Three-Phase Freak Waves

Aleksandr O. SMIRNOV, Sergei G. MATVEENKO, Sergei K. SEMENOV
and Elena G. SEMENOVA

St.-Petersburg State University of Aerospace Instrumentation (SUAI),
67 Bolshaya Morskaya Str., St.-Petersburg, 190000, Russia
E-mail: *alsmir@guap.ru, MatveiS239@gmail.com, sksemenov@mail.ru, egsemenova@mail.ru*

Abstract. In the article, we describe three-phase finite-gap solutions of the focusing nonlinear Schrödinger equation and Kadomtsev–Petviashvili and Hirota equations that exhibit the behavior of almost-periodic "freak waves". We also study the dependency of the solution parameters on the spectral curves.

Key words: nonlinear Schrödinger equation; Hirota equation; freak waves; theta function; reduction; covering; spectral curve

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1 Introduction

This study was motivated by the intention to demonstrate the behavior of three-phase extreme waves. Most recent scientific research shows that the simplest and most universal model for such waves is the focusing nonlinear Schrödinger equation (NLS)

$$ip_t + p_{xx} + 2\,|p|^2\,p = 0, \qquad i^2 = -1, \tag{1}$$

Since 1968 the equation (1) has been describing distribution on the surface of the ocean of weakly nonlinear quasi-monochromatic wave packets with relatively steep fronts [44]. An application of this equation to the problems of nonlinear optics was known earlier [7]. Since the equation (1) is a model of first approximation, it appears in simulations of many weakly nonlinear phenomena. This equation has a wide range of applications ranging from plasma physics [28] to financial markets [43].

Among the properties of equation (1) there is a modulation instability that leads to the appearance of the so-called "freak waves" (in hydrodynamics known as "rogue waves") [2]. These waves represent amplitude peaks localized in space and time. In the last 20 years, first in hydrodynamics and then in nonlinear optics, these waves have been the object of numerous theoretical and experimental studies [3]. Such attention to the problem of the "freak waves" is due to the losses at oil platforms, tankers, container ships and other large vessels caused by the "rogue waves".

There are many more precise and more complex models, which give a more exact description of the "freak waves" [3]. These models can be divided into two classes. In the first class one can solve them analyticaly while in the second class one can use numerical methods only. Analytical methods include: inverse scattering transform method; finite-gap integration method; Bäcklund transform method; Darboux transform method; Hirota method.

In the present work, we use a finite-gap integration method. The works of Dubrovin, Novikov, Marchenko, Lax, McKean, van Moerbeke, Matveev, Its, Krichever [9, 10, 11, 12, 13, 21, 22, 26, 29, 31, 33, 35] give a description of this method (see also the review [32]). However, another method of constructing finite-gap solutions of integrable nonlinear equations exists [23, 24, 34, 36]. Let

us remark that first method is based on Baker–Akhiezer function but the second one is based on some Fay's identities [14]. In our paper, we use the first method and Its and Kotlyarov's classic formulas [18, 20] (see also [6]).

Our goal here is to show the behavior of three-phase algebro-geometric solutions of NLS, KP-I and Hirota equations. Section 2 of this paper contains the basic notations and classic formulas for algebro-geometric solutions of integrable nonlinear equations under consideration. Section 3 is devoted to the periodicity of three-phase solutions of NLS, KP-I and Hirota equations. In Section 4 we consider an example of three-phase algebro-geometric solutions of KP-I and Hirota equations for different values of parameters.

2   Finite-gap multi-phase solutions of the NLS equation

The nonlinear differential equations that are integrated by methods of the algebraic geometry, can be obtained as a compatibility condition of the system of ordinary linear differential equations with a spectral parameter [6, 15, 16]. In particular, let us consider the following equations [15, 19, 38]

$$Y_x = \mathfrak{U}Y, \qquad Y_z = \mathfrak{V}Y, \qquad Y_t = \mathfrak{W}Y, \tag{2}$$

where

$$\mathfrak{U} = -\lambda \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} 0 & i\psi \\ -i\phi & 0 \end{pmatrix}, \qquad \mathfrak{V} = 2\lambda\mathfrak{U} + \mathfrak{V}_0, \qquad \mathfrak{W} = 4\lambda^2\mathfrak{U} + 2\lambda\mathfrak{W}_0 + \mathfrak{W}_1,$$

λ is a spectral parameter. Using these equations and additional relations

$$(Y_x)_z = (Y_z)_x, \qquad (Y_x)_t = (Y_t)_x$$

one can easy obtain the so-called equations of zero curvature

$$\mathfrak{U}_z - \mathfrak{V}_x + \mathfrak{U}\mathfrak{V} - \mathfrak{V}\mathfrak{U} = 0 \qquad \text{and} \qquad \mathfrak{U}_t - \mathfrak{W}_x + \mathfrak{U}\mathfrak{W} - \mathfrak{W}\mathfrak{U} = 0, \tag{3}$$

which should be valid for all values of spectral parameter λ. Respectively, it follows from equations (3) that matrixes $\mathfrak{V}_0$, $\mathfrak{W}_0$, $\mathfrak{W}_1$ take the forms

$$\mathfrak{W}_0 = \mathfrak{V}_0 = \begin{pmatrix} -i\psi\phi & -\psi_x \\ -\phi_x & i\psi\phi \end{pmatrix}, \qquad \mathfrak{W}_1 = \begin{pmatrix} \psi_x\phi - \psi\phi_x & 2i\psi^2\phi - i\psi_{xx} \\ -2i\psi\phi^2 + i\phi_{xx} & \psi\phi_x - \psi_x\phi \end{pmatrix},$$

