

Bispectrality of N -Component KP Wave Functions: A Study in Non-Commutativity

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Abstract. A wave function of the N -component KP Hierarchy with continuous flows determined by an invertible matrix H is constructed from the choice of an MN -dimensional space of finitely-supported vector distributions. This wave function is shown to be an eigenfunction for a ring of matrix differential operators in x having eigenvalues that are matrix functions of the spectral parameter z . If the space of distributions is invariant under left multiplication by H , then a matrix coefficient differential-translation operator in z is shown to share this eigenfunction and have an eigenvalue that is a matrix function of x . This paper not only generates new examples of bispectral operators, it also explores the consequences of non-commutativity for techniques and objects used in previous investigations.

Key words: bispectrality; multi-component KP hierarchy; Darboux transformations; non-commutative solitons

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1 Introduction

The “bispectral problem” seeks to identify linear operators L acting on functions of the variable x and Λ acting on functions of the variable z such that there exists an eigenfunction $\psi(x, z)$ satisfying the equations

$$L\psi = p(z)\psi \quad \text{and} \quad \Lambda\psi = \pi(x)\psi.$$

In other words, the components of the *bispectral triple* $(L, \Lambda, \psi(x, z))$ satisfy two different eigenvalue equations, but with the roles of the spacial and spectral variables exchanged.

The search for bispectral triples was originally formulated and investigated by Duistermaat and Grünbaum [8] in a paper which completely resolved the question in the special case in which the operators were scalar differential operators with one being a Schrödinger operator. Since then, the bispectrality of many different sorts of operators have been considered and many connections to different areas of math and physics have also been discovered. (See [20] and the articles referenced therein.)

The present paper will be considering a type of bispectrality in which both the operators and eigenvalues behave differently when acting from the left than from the right. It is necessary to introduce some notation and reorder the terms in the eigenvalue equations in order to properly describe the main results.

Throughout the paper, M and N should be considered to be fixed (but arbitrary) natural numbers. In addition, H is a fixed (but arbitrary) invertible constant $N \times N$ matrix. Let $\Delta_{n,\lambda}$ denote the linear functional acting from the right on functions of z by differentiating n times and evaluating at $z = \lambda$:

$$(f(z))\Delta_{n,\lambda} = f^{(n)}(\lambda).$$

The set of linear combinations of these finitely-supported distributions with coefficients from \mathbb{C}^N will be denoted \mathbb{D} :

$$\mathbb{D} = \left\{ \sum_{j=1}^m \Delta_{n_j, \lambda_j} C_j : m, (n_j + 1) \in \mathbb{N}, \lambda_j \in \mathbb{C}, C_j \in \mathbb{C}^N \right\}.$$

Finally, let $W^* \subset \mathbb{D}$ be an MN -dimensional space of distributions.

The goal of this paper is to produce from the selection of H and W^* a bispectral triple (L, Λ, ψ) such that

- $\psi(x, z) = (I + O(z^{-1}))e^{xzH}$ is an $N \times N$ matrix function of x and z with the specified asymptotics in z ,
- $\psi(x, z)(zH)^M$ is holomorphic in z and in the kernel of every element of W^* (i.e., ψ “satisfies the conditions”),
- $L\psi(x, z) = \psi(x, z)p(z)$ for the matrix differential operator in x acting from the left and some matrix function $p(z)$,
- and $\psi(x, z)\Lambda = \pi(x)\psi(x, z)$ for a matrix differential-translation operator in z acting from the right and some matrix function $\pi(x)$.

In the case $N = 1$, this goal is already achieved constructively for *any* choice of distributions [23]. One interesting result of the present paper is that for $N > 1$, lack of commutativity with H may impose an obstacle to finding such a bispectral triple in that no such triple exists for certain choices of W^* . The three subsections below each offer some motivation for interest in the existence of such triples.

1.1 New bispectral triples

One source of interest in the present paper is simply the fact that it generates examples of bispectral triples that have not previously been studied. The construction outlined below produces many bispectral triples (L, Λ, ψ) , where L is a matrix coefficient differential operator in x , Λ is an operator in z which acts by both differentiation and *translation* in z , and $\psi(x, z)$ is a matrix function which asymptotically approaches e^{xzH} for a chosen invertible matrix H .

This can be seen either as a matrix generalization of the paper [23] which considered exactly this sort of bispectrality in the scalar case or as a generalization of the matrix bispectrality in [2, 4, 32] to the a more general class of eigenfunctions and operators.

1.2 Bispectral duality of integrable particle systems

Among the applications found for bispectrality is its surprising role in the duality of integrable particle systems. Two integrable Hamiltonian systems are said to be “dual” in the sense of Ruijsenaars if the action-angle maps linearizing one system is simply the inverse of the action-angle map of the other [26]. When the Hamiltonians of the systems are quantized, the Hamiltonian operators themselves share a common eigenfunction and form a bispectral triple [11].

Moreover, the duality of the classical particle systems can also be manifested through bispectrality in that the dynamics of the two operators in a bispectral triple under some integrable hierarchy can be seen to display the particle motion of the two dual systems respectively. This classical bispectral duality was observed first in the case of the self-duality of the Calogero–Moser system [21, 31]. In [23], it was conjectured that certain bispectral triples involving scalar operators that translate in z would similarly be related to the duality of the rational Ruijsenaars–Schneider and hyperbolic Calogero–Moser particle systems. This was later confirmed by Haine [19].

The spin generalization of the Calogero–Moser system similarly exhibits classical bispectral duality [2, 32]. Achieving this result essentially involved generalizing the scalar case [21, 31] to bispectrality for matrix coefficient differential operators. It is hoped that the construction presented in this paper which generalize that in [23] will similarly find application to classical bispectral duality in some future matrix generalization of the results in [19].

1.3 Non-commutative bispectral Darboux transformations

In the original context of operators on scalar functions, the eigenvalue equations defining bispectrality were originally written with all operators and eigenvalues acting from the left. However, research into non-commutative versions of the discrete-continuous version of the bispectral problem in non-commutative contexts found it necessary to have operators in different variables acting from opposite sides in order to ensure that they commute [5, 14, 15, 16, 17, 18] (cf. [9]). Continuous-continuous bispectrality in the non-commutative context is also a subject of interest [6, 13, 27, 33]. As in the discrete case, it was found that the generalizing the results from the scalar case to the matrix case required letting the operators act from opposite sites [2, 4, 32]. Building on this observation, the present paper seeks to further consider the influence of non-commutativity on constructions and results already known for scalar bispectral triples.

In this regard, the wave functions of the N -component KP hierarchy are of interest since they asymptotically look like matrices of the form e^{xzH} , where H is an $N \times N$ matrix [3, 7]. Consequently, unlike the scalar case or the case $H = I$ considered in [2, 32], the vacuum eigenfunction itself may not commute with the coefficients of the operators. In fact, for the purpose of more fully investigating the consequences of non-commutativity for bispectral Darboux transformations, this paper will go beyond the standard formalism for the N -component KP hierarchy by considering the case in which H is not even diagonalizable and therefore has a centralizer with more interesting structure. Furthermore, following the suggestion of Grünbaum [13], the present paper will consider the case in which both of the *eigenvalues* are matrix-valued.

By generalizing the construction from [23] to the context in which the vacuum eigenfunction, eigenvalues, and operator coefficients all generally fail to commute with each other, this investigation has identified some results that are surprisingly different than in the commutative case. For example, it is shown that in this context there exist rational Darboux transformations that do *not* preserve the bispectrality of the eigenfunction (see Section 6.1) and that bispectral triples do not always exhibit ad-nilpotency (see Remark 6.1). These will be summarized in the last section of the paper.

2 Additional notation

2.1 Distributions and matrices

Let M , N , H and W^* be as in the Introduction. The set of constant N -component column vectors will be denoted by \mathbb{C}^N and $\mathbb{C}_{N \times N}$ is the set of $N \times N$ constant matrices. Associated to the selection of H one has

$$\mathcal{C} = \{Q \in \mathbb{C}_{N \times N} : [Q, H] = 0\},$$

the centralizer of H in $\mathbb{C}_{N \times N}$.

Let $\{\delta_1, \dots, \delta_{MN}\}$ be a basis for $W^* \subset \mathbb{D}$. Unlike the selection of N , M and H which were indeed entirely arbitrary, two additional assumptions regarding the choice of W^* will have to be made so that a bispectral triple may be produced from it. However, rather than making those assumptions here at the start, the additional assumptions will be introduced only when they

become necessary. This should help to clarify which results are independent of and which rely on the assumptions.

Nearly all of the objects and constructions below depend on the choice of the number N , the matrix H and the distributions W^* that have been selected and fixed above, but to avoid complicating the notation the dependence on these selections will not be written explicitly. (For instance, the matrix Φ in (3.1) could be called Φ_{N,M,H,W^*} because it does depend on these selections, but it will simply be called Φ .)

2.2 Operators and eigenvalues

The operators in x to be considered in this paper will all be differential operators in the variable x (also sometimes called t_1) which are polynomials in $\partial = \frac{\partial}{\partial x}$ having coefficients that are $N \times N$ matrix functions of x . The operators in z will be written in terms of $\partial_z = \frac{\partial}{\partial z}$ or the translation operator $\tau_\alpha: f(z) \mapsto f(z + \alpha)$. More generally, they will be polynomials in these having coefficients that are $N \times N$ matrix rational functions of z .

