-wave structures. The model with "short-range coupling" is the most widely used, for which a separate oscillator is coupled with two adjacent ones only. There are two main types of oscillators coupling: "capacitive" ("electric", "power") and "inductive" ("magnetic", "inertial"). In the first case, the coupling is proportional to the amplitudes of oscillations in the adjacent cells, in the second one – to the second derivative of these amplitudes. For numerical study of dynamics of a system that can be described by a model of coupled oscillators with an "inductive" coupling, it is necessary to find explicit expressions for the second derivatives of the amplitudes. To find these expressions, we propose to use the methods of solving of difference equations. The results of the analysis of this method are given in the paper.

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INTRODUCTION

The model of coupled oscillators plays an important role in modern physics. It is used for description of various processes: from oscillations of atoms in solid states to electromagnetic oscillations in slow-wave structures. The model of coupled oscillators (for slow-wave structures the term "coupled resonators" is often used) is very simple. An equation for each oscillator contains the oscillator amplitude, multiplied by the square of the resonant frequency, second derivative of the oscillator amplitude and coupling terms which are proportional to the amplitudes of the neighbouring oscillators multiplied by coupling coefficients. The model with "short-range coupling" (nearest neighbouring coupling), when each oscillator in the chain is coupled with two adjacent ones only, is used most widely. There are two main types of oscillators coupling: "capacitive" ("electric", "power") and "inductive" ("magnetic", "inertial").

For using the numerical methods in investigation of the non-stationary (transient) behaviour of the coupled oscillators, each equation for complex amplitude must contain only one second derivative of the oscillator amplitudes. This condition is automatically fulfilled for the chain of electrically coupled oscillators, because in this case the coupling terms contain the amplitudes of neighbouring oscillators. When oscillators in the chain are coupled magnetically, the coupling terms contain the second derivatives of the amplitudes of neighbouring oscillators and in one equation we have several second derivatives. Direct use of numerical methods is impossible in this case. If the coupling coefficients are small, these second derivatives of the amplitudes of neighbouring oscillators can be replaced by the amplitudes multiplied by the square of the resonant frequency. But the issue of using the numerical methods in general case has not been clarified so far [1 - 5].

We propose a method of solving this problem. The results of using this method are given for two systems: an infinite chain of magnetically-coupled cells and the backward travelling wave (BTW) structure.

1. INFINITE CHAIN OF OSCILLATORS

Let's consider an infinite chain of lossless magnetically coupled oscillators[1]. The chain is described by the following system of the second-order differential equations

$$(1+2\varepsilon)\frac{d^2A_n}{dt^2}+\omega_0^2A_n-\varepsilon\left(\frac{d^2A_{n-1}}{dt^2}+\frac{d^2A_{n+1}}{dt^2}\right)=F(t)\delta_{n,p}. \quad (1)$$

Here A_n is the amplitude of the n-th oscillator; ω_0 is the oscillator resonant frequency (all the oscillators are identical); ε is the coupling coefficient; $-\infty < n < \infty$, $F(t)$ is external force that acts on the *p*-th oscillator.

For the case $F(t)\equiv 0$ and the time dependence of the amplitudes as $\exp(-i\omega t)$, the solution of the infinite system (1) can be written as

$$A_n = A_0\rho^n, \quad (2)$$

where ρ is the solution of a characteristic equation

$$\rho^2-\rho\frac{\omega^2(1+2\varepsilon)-\omega_0^2}{\omega^2\varepsilon}+1=0. \quad (3)$$

By introducing

$$x_n=\frac{d^2A_n}{dt^2}, \quad (4)$$

the system (1) can be rewritten as follows:

$$x_n(1+2\varepsilon)-\varepsilon(x_{n-1}+x_{n+1})=-\omega_0^2A_n+F(t)\delta_{n,p}. \quad (5)$$

This is an inhomogeneous second-order difference equation with constant coefficients. The Green's function solution of this equation is

$$x_n=-\omega_0^2\sum_{k=-\infty}^{\infty}X_{n-k}\cdot A_k+X_{n-p}F(t), \quad (6)$$

where

$$X_{n-k}=\begin{cases}\dfrac{g_1^{n-k}}{1+2\varepsilon(1-g_1)}, & n\ge k,\\[2ex] \dfrac{g_2^{n-k}}{1+2\varepsilon(1-g_1)}, & n\le k,\end{cases} \quad (7)$$

$$g_1=\frac{1+2\varepsilon}{2\varepsilon}-\sqrt{\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^2-1}. \quad (8)$$

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[1] For example, magnetic coupling in slow-wave disk-loaded structures is realized through holes or slots out of disk axis.