Also, $\mathfrak{W} = 2\lambda\mathfrak{V} + \mathfrak{W}_1$. Conditions (3) lead to additional system of equations (parities). The first system is the coupled nonlinear Schrödinger equation

$$\begin{aligned} i\psi_z + \psi_{xx} - 2\psi^2\phi &= 0, \\ i\phi_z - \phi_{xx} + 2\psi\phi^2 &= 0, \end{aligned} \tag{4}$$

and the second system is the coupled modified Korteweg–de Vries equation

$$\begin{aligned} \psi_t + \psi_{xxx} - 6\psi\phi\psi_x &= 0, \\ \phi_t + \phi_{xxx} - 6\psi\phi\phi_x &= 0. \end{aligned} \tag{5}$$

These two systems of the nonlinear differential equations are closely related to two other ones. Specifically, differentiating equations (4) with respect to x and substituting them in (5), one obtains the coupled modified two-dimensional nonlinear Schrödinger equation in cone coordinates [27]

$$\begin{aligned} i\psi_t + \psi_{xz} + 2i(\psi\phi_x - \phi\psi_x)\psi &= 0, \\ i\phi_t - \phi_{xz} + 2i(\phi\psi_x - \psi\phi_x)\phi &= 0, \end{aligned}$$

Also, the functions $\psi(x, t, -\alpha t)$ and $\phi(x, t, -\alpha t)$ are solutions to the coupled integrable Hirota equation ($\alpha \in \mathbb{R}$)

$$
\begin{aligned}
i\psi_t + \psi_{xx} - 2\psi^2\phi - i\alpha(\psi_{xxx} - 6\psi\phi\psi_x) = 0, \\
i\phi_t - \phi_{xx} + 2\psi\phi^2 - i\alpha(\phi_{xxx} - 6\psi\phi\phi_x) = 0,
\end{aligned}
\tag{6}
$$

if $\psi(x, z, t)$ and $\phi(x, z, t)$ are solutions of (4) and (5).

Systems of the nonlinear differential equations (4), (5) are the first two integrable systems from the AKNS hierarchy [15]. One of the features of finite-gap multi-phase solutions of the integrable nonlinear equations is that fact that in some sense they are the solutions of all hierarchy. Particulary, our solutions can be used for constructing solutions of generalized nonlinear Schrödinger equation [42]. By substituting $\phi = \pm\overline{\psi}$ into equation (4) we get a standard form of the nonlinear Schrödinger equation. Particularly, for $\phi = -\overline{\psi}$ equations (4) transform to (1) [11, 18, 19] and equations (6) transform to the integrable Hirota equation [4, 8, 17, 30]

$$
i\psi_t + \psi_{xx} + 2\left|\psi\right|^2\psi - i\alpha\left(\psi_{xxx} + 6\left|\psi\right|^2\psi_x\right) = 0.
\tag{7}
$$

It is also easy to check that for any ψ and ϕ, that satisfy both (4) and (5) simultaneously, the function $u(x, z, t) = -2\psi\phi$ is a solution of the Kadomtsev–Petviashvili-I equation (KP-I)

$$
3u_{zz} = (4u_t + u_{xxx} + 6uu_x)_x.
\tag{8}
$$

In case $\phi = \pm\overline{\psi}$ this solution is a real function.

Finite-gap solutions of systems (4), (5) are parameterized by the hyperelliptic curve $\Gamma = \{(\chi, \lambda)\}$ of the genus g [15, 38]:

$$
\Gamma: \ \chi^2 = \prod_{j=1}^{2g+2}(\lambda - \lambda_j),
$$

The branch points ($\lambda = \lambda_j$, $j = 1, \ldots, 2g+2$) of this curve are the endpoints of the spectral arcs of continuous spectrum of Dirac operator (2). Infinitely far point of the spectrum corresponds two different points $\mathcal{P}_\infty^\pm$ on the curve Γ. In case $\phi = -\overline{\psi}$ the curve Γ has the form

$$
\Gamma: \ \chi^2 = \prod_{j=1}^{g+1}(\lambda - \lambda_j)(\lambda - \overline{\lambda}_j) = \lambda^{2g+2} + \sum_{j=1}^{2g+2}\chi_j\lambda^{2g+2-j}, \qquad \Im\chi_j = 0, \qquad \Im(\lambda_j) \neq 0. \tag{9}
$$

Following a standard procedure of constructing finite-gap solutions [6, 11, 38], for Γ let us choose a canonical basis of cycles $\gamma^t = (a_1, \ldots, a_g, b_1, \ldots, b_g)$ with matrix of intersection indices

$$
C_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
$$

To satisfy the condition $\phi = -\overline{\psi}$, it is necessary [6, 11] that this basis of cycles is transformed according to the rules

$$
\widehat{\tau}_1\mathbf{a} = -\mathbf{a}, \qquad \widehat{\tau}_1\mathbf{b} = \mathbf{b} + K\mathbf{a},
\tag{10}
$$

where τ_1 is anti-holomorphic involution, $\tau_1 : (\chi, \lambda) \to (\overline{\chi}, \overline{\lambda})$.

Let us also consider normalized holomorphic differentials $d\mathcal{U}_j$:

$$
\oint_{a_k} d\mathcal{U}_j = \delta_{kj}, \qquad k, j = 1, \ldots, g,
$$

and a matrix of periods B of the curve Γ:

$$B_{kj} = \oint_{b_k} d\mathcal{U}_j, \qquad k,j = 1,\ldots,g.$$

It is well known (see, for example, [5, 11]) that the matrix B is a symmetric matrix with positively defined imaginary part.

Let us introduce g-dimensional Riemann theta function with characteristics $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{R}^g$ [5, 11, 14]:

$$\Theta\big[\boldsymbol{\eta}^t; \boldsymbol{\zeta}^t\big](\mathbf{p}|B) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp\big\{\pi i(\mathbf{m} + \boldsymbol{\eta})^t B(\mathbf{m} + \boldsymbol{\eta}) + 2\pi i(\mathbf{m} + \boldsymbol{\eta})^t(\mathbf{p} + \boldsymbol{\zeta})\big\},$$

$$\Theta\big[\mathbf{0}^t; \mathbf{0}^t\big](\mathbf{p}|B) \equiv \Theta(\mathbf{p}|B),$$

where B is a matrix of periods, $\mathbf{p} \in \mathbb{C}^g$ and summation passes over an integer g-dimensional lattice.