Because operator coefficients and eigenvalues will be matrix-valued, the action of an operator will depend on whether its coefficients multiply from the right or the left. It also matters whether the eigenvalue acts by multiplication on the right or the left of the eigenfunction. This paper will adopt the convention that all operators in x that are independent of z (whether they are differential operators or simply functions acting by multiplication) act from the left and that all operators in z that are independent of x (including functions, translation operators, finitely-supported distributions and differential operators) act from the right. The action of an operator in z will be denoted simply by writing the operator to the right of the function it is acting on. So, for instance, the function e^{xzH} satisfies the eigenvalue equations

$$\partial e^{xzH} = e^{xzH}(zH) \quad \text{and} \quad e^{xzH}\partial_z = (xH)e^{xzH}.$$

The decision to have operators in x and z acting from different sides is not merely a matter of notation. The need for such an assumption for the differential operators in x and z respectively was already noted in prior work on matrix bispectrality [2, 4, 14, 15, 16, 17, 18, 32]. The present work extends this convention to the eigenvalues and finitely-supported distributions as well, and does so because the theorems fail to be true otherwise.

Remark 2.1. Note that one needs to be cautious about applying intuition about eigenfunctions in a commutative setting without considering how non-commutativity may affect it. For example, although a non-zero multiple of an eigenfunction in the commutative setting always remains an eigenfunction with the same eigenvalue, here there are two other possibilities. Suppose $L\psi(x, z) = \psi(x, z)p(z)$, so that ψ is an eigenfunction for L with eigenvalue p , and that g is an invertible constant matrix. Then $g\psi$ may not be an eigenfunction for L if $[L, g] \neq 0$ and more surprisingly even though ψg is an eigenfunction for L , the corresponding eigenvalue changes to $g^{-1}pg$.

3 Dual construction for N -component KP

The purpose of this section is to produce a matrix coefficient pseudo-differential operator satisfying the Lax equations of the multicomponent KP hierarchy introduced by Date–Jimbo–Kashiwara–Miwa [7] and mostly follows the approach of Segal–Wilson [29]. The proof methods utilized here are rather standard in the field of integrable systems. However, one of the main points of this paper is that some of the novel features of this situation pose unexpected obstacles to the standard methods used to study bispectrality. So, although it is not surprising that the operators produced in this way satisfy these Lax equations, the proofs are presented with

sufficient detail to ensure that they work despite the non-diagonalizability of H and the fact that the distributions here are acting from the right.

3.1 The N -component Sato Grassmannian

Let $\mathcal{H}^{(N)}$ denote the Hilbert space of square-integrable vector-valued functions $S^1 \rightarrow (\mathbb{C}^N)^\top$ where $S^1 \subset \mathbb{C}$ is the unit circle $|z| = 1$ and $(\mathbb{C}^N)^\top$ is the set of complex valued row¹ N -vectors. Denote by e_i for $0 \leq i \leq N - 1$ the $1 \times N$ matrix which has the value 1 in column $i + 1$ and zero in the others. This extends to a basis $\{e_i : i \in \mathbb{Z}\}$ of $\mathcal{H}^{(N)}$ for which $e_i = z^a e_b$ when $i = aN + b$ for $0 \leq b \leq N - 1$. The Hilbert space has the decomposition

$$\mathcal{H}^{(N)} = \mathcal{H}_+^{(N)} \oplus \mathcal{H}_-^{(N)},$$

where $\mathcal{H}_+^{(N)}$ is the Hilbert closure of the subspace spanned by e_i for $0 \leq i$ and $\mathcal{H}_-^{(N)}$ is the Hilbert closure of the subspace spanned by e_i for $i < 0$.

Definition 3.1. The Grassmannian $\text{Gr}^{(N)}$ is set of all closed subspaces $V \subset \mathcal{H}^{(N)}$ such that the orthogonal projections $V \rightarrow \mathcal{H}_-^{(N)}$ is a compact operator and such that the orthogonal projection $V \rightarrow \mathcal{H}_+^{(N)}$ is Fredholm of index zero [3, 28, 29].

The notion of N -component KP hierarchy to be considered in this paper is compatible with, but somewhat different from that addressed by previous authors as the following remark explains.

Remark 3.2. For the N -component KP hierarchy, the construction of solutions from a point in the Grassmannian usually involves a collection of diagonal constant matrices H_α ($1 \leq \alpha \leq N$) such that powers of zH_α infinitesimally generate the continuous flows and z -dependent matrices T_β ($1 \leq \beta \leq N - 1$) that generate discrete flows (sometimes called ‘‘Schlesinger transformations’’) of the hierarchy [3, 7]. In the present paper, however, only the continuous flows generated infinitesimally by powers of z times the (not necessarily diagonal) matrix H selected earlier will be considered.

3.2 A point of $\text{Gr}^{(N)}$ associated to the selection of distributions

As usual, one associates a subspace of $\mathcal{H}^{(N)}$ to the choice of W^* by taking its dual in $\mathcal{H}_+^{(N)}$ and multiplying on the right by the inverse of a matrix polynomial in z whose degree depends on the dimension of W^* (cf. [21, 22, 29, 30] where the analogous procedure involved dividing by a scalar polynomial):

Definition 3.3. Let $W \subset \mathcal{H}^{(N)}$ be defined by

$$W = \{p(z)(zH)^{-M} : p(z) \in \mathcal{H}_+^{(N)}, (p)\delta = 0 \text{ for } \delta \in W^*\}.$$

Lemma 3.4. $W \in \text{Gr}^{(N)}$.

Proof. The image of W under the projection map onto $\mathcal{H}_-^{(N)}$ is contained in the finite-dimensional subspace spanned by the basis elements e_i for $-MN \leq i \leq -1$. This is sufficient to conclude that the projection map is compact. The map $w \mapsto w(zH)^M$ from W to $\mathcal{H}_+^{(N)}$ has Fredholm index MN because it has no kernel and the image is the common solution set of MN -linearly independent conditions. The map from $\mathcal{H}_+^{(N)}$ which first right multiplies by $(zH)^{-M}$

¹In previous papers the elements of $\mathcal{H}^{(N)}$ have been written as column vectors. However, because the construction of bispectral operators below is most easily described in terms of matrix finitely-supported distributions in z acting from the *right*, they will be written here as $1 \times N$ matrices.

and then projects onto $\mathcal{H}_+^{(N)}$ has index $-MN$ since its kernel is spanned by the basis vectors e_i with $0 \leq i \leq MN - 1$ but the image is all of $\mathcal{H}_+^{(N)}$. The composition of these maps is the projection from W to $\mathcal{H}_+^{(N)}$ and so its index is the sum of the indices which is zero. ■

3.3 N -component KP wave function

Definition 3.5. Let $\psi_0 = \exp\left(\sum_{i=1}^{\infty} t_i z^i H^i\right)$ where $\mathbf{t} = (t_1, t_2, \dots)$ are the continuous KP time variables, with the variables t_1 and x considered to be identical. Let $\phi_i(\mathbf{t}) = (\psi_0)\delta_i$ ($1 \leq i \leq MN$) be the \mathbb{C}^N -valued functions obtained by applying each element of the basis of W^* to ψ_0 . Combine them as blocks into the $N \times MN$ matrix $\phi = (\phi_1 \cdots \phi_{MN})$ and define the matrix Φ as the $MN \times MN$ block Wronskian matrix

$$\Phi(\mathbf{t}) = \begin{pmatrix} \phi \\ \frac{\partial}{\partial x} \phi \\ \vdots \\ \frac{\partial^{M-1}}{\partial x^{M-1}} \phi \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_{MN} \\ \phi'_1 & \phi'_2 & \cdots & \phi'_{MN} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(M-1)} & \phi_2^{(M-1)} & \cdots & \phi_{MN}^{(M-1)} \end{pmatrix}. \quad (3.1)$$

Assumption 3.6. Henceforth, assume that W^* was chosen so that the matrix Φ in (3.1) is invertible for some values of $x = t_1$ (i.e., so that $\det \Phi \neq 0$).

The following remarks offer two different interpretations of the fact that Assumption 3.6 is necessary here but not for the analogous result in the scalar case [23].

Remark 3.7. When $N = 1$, the requirement that $\det(\Phi) \neq 0$ is equivalent to the requirement that $\{\phi_1, \dots, \phi_M\}$ is a linearly independent set of functions. Then, the independence of the basis of distributions would already ensure that Assumption 3.6 is satisfied. However, when $N > 1$ the block Wronskian matrix Φ can be singular even if the functions ϕ_i are linearly independent as functions of x . (For example, consider the case $M = 1$, $N = 2$, $\phi_1 = (1 \ 1)^\top$, $\phi_2 = (x \ x)^\top$.)

Remark 3.8. The determinant of Φ can be interpreted as the determinant of the projection map from $\psi_0^{-1}(\mathbf{t})W$ to $\mathcal{H}_+^{(N)}$. In other words, it is the τ -function of W . (The proof of this claim is essentially the same as the proof of Theorem 7.5 in [22].) The τ -function is non-zero when $\psi_0^{-1}(\mathbf{t})W$ is in the ‘‘big cell’’ of the Grassmannian. In the case $N = 1$, the orbit of any point W in the Grassmannian under the action of ψ_0^{-1} intersects the big cell [29]. In contrast, for the N -component KP hierarchy it is known that there are points in the Grassmannian whose orbit under the continuous flows never intersect the big cell [3].