For small ε ($\varepsilon << 1$) $g_1 \simeq \varepsilon$.

Using the definition for x_n , we can write

$$\left[1+2\varepsilon\left(1-g_1\right)\right]\frac{d^2A_n}{dt^2}+\omega_0^2A_n+ \\ +\omega_0^2\sum_{k=1}^{\infty}g_1^k\left(A_{n-k}+A_{n+k}\right)=g_1^{|n-p|}F(t). \tag{9}$$

The system of equations (1) and the system of equations (9) describe the same object: the infinite chain of identical magnetically coupled oscillators.

It can be shown that the homogeneous system of equations (9) has the solution of the form (2) with the same characteristic multiplier ρ.

It is useful to pay attention on several characteristic features of the system of equations (9).

Analysis of this system shows that instead of magnetic neighbouring coupling (“short connection”) in the system (1) we obtained the electrically coupled oscillators with “long connection” (each oscillator is connected with all the others).

The external force that acts on the p -th oscillator in the chain of oscillators with magnetic neighbouring coupling transformed into the force that acts on all elements of the chain.

If we can formulate the rule for truncating the sum, the system of equation (9) is suitable to carry out the numerical analysis of non-stationary behaviour of the oscillator chain with magnetic coupling. As for small ε ($\varepsilon << 1$) $g_1 \simeq \varepsilon$, then for $\varepsilon < 1$ we can expect that the sum in the system (9) converges and can be truncated

$$\sum_{k=1}^{\infty}g_1^k\left(A_{n-k}+A_{n+k}\right)\simeq\sum_{k=1}^{M_c}g_1^k\left(A_{n-k}+A_{n+k}\right). \tag{10}$$

Bellow, on the example of the backward travelling wave section, we shall show that the sum in the system (9) really converges and the number of couplings M_c that should be taken into account is determined by the value of ε.

2. BACKWARD TRAVELLING WAVE STRUCTURE

Let’s consider the N cells of disk-loaded waveguide (Fig. 1). The coupling between cells is magnetic, so the coupling slots (or holes) are located at the disks periphery.

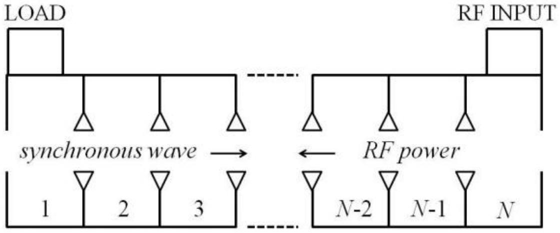

Fig. 1. Backward traveling wave structure

Considered structure is described by the following set of N equations

$$\begin{cases} \dfrac{d^2A_1}{d\tau^2}\left(1+\varepsilon\right)+A_1\dfrac{\omega_1^2}{\omega_p^2}-\varepsilon\dfrac{d^2A_2}{d\tau^2}+\dfrac{\omega_1\beta_1}{\omega_pQ_1}\dfrac{dA_1}{d\tau}=0, \\ \vdots \\ \dfrac{d^2A_n}{d\tau^2}\left(1+2\varepsilon\right)+A_n\dfrac{\omega_0^2}{\omega_p^2}-\varepsilon\dfrac{d^2A_{n-1}}{d\tau^2}- \\ \qquad -\varepsilon\dfrac{d^2A_{n+1}}{d\tau^2}=0, \\ \vdots \\ \dfrac{d^2A_N}{d\tau^2}\left(1+\varepsilon\right)+A_N\dfrac{\omega_N^2}{\omega_p^2}-\varepsilon\dfrac{d^2A_{N-1}}{d\tau^2}+ \\ \qquad +\dfrac{\omega_N\beta_N}{\omega_pQ_N}\dfrac{dA_N}{d\tau}=-\dfrac{2\omega_N\beta_N}{\omega_pQ_N\theta}\dfrac{dU}{d\tau}, \end{cases} \tag{11}$$