Let us also define on Γ normalized Abelian integrals of the second kind $(\Omega_j(\mathcal{P}), j = 1, 2, 3)$ and the third kind $(\omega_0(\mathcal{P}))$ with the following asymptotic at infinitely distant points $\mathcal{P}_\infty^\pm$:

$$\oint_{a_k} d\Omega_1 = \oint_{a_k} d\Omega_2 = \oint_{a_k} d\Omega_3 = \oint_{a_k} d\omega_0 = 0, \qquad k = 1,\ldots,g,$$

$$\Omega_1(\mathcal{P}) = \mp i\left(\lambda - K_1 + O\left(\lambda^{-1}\right)\right), \qquad \mathcal{P} \to \mathcal{P}_\infty^\pm,$$

$$\Omega_2(\mathcal{P}) = \mp i\left(2\lambda^2 - K_2 + O\left(\lambda^{-1}\right)\right), \qquad \mathcal{P} \to \mathcal{P}_\infty^\pm,$$

$$\Omega_3(\mathcal{P}) = \mp i\left(4\lambda^3 - K_3 + O\left(\lambda^{-1}\right)\right), \qquad \mathcal{P} \to \mathcal{P}_\infty^\pm,$$

$$\omega_0(\mathcal{P}) = \mp\left(\ln\lambda - \ln K_0 + O\left(\lambda^{-1}\right)\right), \qquad \mathcal{P} \to \mathcal{P}_\infty^\pm,$$

$$\chi = \pm\left(\lambda^{g+1} + O\left(\lambda^g\right)\right), \qquad \mathcal{P} \to \mathcal{P}_\infty^\pm.$$

Let us denote the vectors of b-periods of Abelian integrals of the second kind $\Omega_1(\mathcal{P})$, $\Omega_2(\mathcal{P})$, $\Omega_3(\mathcal{P})$ by $2\pi i\mathbf{U}$, $2\pi i\mathbf{V}$, $2\pi i\mathbf{W}$ respectively.

Theorem 1 ([6, 38]). *Function*

$$Y(\mathcal{P}, x, z, t) = \begin{pmatrix} y_1(\mathcal{P}, x, z, t) & y_1(\tau_0\mathcal{P}, x, z, t) \\ y_2(\mathcal{P}, x, z, t) & y_2(\tau_0\mathcal{P}, x, z, t) \end{pmatrix},$$

where τ_0 is hyperelliptic involution, $\tau_0 : (\chi, \lambda) \to (-\chi, \lambda)$,

$$y_1(\mathcal{P}, x, z, t) = \frac{\Theta(\mathcal{U}(\mathcal{P}) + \mathbf{U}x + \mathbf{V}z + \mathbf{W}t - \mathbf{X})\Theta(\mathbf{Z})}{\Theta(\mathcal{U}(\mathcal{P}) - \mathbf{X})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z})}$$
$$\times \exp\{\Omega_1(\mathcal{P})x + \Omega_2(\mathcal{P})z + \Omega_3(\mathcal{P})t + i\Phi(x,z,t)\},$$

$$y_2(\mathcal{P}, x, z, t) = \rho\frac{\Theta(\mathcal{U}(\mathcal{P}) + \mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \boldsymbol{\Delta} - \mathbf{X})\Theta(\mathbf{Z} - \boldsymbol{\Delta})}{\Theta(\mathcal{U}(\mathcal{P}) - \mathbf{X})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z})}$$
$$\times \exp\{\Omega_1(\mathcal{P})x + \Omega_2(\mathcal{P})z + \Omega_3(\mathcal{P})t - i\Phi(x,z,t) + \omega_0(\mathcal{P})\},$$

is the eigenfunction of the Dirac operator (2) *with functions*

$$\psi(x,z,t) = \frac{2K_0}{\rho}\frac{\Theta(\mathbf{Z})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z} - \boldsymbol{\Delta})}{\Theta(\mathbf{Z} - \boldsymbol{\Delta})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z})}\exp\{2i\Phi(x,z,t)\},$$

$$\phi(x,z,t) = 2\rho K_0\frac{\Theta(\mathbf{Z} - \boldsymbol{\Delta})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z} + \boldsymbol{\Delta})}{\Theta(\mathbf{Z})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z})}\exp\{-2i\Phi(x,z,t)\}, \tag{11}$$

for any z, t and $\rho \neq 0$. The functions (11) *satisfy the equations* (4) *and* (5). *Here* $\boldsymbol{\Delta}$ *is the vector of holomorphic Abelian integrals, calculated along a path connecting* $\mathcal{P}_\infty^-$ *and* $\mathcal{P}_\infty^+$ *without crossing any of the basic cycles,*

$$\boldsymbol{\Delta} = \mathcal{U}(\mathcal{P}_\infty^+) - \mathcal{U}(\mathcal{P}_\infty^-), \qquad \Phi(x,z,t) = K_1 x + K_2 z + K_3 t,$$

$$\mathbf{X} = \mathcal{K} + \sum_{j=1}^{g} \mathcal{U}(\mathcal{P}_j), \qquad \mathbf{Z} = \mathcal{U}(\mathcal{P}_\infty^+) - \mathbf{X},$$

$\mathcal{K}$ *is a vector of Riemann constants* [5, 11, 14, 25]; $\mathcal{P}_j$, $j = 1, \ldots, g$ *is a non-special divisor. If the spectral curve* Γ *satisfies the condition* (9), *then the following equalities hold*

$$|\psi|^2 = -4K_0^2 \frac{\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z} - \boldsymbol{\Delta})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z} + \boldsymbol{\Delta})}{\Theta^2(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z})}, \tag{12}$$

$$\Im\mathbf{U} = \Im\mathbf{V} = \Im\mathbf{W} = \Im\mathbf{Z} = \mathbf{0}, \qquad K_0^2 < 0.$$