Definition 3.9. Due to Assumption 3.6, we may define the differential operator K as

$$K = \partial^M I - \begin{pmatrix} \phi_1^{(M)} & \cdots & \phi_{MN}^{(M)} \end{pmatrix} \Phi^{-1} \begin{pmatrix} I \\ \partial I \\ \vdots \\ \partial^{M-1} I \end{pmatrix}. \quad (3.2)$$

Lemma 3.10.

- (a) *The operator K defined in (3.2) is the unique monic $N \times N$ differential operator of order M such that² $K\phi = 0$.*

²The expression $D\phi = 0$ is a convenient way to write that applying the $N \times N$ differential operator D to each function ϕ_i ($1 \leq i \leq MN$) results in the zero vector of \mathbb{C}^N .

(b) If L is any $N \times N$ matrix differential operator satisfying $L\phi = 0$ then $L = Q \circ K$ for some differential operator Q .

It is easy to check that $K\phi = 0$. Alternatively, Lemma 3.10(a) follows from results of Etingof, Gelfand and Retakh on quasi-determinants [10]. However, Lemma 3.10(b) is apparently a new result. Although the lemma was originally formulated for this paper³, a self-contained proof is being published separately as [24].

Remark 3.11. The theory of quasi-determinants is not explicitly being used here but the operator K defined in (3.2) could alternatively be computed as a quasi-determinant of a Wronskian matrix with $N \times N$ matrix entries [10]. The method of quasi-determinants was applied to the bispectral problem for matrix coefficient operators in [4]. So, in this sense, the use of this operator K here is a continuation of the approach adopted there.

Definition 3.12. Let $\psi(\mathbf{t}, z) = K(\psi_0)(zH)^{-M}$. This function will play an important role as the eigenfunction for the operators in x and (given one additional assumption) z to be introduced below.

Theorem 3.13. *The function ψ defined above has the following properties:*

- $\psi(\mathbf{t}, z) = (I + O(z^{-1}))\psi_0$ where I is the $N \times N$ identity matrix,
- $\psi(\mathbf{t}, z) \in W$ for all \mathbf{t} in the domain of ψ .

Consequently, $\psi = \psi_W$ is the N -component KP wave function of the point $W \in \text{Gr}^{(N)}$.

Proof. Because $\partial(\psi_0) = \psi_0 z H$, applying the monic differential operator $K = \partial^M + \dots$ to ψ_0 produces a function of the form $K\psi_0 = P(\mathbf{t}, z)\psi_0$ where P is a polynomial of degree M in z with leading coefficient H^M . Then $K\psi_0 H^{-M} z^{-M} = (I + O(z^{-1}))\psi_0$ as claimed. It remains to be shown that ψ is an element of W . By Definition 3.3 it is sufficient to note that for each $1 \leq i \leq MN$, $\psi z^M H^M = K\psi_0$ satisfies

$$(K\psi_0)\delta_i = K((\psi_0)\delta_i) = K\phi_i = 0. \quad \blacksquare$$

3.4 Lax equations of the N -component KP hierarchy

Definition 3.14. Let K^{-1} denote the unique multiplicative inverse of the monic differential operator K in the ring of matrix-coefficient pseudo-differential operators and let $\mathcal{L} = K \circ \partial \circ K^{-1}$ be the pseudo-differential operator obtained by conjugating ∂ by K .

Theorem 3.15. \mathcal{L} satisfies the Lax equations

$$\frac{\partial}{\partial t_i} \mathcal{L} = [(\mathcal{L}^i)_+, \mathcal{L}]$$

for each $i \in \mathbb{N}$, where $(\sum_{i=-\infty}^n \alpha_i(\mathbf{t})\partial^i)_+ = \sum_{i=0}^n \alpha_i(\mathbf{t})\partial^i$.

Proof. Let $\phi = (\psi_0)\delta$ for some $\delta \in W^*$. A key observation is that for each $i \in \mathbb{N}$:

$$\frac{\partial}{\partial t_i} \phi = \frac{\partial}{\partial t_i} (\psi_0)\delta = \left(\frac{\partial}{\partial t_i} \psi_0 \right) \delta = ((zH)^i \psi_0) \delta = \left(\frac{\partial^i}{\partial x^i} \psi_0 \right) \delta = \partial^i (\psi_0) \delta = \frac{\partial^i}{\partial x^i} (\phi).$$

³Lemma 3.10(b) is used in the proofs of Theorem 3.15, Lemma 5.10 and Theorem 4.7.

Using the fact that ϕ satisfies these “dispersion relations” and the intertwining relationship $\mathcal{L}^i \circ K = K \circ \partial^i$ we differentiate the identity $K(\phi) = 0$ (which follows from Lemma 3.10) by t_i to get

$$\begin{aligned} 0 &= K_{t_i}(\phi) + K(\phi_{t_i}) = K_{t_i}(\phi) + K(\partial^i \phi) = K_{t_i}(\phi) + \mathcal{L}^i \circ K(\phi) \\ &= K_{t_i}(\phi) + (\mathcal{L}^i)_- \circ K(\phi) + (\mathcal{L}^i)_+ \circ K(\phi). \end{aligned}$$

Since both K and $(\mathcal{L}^i)_+$ are ordinary differential operators (as “+” denotes the projection onto the subring of ordinary differential operators), $(\mathcal{L}^i)_+ \circ K$ is an ordinary differential operator with a right factor of K . Therefore, ϕ is in its kernel and the last term in the sum above is zero. We may therefore conclude that

$$K_{t_i}(\phi) + (\mathcal{L}^i)_- \circ K(\phi) = 0. \quad (3.3)$$

(Note that the second term in this sum need not be zero since $(\mathcal{L}^i)_-$ is not an ordinary differential operator.)

Using $\mathcal{L}^i \circ K = K \circ \partial^i$ we can split \mathcal{L}^i into its positive and negative parts to get

$$(\mathcal{L}^i)_- \circ K = K \circ \partial^i - (\mathcal{L}^i)_+ \circ K.$$

Since the object on the right is just a difference of differential operators we know that $(\mathcal{L}^i)_- \circ K$ is a differential operator.

According to (3.3), $\Gamma(\phi) = 0$ where Γ is the ordinary differential operator

$$\Gamma = K_{t_i} + (\mathcal{L}^i)_- \circ K.$$

Then by Lemma 3.10, there exists a differential operator Q so that $\Gamma = Q \circ K$. However, Γ has order strictly less than M since the coefficient of the M^{th} order term of K is constant by construction and since multiplying by $(\mathcal{L}^i)_-$ will necessarily lower the order. This is only possible if $Q = 0$ and Γ is the zero operator. Hence, $K_{t_i} = -(\mathcal{L}^i)_- \circ K$. The Lax equation follows because

$$\begin{aligned} \mathcal{L}_{t_i} &= K_{t_i} \circ \partial \circ K^{-1} - K \circ \partial \circ K^{-1} \circ K_{t_i} \circ K^{-1} \\ &= -(\mathcal{L}^i)_- \circ K \circ \partial \circ K^{-1} + K \circ \partial \circ K^{-1} \circ (\mathcal{L}^i)_- \circ K \circ K^{-1} \\ &= [\mathcal{L}, (\mathcal{L}^i)_-] = [(\mathcal{L}^i)_+, \mathcal{L}]. \end{aligned} \quad \blacksquare$$

4 Operators in x having ψ as eigenfunction

Remark 4.1. From this point onwards, the goal is to determine whether the wave function $\psi(\mathbf{t}, z)$ is an eigenfunction for an operator in $x = t_1$ with z -dependent eigenvalue and vice versa. The higher indexed time variables will only complicate the notation. So, it will henceforth be assumed that $t_i = 0$ for $i \geq 2$. Then, ϕ and the coefficients of K can be considered to be functions of only the variable x , and $\psi_0(x, z) = e^{xzH}$ and $\psi(x, z)$ (sometimes called the “stationary wave function”) are functions of x and z . Unlike the numbered assumptions, this one is made for notational simplicity only. Dependence on the KP time variables can be added to the objects to be discussed below so that all claims remain valid.

Definition 4.2. A distribution $\delta \in \mathbb{D}$ can be composed with $p(z) \in \mathbb{C}_{N \times N}[z]$ by defining $p(z) \circ \delta$ to be the distribution whose value when applied to $f(z)$ is the same as that of δ applied to the product $f(z)p(z)$ for any $f(z) \in \mathbb{C}_{N \times N}[z]$. Associate to the choice W^* of distributions the ring of polynomials *with coefficients in \mathcal{C}* which turn elements of W^* into elements of W^* :

$$\mathcal{A} = \{p(z) \in \mathcal{C}[z] : p(z) \circ \delta \in W^* \forall \delta \in W^*\}.$$

In particular, for each $p \in \mathcal{A}$ and each basis element δ_i there exist numbers c_j such that $p \circ \delta_i = \sum_{j=1}^{MN} \delta_j c_j$.