where U is the amplitude of input RF pulse; A_n is the amplitude of electric field in the nth cell, $\tau=\omega_pt$, ω_p is the operating frequency; ω_1 and ω_N are the eigen frequencies of the couplers; ω_0 is the eigen frequency of the cells, ε is the coupling coefficient; $\beta_N/Q_N=\varepsilon\sin\varphi\,\omega_p/\omega_N$, $\beta_1/Q_1=\varepsilon\sin\varphi\,\omega_p/\omega_1$, φ is the phase advance per cell.

For the time dependence of the amplitudes and external signal as $\exp(-i\omega t)$ on the basis of the system (11) the parameters of resonators and its coupling were chosen to provide the phase shift between resonators $\varphi=4\pi/5$ at the operating frequency $f_p=2856$ MHz. Moreover, the couplers parameters were chosen to provide the absence of reflections from terminal cells at this frequency [6, 7].

Denoting $d^2A_n/d\tau^2=\tilde{x}_n$ we can rewrite the set of equations (11) in the following form

$$\begin{cases} \left(1+\varepsilon\right)\tilde{x}_1-\varepsilon\tilde{x}_2=-A_1\dfrac{\omega_1^2}{\omega_p^2}-\dfrac{\omega_1\beta_1}{\omega_pQ_1}\dfrac{dA_1}{d\tau}, \\ \vdots \\ \left(1+2\varepsilon\right)\tilde{x}_n-\varepsilon\left(\tilde{x}_{n-1}+\tilde{x}_{n+1}\right)=-A_n\dfrac{\omega_0^2}{\omega_p^2}, \\ \vdots \\ \left(1+\varepsilon\right)\tilde{x}_N-\varepsilon\tilde{x}_{N-1}= \\ \quad =-A_N\dfrac{\omega_N^2}{\omega_p^2}-\dfrac{\omega_N\beta_N}{\omega_pQ_N}\dfrac{dA_N}{d\tau}+\dfrac{2\omega_N\beta_N}{\omega_pQ_N\theta}\dfrac{dU}{d\tau}. \end{cases} \tag{12}$$

By analogy with the infinite structure, the solution of the system of equations (12) can be expressed through the Green’s function

$$\frac{d^2A_n}{d\tau^2}=-\tilde{X}_{n,1}\left(A_1\frac{\omega_1^2}{\omega_p^2}+\frac{\omega_1\beta_1}{\omega_pQ_1}\frac{dA_1}{d\tau}\right)- \\ -\frac{\omega_0^2}{\omega_p^2}\sum_{k=2}^{N-1}\tilde{X}_{n,k}\cdot A_k- \\ -\tilde{X}_{n,N}\left(A_N\frac{\omega_N^2}{\omega_p^2}+\frac{\omega_N\beta_N}{\omega_pQ_N}\frac{dA_N}{d\tau}-\frac{2\omega_N\beta_N}{\omega_pQ_N\theta}\frac{dU}{d\tau}\right). \tag{13}$$

Here the matrix $\tilde{X}$ is the Green’s function of the system (12)

$$\begin{cases}(1+\varepsilon)\tilde{X}_{1,k}-\varepsilon\tilde{X}_{2,k}=\delta_{1,k}\,,\\ \vdots\\ (1+2\varepsilon)\tilde{X}_{n,k}-\varepsilon\left(\tilde{X}_{n-1,k}+\tilde{X}_{n+1,k}\right)=\delta_{n,k}\,,\\ \vdots\\ (1+\varepsilon)\tilde{X}_{N,k}-\varepsilon\tilde{X}_{N-1,k}=\delta_{N,k}\,,\end{cases} \qquad (14)$$

where $1 \le k \le N$.