It is easy to see that the corresponding solution of KP-I equation (8) has the form

$$u(x,z,t) = -8K_0^2 \frac{\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z} - \boldsymbol{\Delta})\Theta(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z} + \boldsymbol{\Delta})}{\Theta^2(\mathbf{U}x + \mathbf{V}z + \mathbf{W}t + \mathbf{Z})}, \tag{13}$$

and that the square of amplitude of solution of Hirota equation (7) equals

$$|\psi_H|^2(x,t) = -4K_0^2 \frac{\Theta(\mathbf{U}x + (\mathbf{V} - \alpha\mathbf{W})t + \mathbf{Z} - \boldsymbol{\Delta})\Theta(\mathbf{U}x + (\mathbf{V} - \alpha\mathbf{W})t + \mathbf{Z} + \boldsymbol{\Delta})}{\Theta^2(\mathbf{U}x + (\mathbf{V} - \alpha\mathbf{W})t + \mathbf{Z})}.$$

3   Features of three-phase solutions

In case $g = 3$, the basis of normalized holomorphic differentials is defined by the formula [6, 11]:

$$d\mathcal{U}_k = \left(c_{k1}\lambda^2 + c_{k2}\lambda + c_{k3}\right)\frac{d\lambda}{\chi},$$

where

$$C = \left(A^t\right)^{-1}, \qquad A_{jm} = \oint_{a_j} \lambda^{3-m}\frac{d\lambda}{\chi}.$$

It follows from equation (ℓ is an arbitrary path on Γ)

$$\int_{\hat{\tau}\ell} d\omega = \int_\ell \tau^* d\omega,$$

that

$$\overline{A_{jm}} = \oint_{a_j} \overline{\lambda^{3-m}\frac{d\lambda}{\chi}} = \oint_{a_j} \tau_1^*\left(\lambda^{3-m}\frac{d\lambda}{\chi}\right) = \oint_{\hat{\tau}_1 a_j} \lambda^{3-m}\frac{d\lambda}{\chi} = -\oint_{a_j} \lambda^{3-m}\frac{d\lambda}{\chi} = -A_{jm}.$$

Therefore $\overline{A} = -A$ and $\overline{C} = -C$. Similarly, with integrals on b-cycles, we obtain

$$\overline{B} = -B - K \qquad \text{or} \qquad \Re B = -\frac{1}{2}K.$$

It follows from bilinear relations of Riemann (see, for example, [5, 6, 11]) that the coordinates of the vectors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ can be written as

$$U_m = -i\left(\left.\frac{d\mathcal{U}_m}{d\xi_-}\right|_{\xi_-=0} - \left.\frac{d\mathcal{U}_m}{d\xi_+}\right|_{\xi_+=0}\right), \qquad V_m = -2i\left(\left.\frac{d^2\mathcal{U}_m}{d\xi_-^2}\right|_{\xi_-=0} - \left.\frac{d^2\mathcal{U}_m}{d\xi_+^2}\right|_{\xi_+=0}\right),$$

$$W_m = -2i\left(\left.\frac{d^3\mathcal{U}_m}{d\xi_-^3}\right|_{\xi_-=0} - \left.\frac{d^3\mathcal{U}_m}{d\xi_+^3}\right|_{\xi_+=0}\right),$$

where $\xi_\pm = 1/\lambda$ are local parameters in the neighborhood of infinitely distant points $\mathcal{P}_\infty^\pm$. Calculating the derivatives we obtain the relations

$$U_m = -2ic_{m1}, \qquad V_m = 2i\chi_1 c_{m1} - 4ic_{m2}, \qquad W_m = i\big(4\chi_2 - 3\chi_1^2\big)c_{m1} + 4i\chi_1 c_{m2} - 8ic_{m3},$$

or

$$(\mathbf{U}, \mathbf{V}, \mathbf{W}) = iC\begin{pmatrix} -2 & 2\chi_1 & 4\chi_2 - 3\chi_1^2 \\ 0 & -4 & 4\chi_1 \\ 0 & 0 & -8 \end{pmatrix}. \tag{14}$$

It follows from (14) that the vectors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are real and linearly independent. Therefore, $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are the basis vectors in $\mathbb{R}^3$. Hence, any vector from $\mathbb{R}^3$ can be presented in the form of the linear combinations of these vectors. In particular, for the vectors of the periods of the three-dimensional theta-functions $\mathbf{e}_1^t = (1, 0, 0)$, $\mathbf{e}_2^t = (0, 1, 0)$, $\mathbf{e}_3^t = (0, 0, 1)$ we can write the following relations

$$\mathbf{e}_k = \mathcal{X}_k\mathbf{U} + \mathcal{Z}_k\mathbf{V} + \mathcal{T}_k\mathbf{W}.$$

Therefore, three-phase solutions (13) of equation KP-I (8) are the periodic functions in a three-dimensional space

$$u(x + \mathcal{X}_k, z + \mathcal{Z}_k, t + \mathcal{T}_k) = u(x, z, t).$$

If a three-phase solution of (8) has a form of freak waves, then the maxima of its amplitude are located in nodes of a three-dimensional lattice with edges $(\mathcal{X}_k, \mathcal{Z}_k, \mathcal{T}_k)$. These edges can be found by an inversion of the matrix $(\mathbf{U}, \mathbf{V}, \mathbf{W})$:

$$\begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 & \mathcal{X}_3 \\ \mathcal{Z}_1 & \mathcal{Z}_2 & \mathcal{Z}_3 \\ \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 \end{pmatrix} = (\mathbf{U}, \mathbf{V}, \mathbf{W})^{-1} = i\begin{pmatrix} 1/2 & \chi_1/4 & \chi_2/4 - \chi_1^2/16 \\ 0 & 1/4 & \chi_1/8 \\ 0 & 0 & 1/8 \end{pmatrix}A^t.$$

Therefore, for three-phases solutions of equation KP-I (8) it is possible to describe their behavior in the following way: after a time interval $\Delta t = \mathcal{T}_k$ a surface of solution $u(x, z)$ reproduces itself with a shift on the XOZ plane by the $(\mathcal{X}_k, \mathcal{Z}_k)$ vector.