As in the scalar case, elements of \mathcal{A} are stabilizers of the point in the Grassmannian: if $p \in \mathcal{A}$ then $Wp \subset W$. The main significance of the ring \mathcal{A} is that there is a differential operator L_p of positive order satisfying $L_p \psi = \psi p(z)$ for every non-constant $p \in \mathcal{A}$. Interestingly, unlike the scalar case, it will be shown that the question of which *constant* matrices are in \mathcal{A} is also of interest in that whether ψ is part of a bispectral triple is related to whether H is an element of \mathcal{A} . Before those facts are established, however, the following definitions and results show that \mathcal{A} contains polynomials of every sufficiently high degree.

Definition 4.3. Let $S \subset \mathbb{C}$ denote the support of the distributions in W^* . That is, $\lambda \in S$ if and only if $\Delta_{n,\lambda}$ appears with non-zero coefficient for some n in at least one δ_i . For each $\lambda \in S$ let m_λ denote the *largest* number n such that $\Delta_{n,\lambda}$ appears with non-zero coefficient in at least one element of W^* . The scalar polynomial

$$p_0(z) = \prod_{\lambda \in S} (z - \lambda)^{m_\lambda + 1}$$

will be used in the next lemma, in Definition 5.8 and also in Theorem 5.13 below.

Lemma 4.4. For any $p \in \mathcal{C}[z]$, the product $p_0(z)p(z)$ is in \mathcal{A} . So, $p_0(z)\mathcal{C}[z] \subset \mathcal{A}$.

Proof. Let $p \in \mathcal{C}[z]$ and $\delta \in W^*$. Then applying the distribution $p_0 p \circ \delta$ to any polynomial q is equal to $(qp_0 p) \delta$. The distribution δ will act by differentiating and evaluating at $z = \lambda$ for each $\lambda \in S$. However, p_0 was constructed so that it has zeroes of sufficiently high multiplicity at each λ to ensure that this will be equal to zero. Hence $p_0 p \circ \delta$ is the zero distribution, which trivially satisfies the criterion in the definition of \mathcal{A} . ■

Definition 4.5. For $p \in \mathcal{C}[z]$ where $p = \sum_{i=0}^n C_i z^i$ for $C_i \in \mathcal{C}$ define

$$\flat^{-1}(p) = \sum_{i=0}^n C_i H^{-i} \partial^i.$$

Lemma 4.6. For any $p \in \mathcal{C}[z]$, the constant coefficient differential operator $\flat^{-1}(p)$ has $\psi_0 = e^{xzH}$ as an eigenfunction with eigenvalue $p(z)$.

Proof. If p is the polynomial with coefficients C_i as in Definition 4.5 then

$$\begin{aligned} \flat^{-1}(p)\psi_0 &= \left(\sum_{i=0}^n C_i H^{-i} \partial^i \right) \psi_0 = \sum_{i=0}^n C_i H^{-i} \partial^i \psi_0 = \sum_{i=0}^n C_i H^{-i} \psi_0 (zH)^i \\ &= \psi_0 \left(\sum_{i=0}^n C_i H^{-i} (zH)^i \right) = \psi_0 \left(\sum_{i=0}^n C_i z^i \right) = \psi_0 p(z). \end{aligned}$$

Theorem 4.7. For any $p \in \mathcal{A}$ there is an $N \times N$ ordinary differential operator L_p in x satisfying the intertwining relationship

$$L_p \circ K = K \circ \flat^{-1}(p)$$

and the eigenvalue equation $L_p \psi(x, z) = \psi p(z)$.

Proof. Let $p \in \mathcal{A}$. Observe that

$$\begin{aligned} K \circ b^{-1}(p)(\phi_i) &= K \circ b^{-1}(p)((\psi_0)\delta_i) = K((b^{-1}(p)\psi_0)\delta_i) = K((\psi_0)p(z) \circ \delta_i) \\ &= K\left(\sum_{j=1}^{MN} (\psi_0)\delta_j c_j\right) = \sum_{j=1}^{MN} K(\phi_j)c_j = \sum_{j=1}^{MN} 0 \times c_j = 0. \end{aligned}$$

But, that means that each function ϕ_i is in the kernel of $K \circ b^{-1}(p)$ and hence by Lemma 3.10 there is a differential operator L_p such that $K \circ b^{-1}(p) = L_p \circ K$. This establishes the intertwining relationship.

Applying L_p to $\psi = K\psi_0(zH)^{-M}$ one finds

$$\begin{aligned} L_p\psi &= L_p(K\psi_0(zH)^{-M}) = (L_p \circ K)\psi_0(zH)^{-M} = (K \circ b^{-1}(p))\psi_0(zH)^{-M} \\ &= K\psi_0 p(z)(zH)^{-M} = K\psi_0(zH)^{-M} p(z) = \psi p(z). \end{aligned} \quad \blacksquare$$

5 Operators in z having ψ as eigenfunction

Definition 5.1. For any $\alpha \in \mathbb{C}$ let the translation operator \top_α act on functions of z according to the definition

$$(f(z))\top_\alpha = f(z + \alpha).$$

Further let $\top_\alpha^H = \sum_{i,j} \top_{\alpha\gamma_i}(\alpha\partial_z)^j C_{ij}$ where the matrices C_{ij} and constants γ_i are defined by the formula

$$\exp(xH^{-1}) = \sum_{i,j} \exp(\gamma_i x) x^j C_{ij} \quad (\text{with } \gamma_i \neq \gamma_{i'} \text{ if } i \neq i'). \quad (5.1)$$

Lemma 5.2. *The differential-translation operator \top_α^H has $\psi_0 = e^{xzH}$ as an eigenfunction with eigenvalue $e^{\alpha x}$:*

$$\psi_0 \top_\alpha^H = e^{\alpha x} \psi_0.$$

Proof. By definition,

$$e^{xzH} \top_\alpha^H = e^{xzH} \sum_{i,j} T_{\alpha\gamma_i}(\alpha\partial_z)^j C_{ij} = e^{xzH} \sum_{i,j} e^{x\alpha\gamma_i H} (\alpha x H)^j C_{ij}.$$

Then the claim follows if it can be shown that the sum in the last expression is equal to $e^{\alpha x}$.

However, (5.1) holds not only for any scalar x but also when it is replaced by some matrix which commutes with the matrices H^{-1} and C_{ij} which appear in it. In particular, since $\alpha x H$ commutes with H^{-1} , its commutator with the expression on the left side of equation (5.1) is zero. From this one can determine that its commutator with each coefficient matrix C_{ij} on the right is zero as well. However, replacing x with $\alpha x H$ yields the formula

$$\exp(\alpha x) = \sum_{i,j} \exp(\gamma_i \alpha x H) (\alpha x H)^j C_{ij}$$

as needed. \blacksquare

The goal of this section is to produce operators in z which are matrix differential-translation operators in z having rational coefficients that share the eigenfunction ψ with the differential operators L_p in x constructed in the previous section. In order for the construction to work, an additional assumption is required:

Assumption 5.3. Henceforth, it is assumed that $H \in \mathcal{A}$. Equivalently, assume that for each $1 \leq i \leq MN$ there exist numbers c_j such that

$$H\delta_i = \sum_{j=1}^{MN} c_j \delta_j.$$

5.1 The anti-isomorphism

This section will introduce an anti-isomorphism between rings of operators in x and z respectively such that an operator and its image have the same action on ψ_0 . The use of such an anti-isomorphism as a method for studying bispectrality was pioneered in special cases in [30] and [25] and extended to a very general commutative context in [1]. (Additionally, three months after the present paper was posted and submitted, a new preprint by two of the same authors as [1] appeared which seeks to further generalize those results to the non-commutative context [12].)

Definition 5.4. The two rings of operators of interest to this construction are

$$\mathcal{W} = \bigoplus_{\alpha \in \mathbb{C}} e^{\alpha x} \mathcal{C}[x, \partial] \quad \text{and} \quad \mathcal{W}^b = \bigoplus_{\alpha \in \mathbb{C}} \top_{\alpha}^H \mathcal{C}[z, \partial].$$

Note that operators from both rings involve polynomial coefficient matrix differential operators which commute with the constant matrix H , but elements of \mathcal{W} may also include a finite number of factors of the form $e^{\alpha x}$ while elements of \mathcal{W}^b may similarly include factors of \top_{α}^H for a finite number of complex numbers α .

Definition 5.5. Let $\flat: \mathcal{W} \rightarrow \mathcal{W}^b$ be defined by

$$\flat \left(\sum_{i=0}^l \sum_{j=0}^m \sum_{k=1}^n C_{ijk} e^{\alpha_k x} x^i \partial^j \right) = \sum_{i=0}^l \sum_{j=0}^m \sum_{k=1}^n \partial_z^i \top_{\alpha_k}^H z^j C_{ijk} H^{j-i}, \quad (5.2)$$

where $C_{ijk} \in \mathcal{C}$ are the coefficient matrices.

Lemma 5.6. For any $L_0 \in \mathcal{W}$, the operators L_0 and $\flat(L_0)$ have the same action on $\psi_0 = e^{xzH}$:

$$L_0 \psi_0 = \psi_0 \flat(L_0).$$

Moreover, \flat is an anti-isomorphism.