In general case all elements of the matrix $\tilde{X}$ are nonzero and the system (13) describes the interaction of individual resonators with all the other ones. Using in the system (13) truncated matrix $\bar{X}$ instead of the Green's function $\tilde{X}$, we can restrict the number of interacting resonators. For example, for the matrix

$$\bar{X}=\begin{pmatrix}\tilde{X}_{1,1} & \tilde{X}_{1,2} & \tilde{X}_{1,3} & 0 & 0 & 0 & 0 & \dots\\ \tilde{X}_{2,1} & \tilde{X}_{2,2} & \tilde{X}_{2,3} & \tilde{X}_{2,4} & 0 & 0 & 0 & \dots\\ \tilde{X}_{3,1} & \tilde{X}_{3,2} & \tilde{X}_{3,3} & \tilde{X}_{3,4} & \tilde{X}_{3,5} & 0 & 0 & \dots\\ 0 & \tilde{X}_{4,2} & \tilde{X}_{4,3} & \tilde{X}_{4,4} & \tilde{X}_{4,5} & \tilde{X}_{4,6} & 0 & \dots\\ 0 & 0 & \tilde{X}_{5,3} & \tilde{X}_{5,4} & \tilde{X}_{5,5} & \tilde{X}_{5,6} & \tilde{X}_{5,7} & \dots\\ \dots & & & & & & & \end{pmatrix} \qquad (15)$$

each resonator is coupled with the four neighboring ones[2]. For convenience, we shall mark the case when each resonator is coupled with the two neighboring ones as $M_c = 1$, when each resonator is coupled with the four neighboring ones as $M_c = 2$, when each resonator is coupled with the six neighboring ones as $M_c = 3$ and so on. The case of interacting of individual resonators with all the other ones we shall mark as $M_c = N$.

We used the Runge-Kutta method to find approximate solution of the system (13). As an input signal we used the wave front of the type

$$U(t)=e^{-i\omega_p t}\cdot\begin{cases}\sin\left(\dfrac{\pi}{2}\dfrac{t}{t_p}\right), & 0\le t\le t_p\,,\\ 1, & t>t_p\,.\end{cases} \qquad (16)$$

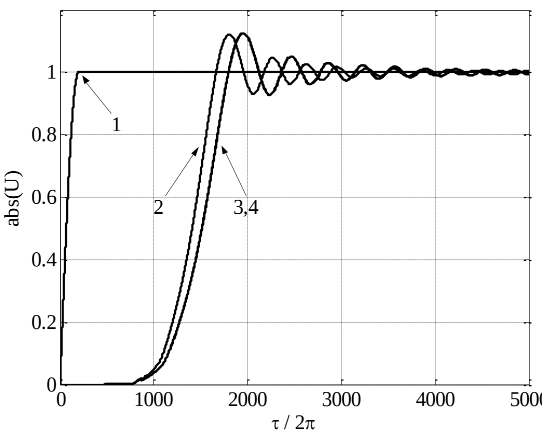

Fig. 2. The time dependence of the input (1) and output amplitudes (2 – M_c=1; 3 – M_c=2; 4 – M_c=N), ε= 0.02

[2] Except for the terminal resonators

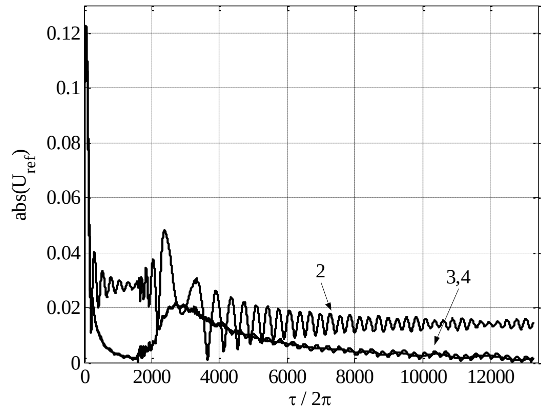

Fig. 3. The time dependence of the reflected signal (2 – M_c=1; 3 – M_c=2; 4 – M_c= N), ε = 0.02