As the three-phase solution of the equations (1) depends on two coordinates, x and z, and the third coordinate t is considered to be a parameter, the value of amplitude of this solution depends on the distance between the nodes of the given three-dimensional lattice and a plane $t = t_0$. Hence, in contrast to the case of the two-phase solution [39, 40, 41], where the change of initial phase $\mathbf{Z}$ led to trivial shift of the solution on XOZ plane, the amplitude of the three-phase solution (11) of equations (1) depends on a choice of initial phase $\mathbf{W}t_0 + \mathbf{Z}$ in a slightly more complicated fashion.

4   An example of three-phase solution

Let us consider a spectral curve $\Gamma_3 = \{\chi, \lambda\}$ of genus $g = 3$:

$$\Gamma_3: \quad \chi^2 = \big((\lambda - \lambda_0)^4 - 2a^2(\lambda - \lambda_0)^2\cos 2\varphi + a^4\big)\big((\lambda - \lambda_0)^4 - 2b^2(\lambda - \lambda_0)^2\cos 2\varphi + b^4\big),$$

where $0 < a < b$, $\pi/4 < \varphi < \pi/2$.
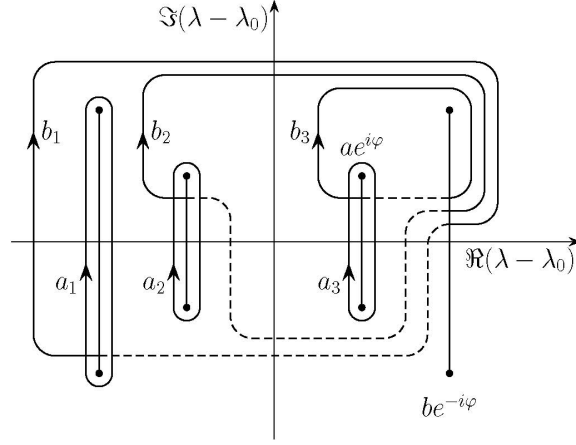
Figure 1. Canonical basis of cycles on Γ_3.

Let us choose the basis of cycles on Γ_3 as it is shown on Fig. 1.

It is easy to check that the anti-holomorphic involution τ_1 transforms the canonical basis of cycles using the rule (10) with the matrix

$$K = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

There are also three holomorphic involutions on Γ_3:

$$\tau_0 : \ (\chi, \lambda) \to (-\chi, \lambda),$$
$$\tau_2 : \ (\chi, \lambda) \to (\chi, 2\lambda_0 - \lambda),$$
$$\tau_3 : \ (\chi, \lambda) \to \left(a^2 b^2 (\lambda - \lambda_0)^{-4} \chi, \lambda_0 + ab(\lambda - \lambda_0)^{-1}\right).$$

As a corollary, the curve Γ_3 covers the following two curves:

1) $\Gamma_1 = \Gamma_3/\tau_2$ of genus $g = 1$

$$\Gamma_1 : \ \chi_+^2 = \left(t^2 - 2a^2 t \cos 2\varphi + a^4\right)\left(t^2 - 2b^2 t \cos 2\varphi + b^4\right),$$

2) $\Gamma_2 = \Gamma_3/(\tau_0 \tau_2)$ of genus $g = 2$

$$\Gamma_2 : \ \chi_-^2 = t\left(t^2 - 2a^2 t \cos 2\varphi + a^4\right)\left(t^2 - 2b^2 t \cos 2\varphi + b^4\right),$$

where $t = (\lambda - \lambda_0)^2$, $\chi_+ = \chi$, $\chi_- = (\lambda - \lambda_0)\chi$, and

$$\frac{dt}{\chi_+} = \frac{2(\lambda - \lambda_0)d\lambda}{\chi}, \qquad \frac{tdt}{\chi_-} = \frac{2(\lambda - \lambda_0)^2 d\lambda}{\chi}, \qquad \frac{dt}{\chi_-} = \frac{2d\lambda}{\chi}.$$

The curves Γ_1 and Γ_2 are shown on Figs. 2 and 3, where $t_1 = b^2 e^{2i\varphi}$, $t_2 = a^2 e^{2i\varphi}$.

The coverings generate the following transformations of cycles:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \to S \begin{pmatrix} a^1 \\ a_1^2 \\ a_2^2 \end{pmatrix} + P \begin{pmatrix} b^1 \\ b_1^2 \\ b_2^2 \end{pmatrix}, \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \to Q \begin{pmatrix} a^1 \\ a_1^2 \\ a_2^2 \end{pmatrix} + R \begin{pmatrix} b^1 \\ b_1^2 \\ b_2^2 \end{pmatrix},$$
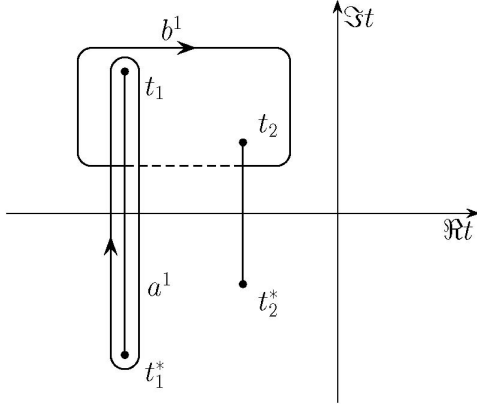
Figure 2. The curve Γ_1.
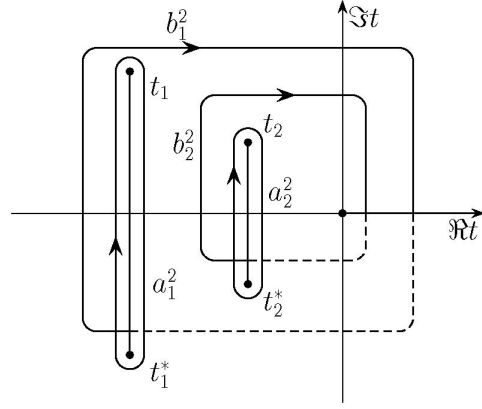
Figure 3. The curve Γ_2.

where

$$S = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix},$$

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Recall that these matrices should satisfy the relations

$$S^t Q = Q^t S, \qquad R^t P = P^t R, \qquad S^t R - Q^t P = nI,$$

where I is identity matrix, $n = 2$ is the number of covering sheets.