Proof. Since

$$\begin{aligned} \sum_{i=0}^m \sum_{j=0}^n \sum_k C_{ij} x^i e^{\alpha_k x} \partial^j (\psi_0) &= \sum_{i=0}^m \sum_{j=0}^n \sum_k C_{ij} e^{\alpha_k x} x^i (zH)^j (\psi_0) \\ &= \sum_{i=0}^m \sum_{j=0}^n \sum_k C_{ij} e^{\alpha_k x} x^i (\psi_0) (zH)^j = \sum_{i=0}^m \sum_{j=0}^n \sum_k C_{ij} e^{\alpha_k x} (\psi_0) (H^{-1} \partial_z)^i (zH)^j \\ &= \sum_{i=0}^m \sum_{j=0}^n \sum_k C_{ij} (\psi_0) \top_{\alpha_k}^H (H^{-1} \partial_z)^i (zH)^j = \sum_{i=0}^m \sum_{j=0}^n \sum_k (\psi_0) (H^{-1} \partial_z)^i \top_{\alpha_k}^H (zH)^j C_{ij} \\ &= (\psi_0) \sum_{i=0}^m \sum_{j=0}^n \sum_k (H^{-1} \partial_z)^i \top_{\alpha_k}^H (zH)^j C_{ij} = \psi_0 \flat \left(\sum_{i=0}^m \sum_{j=0}^n \sum_k C_{ij} x^i e^{\alpha_k x} \partial^j \right) \end{aligned}$$

it follows that L_0 and $\flat(L_0)$ have the same action on ψ_0 .

The inverse map \flat^{-1} is found by simply moving the factor of H^{j-i} to the other side of equation (5.2), which confirms that \flat is a bijection. If $K_1, K_2 \in \mathcal{W}$ we have

$$\begin{aligned} (K_2 \circ K_1)\psi_0 &= K_2(K_1(\psi_0)) = K_2(\psi_0 \flat(K_1)) = (K_2(\psi_0))\flat(K_1) \\ &= (\psi_0 \flat(K_2))\flat(K_1) = \psi_0(\flat(K_2) \circ \flat(K_1)). \end{aligned}$$

However, we also know that $(K_2 \circ K_1)\psi_0 = \psi_0 \flat(K_2 \circ K_1)$. Then, $\flat(K_2 \circ K_1) - \flat(K_2) \circ \flat(K_1)$ has the ψ_0 in its kernel. The only operator in \mathcal{W}^b having ψ_0 in its kernel is the zero operator. Therefore,

$$\flat(K_2 \circ K_1) = \flat(K_2) \circ \flat(K_1) \tag{5.3}$$

and the map is an anti-isomorphism. ■

Remark 5.7. The reader may be surprised to see “anti-isomorphism” being used to describe a map satisfying equation (5.3). Generally, a map having this property is called an *isomorphism*. There are two reasons this terminology is being used here. First, that is the terminology that was used for the analogous map in previous papers in the scalar setting [1, 23, 25]. More importantly, the fact that the order of operators here is preserved is merely a consequence of the fact that the operators in z act from the right while the operators in x act from the left. Note that when $K_2 \circ K_1$ acts on a function it is K_1 that acts first while when $\flat(K_2) \circ \flat(K_1)$ acts it is $\flat(K_2)$ that acts first and it is in this sense that it is an *anti*-isomorphism. Nevertheless, it is interesting to note that in this context the map does preserve the order of operators in a product. It may be that had the operators in different variables been written as acting from opposite sides even in the commutative case from the beginning, the map would have been described as an isomorphism instead.

5.2 Eigenvalue equations for operators in z

Definition 5.8. Let L_0 be the constant coefficient matrix differential operator

$$L_0 = \flat^{-1}(p_0(z)I).$$

Lemma 5.9. $L_0\phi = 0$ and so $L_0 = Q \circ K$ for some Q .

Proof. Because differentiation in x and multiplication on the left commute with the application of the distributions δ_i we have

$$L_0\phi_i = L_0(\psi_0)\delta_i = (L_0\psi_0)\delta_i.$$

By Lemma 5.6, $L_0\psi_0 = \psi_0 p_0(z)$. The polynomial p_0 was chosen so that it has a zero of high enough multiplicity at each point in the support of δ_i to guarantee that

$$(\psi_0 p_0(z))\delta_i = 0.$$

It then follows from Lemma 3.10 that $L_0 = Q \circ K$. ■

In the remainder of this construction, Q will denote the differential operator such that $L_0 = Q \circ K$.

Lemma 5.10. Given Assumption 5.3, the operators L_0 , K and Q commute with H .

Proof. Applying the operator $H^{-1}KH$ to ϕ_i we see that

$$\begin{aligned} H^{-1}KH\phi_i &= H^{-1}KH(e^{xzH})\delta_i = H^{-1}K(e^{xzH})H\delta_i = H^{-1}K\left(\sum_{j=1}^{MN} c_j\phi_j\right) \\ &= H^{-1}\left(\sum_{j=1}^{MN} c_jK(\phi_j)\right) = 0. \end{aligned}$$

However, by Lemma 3.10, K is the unique monic operator of order M having all the basis functions ϕ_i in its kernel so $H^{-1}KH = K$.

We also know that L_0 commutes with H because p_0 is a scalar multiple of the identity and \flat^{-1} which turns p_0 into L_0 only introduces additional powers of H . Then

$$HQ \circ K = HL_0 = L_0H = Q \circ KH = Q \circ HK = QH \circ K.$$

Multiplying this by K^{-1} on the right yields $HQ = QH$. ■

Note that K and Q are probably not in \mathcal{W} , as they may be *rational* functions in x and a finite number of functions of the form $e^{\alpha x}$. However, it is possible to clear their denominators either by multiplying by a function on the left or by composing with a function on the right, which motivates the following definition:

Definition 5.11. Let \mathcal{A}^b be the set of x -dependent $N \times N$ matrix functions defined as follows:

$$\mathcal{A}^b = \left\{ \hat{\pi} \in \bigoplus_{\alpha \in \mathbb{C}} \mathcal{C}[x]e^{\alpha x} : \hat{\pi} = \pi_Q\pi_K, \pi_K(x)K \in \mathcal{W}, Q \circ \pi_K(x) \in \mathcal{W} \right\}.$$

In other words, it is the set of zero order elements of \mathcal{W} which factor as a product such the right factor times K and Q composed with the left factor are both elements of \mathcal{W} .

Lemma 5.12. \mathcal{A}^b is non-empty and contains matrix functions that are non-constant in x .

Proof. Note that K and Q are operators that are rational in x and a finite number of terms of the form e^x . If we let π_K be the least common multiple of the denominators of the coefficients in K then $\pi_K K$ is in \mathcal{W} (since we have simply cleared the denominator by multiplication). Similarly, if we let π_Q be a high enough power of the least common multiple of the denominators of Q then $Q \circ \pi_Q \in \mathcal{W}$. Then, $\hat{\pi} = \pi_Q\pi_K$ is by construction an element of \mathcal{A}^b . Since, $\pi_Q f \pi_K$ is also an element of \mathcal{A}^b for any order zero element of \mathcal{W} , \mathcal{A}^b contains non-constant matrix functions. ■

The main result is the construction of an operator Λ in z with eigenvalue $\hat{\pi}$ when applied to the wave function ψ for every $\hat{\pi} \in \mathcal{A}^b$:

Theorem 5.13. Let $\hat{\pi} = \pi_Q\pi_K \in \mathcal{A}^b$ where $\bar{K} = \pi_K K \in \mathcal{W}$ and $\bar{Q} = Q \circ \pi_Q \in \mathcal{W}$. Define $\Lambda := (zH)^M \circ (p_0(z))^{-1} \circ \flat(\bar{Q}) \circ \flat(\bar{K}) \circ (zH)^{-M}$, then

$$\psi\Lambda = \hat{\pi}(x)\psi.$$

Proof. Write L_0 as $L_0 = \bar{Q} \circ (\pi_K(x)\pi_Q(x))^{-1} \circ \bar{K}$. Applying this operator to ψ_0 and multiplying each side by $p_0^{-1}(z)$ gives

$$(\bar{Q} \circ (\pi_K(x)\pi_Q(x))^{-1} \circ \bar{K})\psi_0 p_0^{-1}(z) = \psi_0.$$

Moving \bar{K} to the other side using the anti-isomorphism and applying $\flat(\bar{Q})$ to both sides this becomes

$$(\bar{Q} \circ (\pi_K(x)\pi_Q(x))^{-1})\psi_0(\flat(\bar{K}) \circ p_0^{-1}(z) \circ \flat(\bar{Q})) = \psi_0\flat(\bar{Q}).$$

Moving the last expression to the other side of the equality and moving $\flat(\bar{Q})$ in it to the other side, we finally get

$$\bar{Q}((\pi_K\pi_Q)^{-1}\psi_0(\flat(\bar{K}) \circ p_0^{-1} \circ \flat(\bar{Q})) - \psi_0) = 0.$$