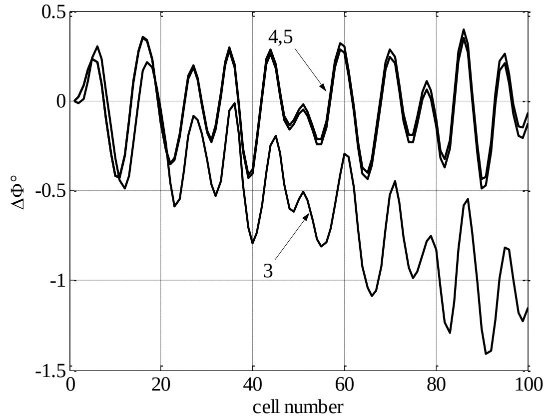

Fig. 4. Deviation from a predetermined phase distribution (3 – M_c=2; 4 – M_c=3; 5 – M_c=N), ε = 0.02, τ = 2π·13000

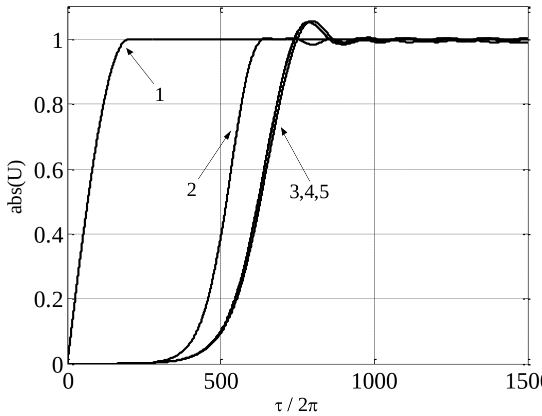

Fig. 5. The time dependence of the input (1) and output amplitudes (2 – M_c=1;3 – M_c=2; 4 – M_c=3; 5 – M_c=N), ε = 0.06

The time dependence of the input and output amplitudes for the structure with $\varphi = 4\pi/5$ and $N = 100$ is shown in Fig. 2 (ε = 0.02, $\beta_g = v_g/c$ ~ -0.03), Fig. 5 (ε = 0.06, $\beta_g = v_g/c$ ~ -0.073) and Fig. 8 (ε = 0.2, $\beta_g = v_g/c$ ~ -0.17). The time dependence of the amplitude of the reflected signal is shown in Figs. 3, 6, 9 for the same parameters. Deviations of phase distributions from a predetermined one ($\varphi_n = \varphi \times n$) for the same values of ε and $\beta_g = v_g/c$ are shown in Fig. 4 ($\tau = 2\pi$·13000), Fig. 7 ($\tau = 2\pi$·5000), and Fig. 10 ($\tau = 2\pi$·2200).

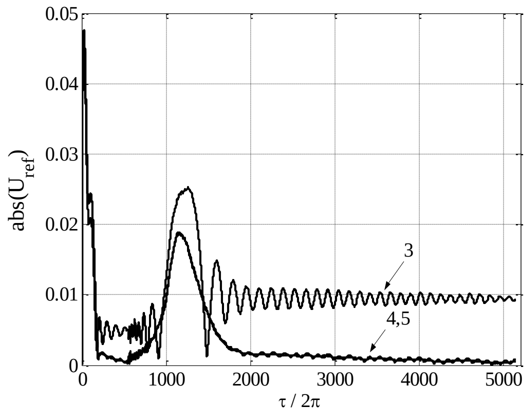

Fig. 6. The time dependence of the reflected signal ($3 - M_c$=2; $4 - M_c$=3; $5 - M_c$=N), ε= 0.06

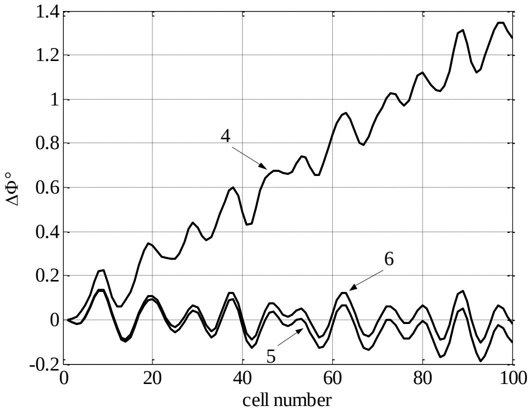

Fig. 7. Deviation from a predetermined phase distribution ($4 - M_c$=3; $5 - M_c$=4; $6 - M_c$=N), ε= 0.06, $\tau = 2\pi \cdot 5000$