Due to involution τ_3, the curve Γ_2 covers two elliptic curves $\Gamma_\pm$ (Figs. 4 and 5):

$$\Gamma_\pm : \; \nu_\pm^2 = (s \pm 2ab)\left(s^2 - 2\left(a^2 + b^2\right)s\cos 2\varphi + a^4 + b^4 + 2a^2b^2\cos 4\varphi\right),$$

where

$$s = t + \frac{a^2 b^2}{t}, \qquad \nu_\pm = \frac{t \pm ab}{t^2}\chi_-, \qquad \frac{ds}{\nu_\pm} = \frac{(t \mp ab)dt}{\chi_-}.$$

The coverings of Γ_2 on $\Gamma_\pm$ generate the following cycles mappings

$$\begin{pmatrix} a_1^2 \\ a_2^2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}, \qquad \begin{pmatrix} b_1^2 \\ b_2^2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} b_+ \\ b_- \end{pmatrix},$$

As a result, we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} a^1 \\ a_+ \\ a_- \end{pmatrix} + \begin{pmatrix} 0 & -2 & -2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_+ \\ b_- \end{pmatrix}, \tag{15}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a^1 \\ a_+ \\ a_- \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_+ \\ b_- \end{pmatrix}. \tag{16}$$
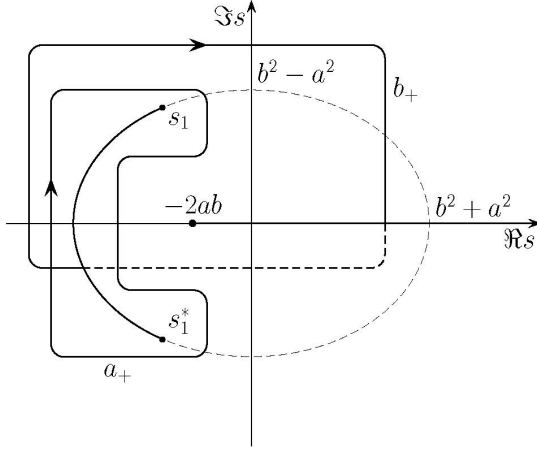
Figure 4. The curve Γ_+.
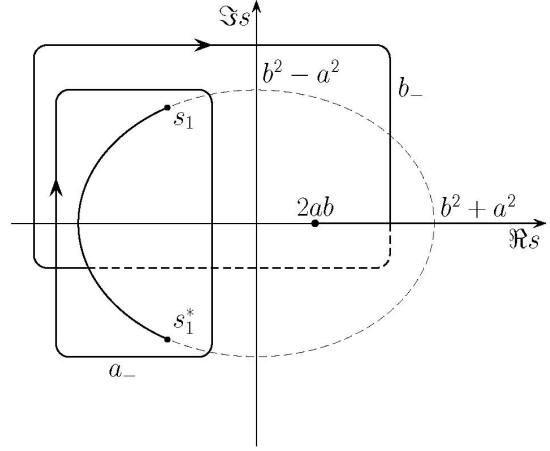
Figure 5. The curve Γ_-.

From (15), (16) and relations

$$\frac{d\lambda}{\chi} = \frac{1}{4ab}\frac{ds}{\nu_-} - \frac{1}{4ab}\frac{ds}{\nu_+}, \qquad \frac{\lambda d\lambda}{\chi} = \frac{1}{2}\frac{dt}{\chi_+} + \frac{\lambda_0}{4ab}\frac{ds}{\nu_-} - \frac{\lambda_0}{4ab}\frac{ds}{\nu_+},$$

$$\frac{\lambda^2 d\lambda}{\chi} = \lambda_0\frac{dt}{\chi_+} + \frac{\lambda_0^2+ab}{4ab}\frac{ds}{\nu_-} - \frac{\lambda_0^2-ab}{4ab}\frac{ds}{\nu_+}$$

it follows that

$$C = \begin{pmatrix} \mathfrak{c}_1+\mathfrak{c}_3 & -2\lambda_0(\mathfrak{c}_1+\mathfrak{c}_3) & (\lambda_0^2-ab)\mathfrak{c}_1+(\lambda_0^2+ab)\mathfrak{c}_3 \\ \mathfrak{c}_1 & \mathfrak{c}_2-2\lambda_0\mathfrak{c}_1 & (\lambda_0^2-ab)\mathfrak{c}_1-\lambda_0\mathfrak{c}_2 \\ \mathfrak{c}_3 & \mathfrak{c}_2-2\lambda_0\mathfrak{c}_3 & (\lambda_0^2+ab)\mathfrak{c}_3-\lambda_0\mathfrak{c}_2 \end{pmatrix},$$

$$B = \begin{pmatrix} i(\mathfrak{b}_1+\mathfrak{b}_3) & i\mathfrak{b}_1-1/2 & i\mathfrak{b}_3-1/2 \\ i\mathfrak{b}_1-1/2 & i(\mathfrak{b}_1+\mathfrak{b}_2) & i\mathfrak{b}_2-1/2 \\ i\mathfrak{b}_3-1/2 & i\mathfrak{b}_2-1/2 & i(\mathfrak{b}_2+\mathfrak{b}_3) \end{pmatrix},$$

where

$$\mathfrak{c}_1 = \frac{1}{2(\alpha_1-2\beta_1)}, \qquad \mathfrak{c}_2 = \frac{1}{2\alpha_2}, \qquad \mathfrak{c}_3 = \frac{1}{2(\alpha_3-2\beta_3)},$$

$$i\mathfrak{b}_1 = \frac{\alpha_1}{2(\alpha_1-2\beta_1)}, \qquad i\mathfrak{b}_2 = \frac{\beta_2}{2\alpha_2}, \qquad i\mathfrak{b}_3 = \frac{\alpha_3}{2(\alpha_3-2\beta_3)},$$