Note that \bar{Q} is a differential operator in x with a non-singular leading coefficient (since it is a factor of the monic differential operator L_0). Hence, its kernel is finite-dimensional. The only way the expression to which it is applied, a polynomial in z multiplied by e^{xzH} , could be in its kernel for all z is if it is equal to zero. From this, we conclude that

$$\flat(\bar{K}) \circ p_0^{-1} \circ \flat(\bar{Q}) = \flat(\pi_K\pi_Q).$$

Using this one finds that the action of Λ on ψ is

$$\begin{aligned} (\psi)\Lambda &= (K\psi_0(zH)^{-M})\Lambda = (\pi_K^{-1}(x)\bar{K}\psi_0(zH)^{-M})\Lambda \\ &= (\pi_K^{-1}(x)(\psi_0)\flat(\bar{K})(zH)^{-M})\Lambda = (\pi_K^{-1}(x)\psi_0)\flat(\bar{K}) \circ (zH)^{-M} \circ \Lambda \\ &= (\pi_K^{-1}(x)\psi_0)\flat(\bar{K}) \circ (p_0(z))^{-1} \circ \flat(\bar{Q}) \circ \flat(\bar{K}) \circ (zH)^{-M} \\ &= (\pi_K^{-1}(x)\psi_0)\Lambda_0 \circ \flat(\bar{K}) \circ (zH)^{-M} = (\pi_K^{-1}(x)\pi_K(x)\pi_Q(x)\psi_0) \circ \flat(\bar{K}) \circ (zH)^{-M} \\ &= (\pi_Q(x)\psi_0) \circ \flat(\bar{K}) \circ (zH)^{-M} = (\pi_Q(x)\pi_K(x)\pi_K^{-1}(x)\psi_0) \circ \flat(\bar{K}) \circ (zH)^{-M} \\ &= \hat{\pi}(x)(\pi_K^{-1}(x)\psi_0)\flat(\bar{K}) \circ (zH)^{-M} = \hat{\pi}(x)(\pi_K^{-1}(x)\bar{K}\psi_0)(zH)^{-M} \\ &= \hat{\pi}(x)(K\psi_0)(zH)^{-M} = \hat{\pi}(x)\psi. \end{aligned} \quad \blacksquare$$

5.3 Ad-nilpotency

We now have eigenvalue equations $L_p\psi = \psi p(z)$ and $\psi\Lambda = \hat{\pi}(x)\psi$. In the commutative case, Duistermaat and Grünbaum noticed that these operators and eigenvalues satisfied an interesting relationship that goes by the name of “ad-nilpotency” [8]. As usual, for operators R and P we recursively define $\text{ad}_R^n P$ by

$$\text{ad}_R^1 P = R \circ P - P \circ R \text{ and } \text{ad}_R^n P = R \circ (\text{ad}_R^{n-1} P) - (\text{ad}_R^{n-1} P) \circ R, \quad n \geq 2.$$

When R is a differential operator of order greater than one, then one generally expects the order of $\text{ad}_R^n P$ to get large as n goes to infinity, but they found that when R is a scalar bispectral differential operator and P is the eigenvalue of a corresponding differential operator in the spectral variable, then surprisingly $\text{ad}_R^n P$ is the zero operator for large enough n . In fact, in that scalar Schrödinger operator case they considered, Duistermaat and Grünbaum found that ad-nilpotency was both a necessary and sufficient condition for bispectrality [8].

As it turns out, in the this new context even the easier of these two statements fails to hold. In particular, the ad-nilpotency in the commutative case is a consequence of the fact that the leading coefficients of $R \circ P$ and $P \circ R$ are equal, but this is not generally true for differential operators with matrix coefficients. Consequently, it is *not* the case that ad-nilpotency holds for all of the bispectral operators produced by the procedure described above. The following results (and an example in Remark 6.1) show how and to what extent the old result generalizes.

Theorem 5.14. *Given L_p , Λ , $\hat{\pi}(x)$ and $p(z)$ as above, the results of applying $\text{ad}_\Lambda^n L_p$ and $\text{ad}_\Lambda^n p$ on ψ are equal for every $n \in \mathbb{N}$.*

Proof. The case $n = 1$ is obtained by subtracting the equations

$$\hat{\pi}(x)L_p\psi = \hat{\pi}(x)\psi p(z) = \psi\Lambda p(z)$$

from the equations with the orders of the operators reversed

$$L_p\hat{\pi}(x)\psi = L_p\psi\Lambda = \psi p(z)\Lambda.$$

If one assumes the claim is true for $n = k$ then the case $n = k + 1$ is proved similarly and the general case follows by induction. \blacksquare

So, the equivalence of the actions of the two operators generated by iterating “ad” remains true regardless of non-commutativity. One cannot conclude from this alone that either of them is zero without making further assumptions, but if $\text{ad}_{\hat{\pi}}^n L = 0$ (which would be the case, for instance, if one could be certain that $\hat{\pi}$ would commute with the coefficients) then because ψ is not in the kernel of any non-zero operator in z alone, the one could conclude the same is true for the corresponding operator written in terms of Λ and p :

Corollary 5.15. *If $\text{ad}_{\hat{\pi}}^n L = 0$ then $\text{ad}_{\Lambda}^n p = 0$.*

Remark 5.16. This may be the first time that ad-nilpotency has been considered for translation operators as well as for matrix operators. It is therefore relevant to note that Corollary 5.15 is valid even when the operator Λ involves shift operators of the form \top_{α} (see Remark 6.1).

6 Examples

6.1 A wave function that is not part of a bispectral triple

Consider the case $W^* = \text{span}\{\delta_1, \delta_2\}$,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} \Delta_{1,0} \\ \Delta_{0,0} \end{pmatrix}, \quad \text{and} \quad \delta_2 = \begin{pmatrix} 0 \\ \Delta_{1,0} \end{pmatrix}.$$

Then Assumption 3.6 is met and

$$\psi(x, z) = \begin{pmatrix} \frac{xz-1}{xz} e^{xz} & 0 \\ \frac{1}{x^2 z} e^{xz} & \frac{xz+1}{xz} e^{-xz} \end{pmatrix}.$$

The matrix polynomial $p(z) = z^2 I$ satisfies $p \circ \delta_i = 0$ for $i = 1$ and $i = 2$ and so $p \in \mathcal{A}$ and as predicted by Theorem 4.7, the corresponding operator

$$L_p = \begin{pmatrix} \partial^2 - \frac{2}{x^2} & 0 \\ \frac{4}{x^3} & \partial^2 - \frac{2}{x^2} \end{pmatrix}$$

satisfies $L_p\psi = \psi p$.

Note that this is a rational Darboux transformation of the first operator and eigenfunction from the bispectral triple $(\partial^2, \partial_z^2, e^{xz}H)$. Since in the scalar case it has been found that rational Darboux transformations preserve bispectrality, one might expect L_p and ψ to be part of a bispectral triple. However, $H\delta_i \notin W^*$ and so Assumption 5.3 is not satisfied. Thus, Theorem 5.13 does not guarantee the existence of a differential-translation operator Λ in z having ψ as an eigenfunction. In fact, in this simple case we can see that no such operator exists.

Briefly, the argument is as follows. Consider a differential-translation operator Λ acting on ψ from the right and suppose there is a function $\pi(x)$ such that $\psi\Lambda = \pi(x)\psi$. By noting the coefficients of e^{xz} and e^{-xz} in the top right entry of each side of this equality, we see immediately that $\Lambda_{12} = \pi_{12} = 0$ (i.e., Λ and π are both lower triangular.). Then, we have the scalar eigenvalue equation $\psi_{11}\Lambda_{11} = \pi_{11}\psi_{11}$ and also $\psi_{21}\Lambda_{11} = \pi_{21}\psi_{11} + \pi_{22}\psi_{21}$. Combining these with $\psi_{11} + x\psi_{21} = e^{xz}$ one concludes that $e^{xz}\Lambda_{11}$ can be written in the form $(f(x) + g(x)/z)e^{xz}$. It follows that Λ_{11} as an operator in z has only coefficients that are constant or are a constant multiplied by $1/z$. In fact, all of the operators in z having ψ_{11} as eigenfunction are known; they are the operators that intertwine by $\partial - \frac{1}{z}$ with a constant coefficient operator. The only ones that meet both the criteria of the previous two sentences are the order zero operators. In other words, Λ_{11} would have to be a number. Then the equation for the action of Λ_{11} on ψ_{21} tells us that $\pi_{21} = 0$ and $\pi_{22} = \pi_{11}$ is also a number. Since the eigenvalue of this operator Λ does not depend on x , the operator does not form a bispectral triple with L and ψ .