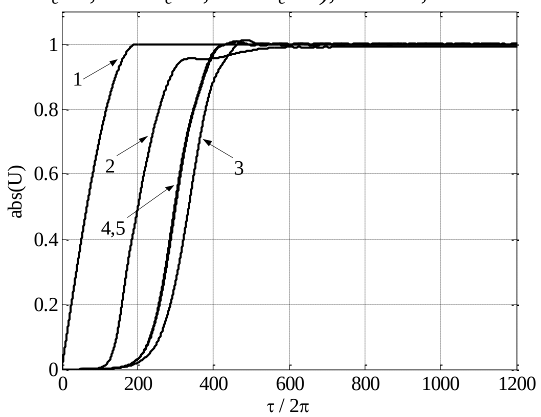

Fig. 8. The time dependence of the input (1) and output amplitudes ($2 - M_c$=1; $3 - M_c$=2; $4 - M_c$=3; $5 - M_c$=N), ε= 0.2

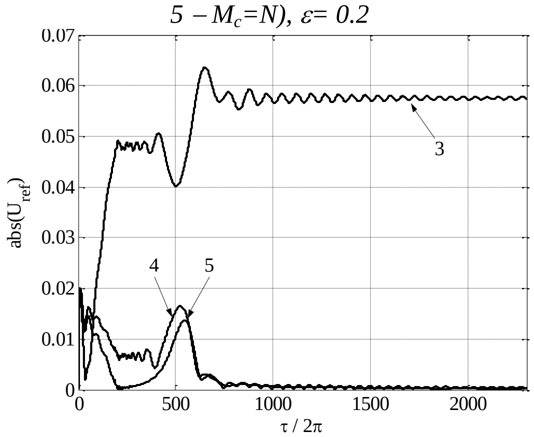

Fig. 9. The time dependence of the reflected signal ($3 - M_c$=2; $4 - M_c$=3; $5 - M_c$=N), ε= 0.2

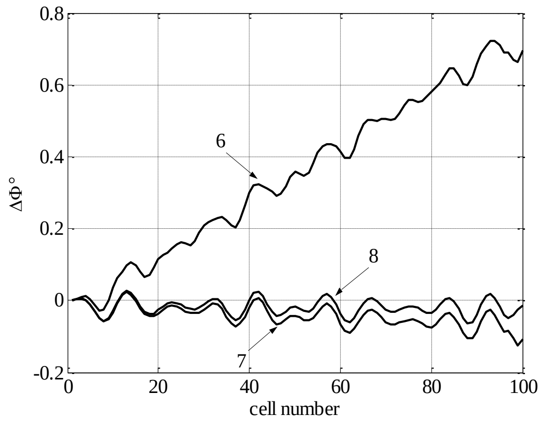

Fig. 10. Deviation from a predetermined phase distribution ($6 - M_c$=5; $7 - M_c$=6; $8 - M_c$=N), ε = 0.2, $\tau = 2\pi \cdot 2200$

Phase oscillations in Figs. 4, 7, 10 indicate that due to reflections from couplers there is no completely steady state at the specified time.

Presented results show that the transients in the chain of magnetically coupled oscillators are sensitive to a mathematical model that used for numerical simulation. Especially strong influence of the value of coupling and the number of coupled resonators is observed for phase distributions (see Figs. 4, 7, 10). Also the influence of coupling characteristics on the reflected signal can not be neglected. It is important for developing methods for tuning couplers and resonators [8, 9].

CONCLUSIONS

For numerical study of dynamics of a system that can be described by a model of coupled oscillators with an "inductive" coupling, we proposed to use the methods of solving of difference equations. Based on this approach we analysed the influence of the value of coupling and the number of coupling resonators on the characteristics of transients in the chain of magnetically coupled oscillators. It was shown that the transients in this chain are sensitive to a mathematical model that used for numerical simulation.