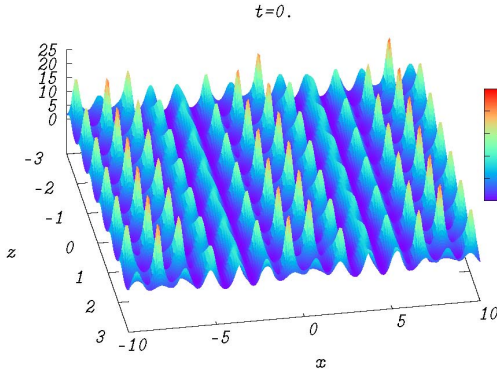$$\alpha_1 = \frac{1}{2}\oint_{a_+}\frac{ds}{\nu_+}, \qquad \alpha_2 = \frac{1}{2}\oint_{a^1}\frac{dt}{\chi_+}, \qquad \alpha_3 = \frac{1}{2}\oint_{a_-}\frac{ds}{\nu_-},$$

$$\beta_1 = \frac{1}{2}\oint_{b_+}\frac{ds}{\nu_+}, \qquad \beta_2 = \frac{1}{2}\oint_{b^1}\frac{dt}{\chi_+}, \qquad \beta_3 = \frac{1}{2}\oint_{b_-}\frac{ds}{\nu_-}.$$
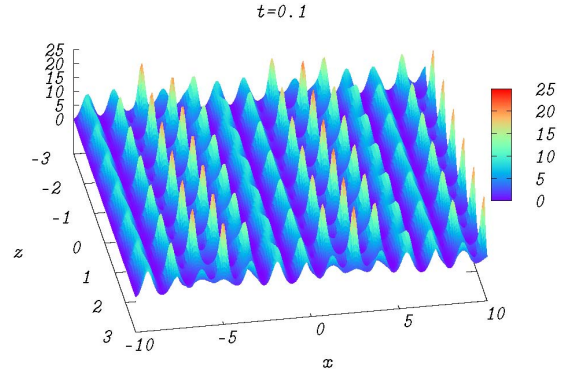
From the structure of the matrix B and from the matrix version of Appel's theorem [37] it follows that the Riemann theta function of curve Γ_3 equals:

$$\Theta(\mathbf{p}|B) = f(\widetilde{p}_1,\widetilde{p}_2,\widetilde{p}_3) = \vartheta_3(\widetilde{p}_1|h_1)\vartheta_3(\widetilde{p}_2|h_2)\vartheta_3(\widetilde{p}_3|h_3) + \vartheta_4(\widetilde{p}_1|h_1)\vartheta_1(\widetilde{p}_2|h_2)\vartheta_1(\widetilde{p}_3|h_3)$$
$$+ \vartheta_1(\widetilde{p}_1|h_1)\vartheta_4(\widetilde{p}_2|h_2)\vartheta_1(\widetilde{p}_3|h_3) + \vartheta_1(\widetilde{p}_1|h_1)\vartheta_1(\widetilde{p}_2|h_2)\vartheta_4(\widetilde{p}_3|h_3), \tag{17}$$
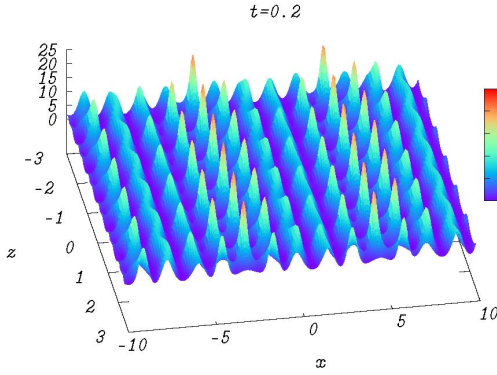
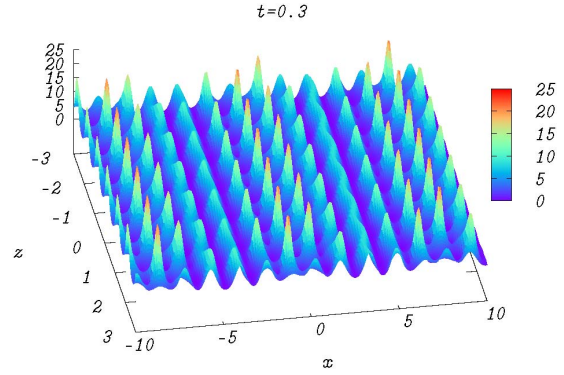where $\widetilde{p}_j = p_j + p_{j+1} - p_{j+2}$, $p_{j+3} \equiv p_j$, $h_j = \exp(-4\mathfrak{b}_j)$.

Figure 6. Three-phase solution of KP-I equation for $\lambda_0 = 0$, $t = 0$.

Figure 7. Three-phase solution of KP-I equation for $\lambda_0 = 0$, $t = 0.1$.

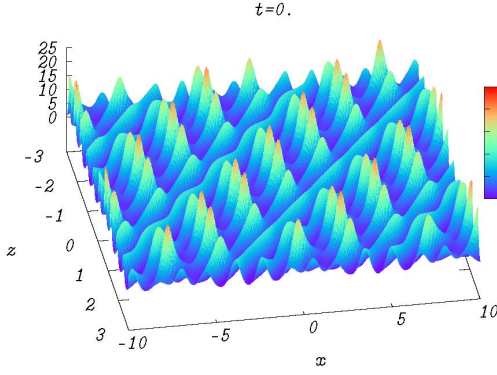**Figure 8.** Three-phase solution of KP-I equation for $\lambda_0 = 0$, $t = 0.2$.