6.2 A rational example with non-diagonalizable H

Consider $H = I + U$, where

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{so that} \quad \psi_0 = e^{xzH} = \begin{pmatrix} e^{xz} & ze^{xz} \\ 0 & e^{xz} \end{pmatrix}.$$

The distributions chosen are $W^* = \text{span}\{\delta_1, \dots, \delta_4\}$ with

$$\delta_1 = \begin{pmatrix} \Delta_{2,1} \\ 0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 \\ \Delta_{2,1} \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} \Delta_{0,1} \\ 0 \end{pmatrix}, \quad \text{and} \quad \delta_4 = \begin{pmatrix} 0 \\ \Delta_{0,1} \end{pmatrix}.$$

Assumption 3.6 is met and the unique monic operator of order 2 satisfying $K(\phi_i) = 0$ where $\phi_i = (\psi_0)\delta_i$ for $1 \leq i \leq 4$ is

$$K = \partial^2 I + \begin{pmatrix} \frac{-2x-1}{x} & -2 \\ 0 & \frac{-2x-1}{x} \end{pmatrix} \partial + \begin{pmatrix} \frac{x+1}{x} & \frac{2x^2+x}{x^2} \\ 0 & \frac{x+1}{x} \end{pmatrix}.$$

Now define

$$\psi = K(\psi_0)(zH)^{-2} = \begin{pmatrix} \frac{xz^2 - 2xz - z + x + 1}{xz^2} & \frac{z-1}{xz^2} \\ 0 & \frac{xz^2 - 2xz - z + x + 1}{xz^2} \end{pmatrix} e^{xzH}.$$

Assumption 5.3 is met because $H\delta_i = \delta_i$, $H\delta_{i+1} = \delta_i + \delta_{i+1}$ for $i = 1, 3$. Consequently, it will be possible to produce operators in z sharing the eigenfunction ψ . The matrix polynomial

$$p(z) = \begin{pmatrix} (z-1)^2 & (z-1)^3 \\ 0 & (z-1)^2 \end{pmatrix} = (z-1)^2 I + (z-1)^3 U$$

is in \mathcal{A} because $p \circ \delta_i = 2\delta_{i+2}$ and $p \circ \delta_{i+2} = 0$ for $i = 1, 2$. So,

$$b^{-1}(p(z)) = \begin{pmatrix} \partial^2 - 2\partial + 1 & \partial^3 - 5\partial^2 + 5\partial - 1 \\ 0 & \partial^3 - 2\partial + 1 \end{pmatrix}$$

satisfies the intertwining relationship

$$K \circ b^{-1}(p(z)) = L_p \circ K$$

with

$$L_p = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \partial^3 + \begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix} \partial^2 + \begin{pmatrix} -2 & 5 - \frac{3}{x^2} \\ 0 & -2 \end{pmatrix} \partial + \begin{pmatrix} 1 - \frac{2}{x^2} & -1 + \frac{7}{x^2} + \frac{3}{x^3} \\ 0 & 1 - \frac{2}{x^2} \end{pmatrix}.$$

This operator has eigenfunction ψ with eigenvalue p :

$$L_p \psi = L_p \circ K \psi_0 = K \circ L_0 \psi_0 = K \psi_0 p(z) = \psi p(z).$$

(As mentioned in Remark 2.1, multiplying ψ by a matrix on the right has the effect of conjugating the eigenvalue. It is therefore worth noting that the eigenvalue p here is not only non-diagonal but in fact non-diagonalizable.)

To produce an operator in z sharing ψ as eigenfunction, we consider the polynomial $p_0(z) = (z-1)^3$ (which has this form because $\lambda = 1$ is the only point in the support of the distributions in W^* and because the highest derivative they take there is $m_\lambda = 2$). Then

$$L_0 = \flat^{-1}(p_0(z)I) = (\partial^3 - 3\partial^2 + 3\partial - 1)I + (-3\partial^3 + 6\partial^2 - 3\partial)U$$

is the operator satisfying $L_0 \psi_0 = \psi_0 p_0(z)$ and L_0 factors as $L_0 = Q \circ K$ with

$$Q = \begin{pmatrix} \partial + \frac{1-x}{x} & -3\partial + \frac{2x-3}{x} \\ 0 & \partial + \frac{1-x}{x} \end{pmatrix}.$$

The next step is to choose two functions from $\mathcal{C}[x]$, so that $\bar{K} = \pi_K K$ and $\bar{Q} = Q \circ \pi_Q$ are in $\mathbb{C}[x, \partial]$. It turns out that the selection

$$\pi_K = \begin{pmatrix} x & x^2 \\ 0 & x \end{pmatrix} = \pi_Q$$

works and we get that

$$\begin{aligned} \Lambda &= (zH)^2 \circ (p_0(z))^{-1} \circ \flat(\bar{Q}) \circ \flat(\bar{K}) \circ (zH)^{-2} = \partial_z^3 I + \partial_z^2 \begin{pmatrix} 1 & -\frac{2(z^2 - z + 6)}{(z-1)z} \\ 0 & 1 \end{pmatrix} \\ &+ \partial_z \begin{pmatrix} \frac{4}{z-z^2} & \frac{2(z^2 + 8z + 6)}{(z-1)^2 z^2} \\ 0 & \frac{4}{z-z^2} \end{pmatrix} + \begin{pmatrix} \frac{-2z^2 + 4z + 2}{(z-1)^2 z^2} & \frac{4z^3 - 6z^2 - 8z - 20}{(z-1)^3 z^2} \\ 0 & \frac{-2z^2 + 4z + 2}{(z-1)^2 z^2} \end{pmatrix} \end{aligned}$$

does indeed satisfy $\psi \Lambda = \pi_Q(x) \pi_K(x) \psi$.

6.3 Computing \mathbb{T}_α^H for non-diagonal H

Suppose $H = \lambda I + U$ where U is still the upper-triangular matrix from the previous example. Since

$$\exp(xH^{-1}) = e^{x/\lambda} I - \frac{x}{\lambda^2} e^{x/\lambda} U,$$

the corresponding operator formed by replacing $e^{x/\lambda}$ with $\mathbb{T}_{\alpha/\lambda}$ and other occurrences of x with $\alpha \partial_z$ would be

$$\mathbb{T}_\alpha^H = \mathbb{T}_{\alpha/\lambda} I - \frac{1}{\lambda^2} \mathbb{T}_{\alpha/\lambda} \alpha \partial_z U.$$

This operator in z that combines differentiation and translation has the property that $(e^{xzH}) \mathbb{T}_\alpha^H = e^{\alpha x} e^{xzH}$.

6.4 An exponential example with $H = I$

Finally, consider $H = I$ and

$$W^* = \text{span} \left\{ \begin{pmatrix} \Delta_{0,0} + \Delta_{0,1} \\ \Delta_{0,1} \end{pmatrix}, \begin{pmatrix} 0 \\ -\Delta_{0,0} + \Delta_{0,1} \end{pmatrix} \right\}.$$

Then

$$K = \begin{pmatrix} \partial - \frac{e^x}{1+e^x} & 0 \\ \frac{e^x}{e^{2x}-1} & \partial + \frac{e^x}{1-e^x} \end{pmatrix} \quad \text{and}$$

$$\psi(x, z) = \begin{pmatrix} \frac{e^{xz}(e^x(z-1)+z)}{(1+e^x)z} & 0 \\ \frac{e^{zx+x}}{(-1+e^{2x})z} & \frac{e^{xz}(e^x(z-1)-z)}{(-1+e^x)z} \end{pmatrix}.$$

Because of the support of the distributions, $p_0(z) = z^2 - z$ and we may choose $p \in \mathcal{A}$ to be a multiple of that by a constant matrix

$$p(z) = \begin{pmatrix} 0 & z^2 - z \\ z^2 - z & 0 \end{pmatrix}.$$

The constant operator $b^{-1}(p) = p(\partial)$, satisfies the intertwining relationship $K \circ p(\partial) = L_p \circ K$ with

$$L_p = \begin{pmatrix} -\frac{e^x(-1+e^x-2e^{2x})}{(-1+e^{2x})^2} & -\frac{2e^{2x}}{(-1+e^x)^2(1+e^x)} \\ \frac{e^{2x}(-3+2e^x)}{(-1+e^{2x})^2} & \frac{e^x}{(-1+e^x)^2(1+e^x)} \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{e^x}{-1+e^{2x}} & \frac{-1-e^x+3e^{2x}-e^{3x}}{(-1+e^x)^2(1+e^x)} \\ \frac{-1+2e^x+2e^{2x}-2e^{3x}-e^{4x}}{(-1+e^{2x})^2} & \frac{e^x}{(-1+e^x)(1+e^x)} \end{pmatrix} \partial + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x^2.$$

Hence $L_p \psi = \psi p(z)$. (Since $p(z)$ and ψ do not commute, it is necessary to write the eigenvalue on the right rather than the left: $L\psi \neq p(z)\psi$.)