REFERENCES

1. T. Nishikawa. Beam Loading Effects in Standing Wave Linacs // *Proc. of the Linear Accel. Conf.* 1966, p. 294-302.
2. E.A. Knapp. Resonantly Coupled Standing Wave Accelerator Structures for Electron and Proton Linac Applications // *IEEE Trans. Nucl. Sci*. NS-16, 1969, p. 329-337.
3. G.R. Swain, L.D. Scott. Calculation of the Transient Response of Long Chains of Slightly Lossy Coupled Resonators // *IEEE Trans. Nucl. Sci.* NS-17, 1970, p. 10-13.
4. H. Henke, M. Filtz. Envelope Equations for Transients in Linear Chains of Resonators // *Proc. of Particle Accel. Conf.* 1993, p. 901-903.
5. L.M. Young, S. Nath. Effect of Transients on the Beam in the Superconducting Supercollider (SSC) Coupled-Cavity Linac // *Proc. of the Linear Accel. Conf.* 1992, p. 208-210.
6. D.H. Whittum. Introduction to Electrodynamics for Microwave Linear Accelerators // S.I. Kurokawa,

M. Month, S. Turner (Eds). *Frontiers of Accelerator Technology*. World Scientific Publishing Co. Pte. Ltd., 1999.
7. M.I. Ayzatsky. A novel approach to the synthesis of the electromagnetic field distribution in a chain of coupled resonators // arXiv:1709.00842, 2017, p. 1-20.
8. T. Khabiboulline, V. Puntus, M. Dohlus, et al. A new tuning method for traveling wave structures // *Proc. of Particle Accel. Conf*. 1995, p. 1666-1668.
9. M.I. Ayzatskiy, V.V. Mytrochenko. Electromagnetic fields in nonuniform disk-loaded waveguides // *Problems of Atomic Science and Technology. Series "Nuclear Physics Investigations"*. 2016, № 3, p. 3-10; // arXiv:1503.05006, 2015, p. 1-19.

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АНАЛИЗ НЕСТАЦИОНАРНОЙ МОДЕЛИ СВЯЗАННЫХ ОСЦИЛЛЯТОРОВ С ИНДУКТИВНОЙ СВЯЗЬЮ

Н.И. Айзацкий, Е.Ю. Крамаренко

Модель связанных осцилляторов играет важную роль в современной физике. Она используется для описания различных процессов: от колебаний атомов в твердых телах до электромагнитных колебаний в замедляющих структурах. Наиболее широко используется модель с «ближней связью», когда конкретный осциллятор связан только с двумя соседними. Существует два основных вида связи осцилляторов: «электрическая» («емкостная», «силовая») и «магнитная» («индуктивная», «инерционная»). В первом случае связь пропорциональна амплитудам колебаний в соседних ячейках, во втором – второй производной этих амплитуд. При численном исследовании динамики системы, описываемой моделью связанных осцилляторов с индуктивной связью, необходимо найти явные выражения для вторых производных амплитуд. Для нахождения этих выражений в данной работе предлагается использовать методы решения разностных уравнений. Приводятся результаты анализа данного метода.

АНАЛІЗ НЕСТАЦІОНАРНОЇ МОДЕЛІ ЗВ'ЯЗАНИХ ОСЦИЛЯТОРІВ З ІНДУКТИВНИМ ЗВ'ЯЗКОМ

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Модель зв'язаних осциляторів відіграє важливу роль у сучасній фізиці. Її використовують для опису різноманітних процесів: від коливань атомів у твердих тілах до електромагнітних коливань в уповільнюючих структурах. Найбільш широко використовують модель з «ближнім зв'язком», коли конкретний осцилятор зв'язаний тільки з двома сусідніми. Існує два основних види зв'язку осциляторів: «електричний» («ємнісний», «силовий») і «магнітний» («індуктивний», «інерційний»). У першому випадку зв'язок є пропорційним амплітудам коливань у сусідніх комірках, у другому – другій похідній цих амплітуд. При чисельному дослідженні динаміки системи, яка описується моделлю зв'язаних осциляторів з індуктивним зв'язком, необхідно знайти явні вирази для других похідних амплітуд. Для знаходження цих виразів у даній роботі пропонується використовувати методи розв'язання різницевих рівнянь. Приводяться результати аналізу даного методу.