Now, to compute an operator in z having the same function ψ as an eigenfunction, we factor $p_0(\partial)I$ as $Q \circ K$ to obtain

$$Q = \partial I + \begin{pmatrix} -\frac{1}{1+e^x} & 0 \\ \frac{1}{1-e^{2x}} & -\frac{1}{-1+e^x} \end{pmatrix}.$$

Letting π_K and π_Q be

$$\pi_K = \begin{pmatrix} e^{2x}-1 & 0 \\ 0 & 1-e^{2x} \end{pmatrix}, \quad \pi_Q = \begin{pmatrix} e^{2x}-1 & 0 \\ 0 & e^{2x}-1 \end{pmatrix}$$

we get

$$\bar{K} = \pi_K K = \begin{pmatrix} -e^x(-1+e^x) & 0 \\ -e^x & e^x+e^{2x} \end{pmatrix} + \begin{pmatrix} (-1+e^x)(1+e^x) & 0 \\ 0 & 1-e^{2x} \end{pmatrix} \partial \in \mathcal{W}$$

and

$$\bar{Q} = Q \circ \pi_Q = \begin{pmatrix} 1 - e^x + 2e^{2x} & 0 \\ -e^x & 1 + e^x + 2e^{2x} \end{pmatrix} + \begin{pmatrix} -1 + e^{2x} & 0 \\ 0 & -1 + e^{2x} \end{pmatrix} \partial \in \mathcal{W}.$$

Replacing $e^{\alpha x}$ by $\top_\alpha^H = \top_\alpha$ and ∂ by z we get the corresponding translation operators

$$b(\bar{K}) = \begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix} + \top_1 \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + \top_2 \begin{pmatrix} z-1 & 0 \\ 0 & 1-z \end{pmatrix}$$

and

$$b(\bar{Q}) = \begin{pmatrix} 1-z & 0 \\ 0 & 1-z \end{pmatrix} + \top_1 \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} + \top_2 \begin{pmatrix} z+2 & 0 \\ 0 & z+2 \end{pmatrix}.$$

As predicted by Theorem 5.13,

$$\begin{aligned} \Lambda &= \frac{z}{p_0(z)} \circ b(\bar{Q}) \circ b(\bar{K}) \circ \frac{1}{z} = \begin{pmatrix} \frac{z^3 + 5z^2 + 6z}{z(z+2)(z+3)} & 0 \\ 0 & \frac{-z^3 - 5z^2 - 6z}{z(z+2)(z+3)} \end{pmatrix} \\ &+ \top_1 \begin{pmatrix} 0 & 0 \\ \frac{2z^2 + 6z + 4}{z(z+1)(z+2)} & 0 \end{pmatrix} + \top_2 \begin{pmatrix} \frac{-2z^3 - 10z^2 - 12z}{z(z+2)(z+3)} & 0 \\ \frac{-2z - 4}{z(z+1)(z+2)} & \frac{2z^3 + 10z^2 + 12z}{z(z+2)(z+3)} \end{pmatrix} \\ &+ \top_3 \begin{pmatrix} \frac{4z + 12}{z(z+2)(z+3)} & 0 \\ \frac{-2z^2 - 4z - 2}{z(z+1)(z+2)} & \frac{4z + 12}{z(z+2)(z+3)} \end{pmatrix} \\ &+ \top_4 \begin{pmatrix} \frac{z^3 + 5z^2 + 2z - 8}{z(z+2)(z+3)} & 0 \\ 0 & \frac{-z^3 - 5z^2 - 2z + 8}{z(z+2)(z+3)} \end{pmatrix} \end{aligned}$$

satisfies $\psi\Lambda = \hat{\pi}\psi$ ($\hat{\pi} = \pi_Q\pi_K$).

Remark 6.1. This is a good example for demonstrating ad-nilpotency and how it has changed in this non-commutative context. Using L , Λ , $p(z)$ and π_K , π_Q as above, it is indeed true that

$$\text{ad}_{\hat{\pi}}^n L\psi = \psi \text{ad}_{\Lambda}^n p$$

(with the operator in z acting from the right as usual). However, contrary to our expectation from the commutative case in which the order of the operator on the left decreases to zero when n gets large, $\text{ad}_{\hat{\pi}}^n L$ is an operator of order 2 for every n . This happens because the action of iterating $\text{ad}_{\hat{\pi}}$ on the leading coefficient of L itself is not nilpotent. On the other hand, if instead of π_K and π_Q we had chosen

$$\pi_K^* = \pi_Q^* = (e^{2x} - 1)I, \quad \hat{\pi}^* = \pi_K^* \pi_Q^* = (e^{2x} - 1)^2 I$$

(intentionally selected so as to commute with the coefficients of any operator) then Theorem 5.13 would have produced a different operator Λ^* satisfying $\hat{\pi}^*\psi = \psi\Lambda^*$. In this case, because the order is lowered at each iteration, $\text{ad}_{\hat{\pi}^*}^3 L = 0$. This is not particularly surprising or special as the same could be said if $\hat{\pi}^*$ were replaced by any function of the form $f(x)I$. However, because

of the correspondence in Theorem 5.14, we can conclude from this that $\text{ad}_{\Lambda^*}^3 p(z) = 0$ also. That is more interesting because in general repeatedly taking the commutator of the operator Λ^* with some function will not produce the zero operator, even if all coefficients are assumed to commute. Nevertheless, with these specific choices everything cancels out leaving exactly zero when ad_{Λ^*} is applied three times to the function $p(z)$ that is the eigenvalue above.

7 Conclusions and remarks

Given a choice of an invertible $N \times N$ matrix H and MN -dimensional space W^* of vector-valued finitely supported distributions, this paper sought to produce a bispectral triple (L, Λ, ψ) where L is a differential operator acting from the left, Λ is a differential-translation operator acting from the right and ψ is a common eigenfunction that is asymptotically of the form e^{xzH} satisfying the “conditions” generated by the distributions. In the scalar case, this was achieved in [23] for *any* choice of W^* . In the matrix generalization above, however, the construction only works given Assumptions 3.6 and 5.3. The bispectral triples produced given those two assumptions include many new examples both in the form of the eigenfunction (asymptotically equal to e^{xzH}) and the fact that the matrix-coefficient operator Λ may involve translation in z as well as differentiation in z . More importantly, this investigation yielded some observations that may be useful in future studies of bispectrality in a non-commutative context.

This paper sought to develop a general construction of bispectral triples with matrix-valued eigenvalues and also to understand what obstructions there might be to generalizing the construction from [23] to the matrix case. It is interesting to note that these seemingly separate goals both turn out to depend on the non-commutativity of the ring \mathcal{A} of functions that stabilize the point in the Grassmannian. Since \mathcal{A} is also the ring of eigenvalues for the operators in x , it is not a surprise that by letting its elements be matrix-valued gives us a non-commutative ring $\{L_p : p \in \mathcal{A}\}$ of operators sharing the eigenfunction ψ . It was not clear at first that the ring \mathcal{A} would also be the source of the obstruction to producing bispectral triples. However, the construction of operators in z sharing ψ as eigenfunction uses the assumption that H is an element of \mathcal{A} . (Specifically, this is used in the proof of Lemma 5.10.) It is interesting to note that this *also* depends on the fact that we considered a matrix-valued stabilizer ring.

Section 5.3 explored the extent to which the property of ad-nilpotency, which has been a feature of papers on the bispectral problem since it was first noted in [8], continues to apply in the case of matrix coefficient operators. It is still the case that the operator formed by iterating the adjoint of one of the operators in a bispectral triple on the eigenvalue of the other operator has the same action on the eigenfunction as the operator formed by iterating the adjoint action of its eigenvalue on the other operator. However, unlike the scalar case, only if additional assumptions about the coefficients of the operators are met will either of those be zero for a large enough iterations.

It was previously observed [2, 4, 32] (see also [14, 15, 16, 17, 18]) that requiring the operators in x and z to act on the eigenfunctions from opposite side resulted in a form of bispectrality whose structure and applications more closely resembled that in the scalar case. Because generalizing the scalar results of [23] necessitated also requiring that the eigenvalues and finitely-supported distributions act from the same side as the operators acting in the same variables, this previous observation can now be extended to those other objects as well.

This is the first time that the bispectral anti-isomorphism was used to construct bispectral operators in a context involving operators with matrix coefficients, see section 5.1 (cf. [12]). Some modifications were necessary since the rather general construction in [1] assumed that there were no zero-divisors so that formal inverses could be introduced. In addition, the use of the method here depended on the convention of considering operators in x and z to be acting from opposite sides and required that the coefficients of the operators on which it acted were

taken from the centralizer of H . The most interesting difference may have been that because the operators in z are acting from the right rather than the left, the map actually *preserves* the order of a product.

Unlike the scalar case, not every choice of distributions W^* corresponds to a bispectral triple. For instance, if Assumption 3.6 fails to be met then there simply is no wave function ψ satisfying the conditions. More interestingly, if Assumption 3.6 is met but Assumption 5.3 is not then there is a wave function that is an eigenfunction for a ring of differential operators in x , but Theorem 5.13 does not produce a corresponding operator in z . In fact, as Section 6.1 demonstrates, in at least some cases there actually is no bispectral triple of the form considered above which includes that wave function. This is very different than the scalar case. Unfortunately, this paper does not entirely answer the question of which choices of matrix H and distributions W^* produce a wave function ψ that is part of a bispectral triple of the type considered here. In particular, this paper does not show or claim that there *cannot be* a differential-difference operator Λ in z having ψ as an eigenfunction with x -dependent eigenvalue when $H \notin \mathcal{A}$.

Arguably, some of the non-commutativity involved in the construction above was “artificially inserted” in the form of the choice of the matrix H . If one chooses to consider only the case $H = I$, then Assumption 5.3 is automatically met and Theorems 4.7 and 5.13 produce a bispectral triple for any W^* satisfying Assumption 3.6. Since any operator L_p produced by the construction above for some choice of H *can* be produced using a different choice of distributions but with $H = I$, one may conclude that each of these operators is part of a bispectral triple. (In other words, the obstruction to bispectrality that is visible when one seeks a bispectral triple for a given wave function ψ disappears if one instead focuses on the operator L_p and seeks a corresponding bispectral triple.) However, it seems plausible that some examples of bispectrality to be considered in the future will involve vacuum eigenfunctions that are necessarily non-commutative (unlike these examples in which the non-commutativity of the vacuum eigenfunctions can always be eliminated through a change of variables), and the observations and results above will prove useful in those contexts.

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