Figure 9. Three-phase solution of KP-I equation for $\lambda_0 = 0$, $t = 0.3$.

The functions $\vartheta_j(p|h)$ are Jacoby elliptic theta functions [1]:

$$\vartheta_1(p|h) = 2 \sum_{m=1}^{\infty} (-1)^{m-1} h^{(m-1/2)^2} \sin[(2m-1)\pi p],$$

$$\vartheta_2(p|h) = 2 \sum_{m=1}^{\infty} h^{(m-1/2)^2} \cos[(2m-1)\pi p],$$

$$\vartheta_3(p|h) = 1 + 2 \sum_{m=1}^{\infty} h^{m^2} \cos(2m\pi p),$$

$$\vartheta_4(p|h) = 1 + 2 \sum_{m=1}^{\infty} (-1)^m h^{m^2} \cos(2m\pi p).$$

Using the reduced form of theta function (17) and values for the vectors of periods, one obtains the following formula for a squared absolute value of the three-phase solution (12) of the focusing NLS equation (1)
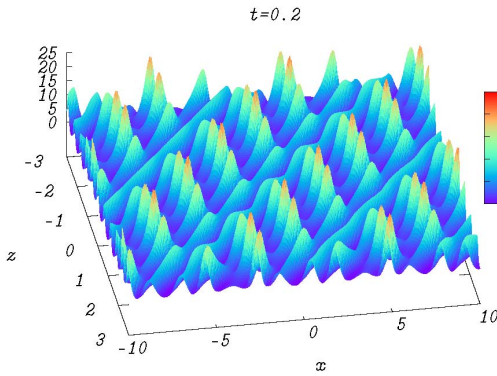
$$|\psi|^2 = -4K_0^2 f(k_1 x + \kappa_1 t + \delta_1, k_2 z + \delta_2, k_3 x + \kappa_3 t + \delta_3) \tag{18}$$
$$\times f(k_1 x + \kappa_1 t - \delta_1, k_2 z - \delta_2, k_3 x + \kappa_3 t - \delta_3) \times \{f(k_1 x + \kappa_1 t, k_2 z, k_3 x + \kappa_3 t)\}^{-2},$$
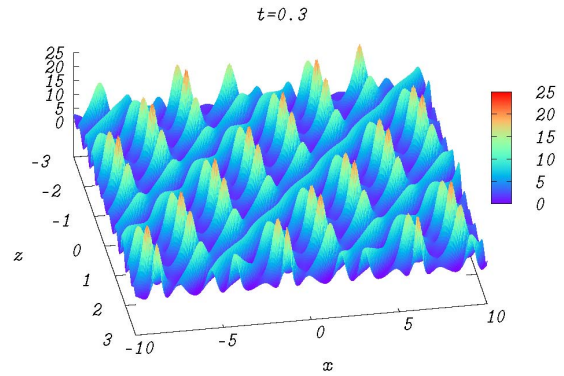
Figure 10. Three-phase solution of KP-I equation for $\lambda_0 = k_2/(4k_1)$, $t = 0$.
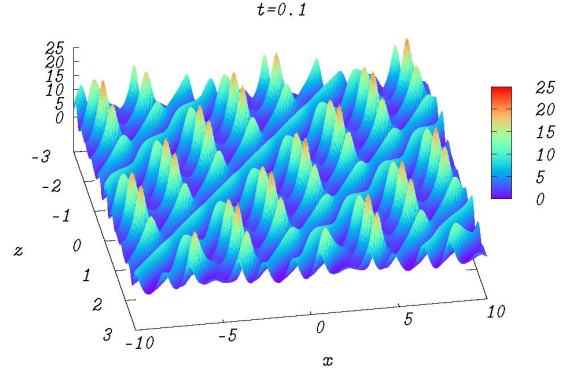
Figure 11. Three-phase solution of KP-I equation for $\lambda_0 = k_2/(4k_1)$, $t = 0.1$.

Figure 12. Three-phase solution of KP-I equation for $\lambda_0 = k_2/(4k_1)$, $t = 0.2$.

Figure 13. Three-phase solution of KP-I equation for $\lambda_0 = k_2/(4k_1)$, $t = 0.3$.

where the function $f(\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3)$ is defined by equation (17), and

$$k_1 = -4i\mathfrak{c}_1, \qquad k_2 = -8i\mathfrak{c}_2, \qquad k_3 = -4i\mathfrak{c}_3,$$
$$\kappa_1 = 4k_1\big(3\lambda_0^2 - ab + \big(a^2 + b^2\big)\cos(2\varphi)\big), \qquad \kappa_3 = 4k_3\big(3\lambda_0^2 + ab + \big(a^2 + b^2\big)\cos(2\varphi)\big).$$

(17), (18) imply that for $\lambda_0 = 0$ the amplitude of the constructed solution of NLS equation (1) is a periodic function of z, and for $\lambda_0 = 0$, $\varphi = \frac{1}{2}\arccos\big(\frac{\pm ab}{a^2+b^2}\big)$ it is a periodic function of z and t.

Recall that the three-phase solution $u(x, z, t)$ of the KP-I equation (8) and the square of amplitude of three-phase solution of Hirota equation (7), $|\psi_{\mathrm{H}}(x, t)|^2$, can be constructed from (18) by using relations $u(x, z, t) = 2|\psi(x, z, t)|^2$ and $|\psi_{\mathrm{H}}(x, t)|^2 = |\psi(x, t, -\alpha t)|^2$.

The three-phase solution of KP-I equation for $ab = 1$, $\sqrt{b/a} = 1.3$, $\varphi = 0.3\pi$ at the different moment of time t and for $\lambda_0 = 0$ is presented on the Figs. 6–9. One can see the same solution for $\lambda_0 = k_2/(4k_1)$ on the Figs. 10–13. It is easy to see all three phases of solution on Figs. 6–13. Two phases are shortwaves and the third phase is a long-wave envelope. One can see also on Figs. 6–9 that the solution for $\lambda_0 = 0$ is periodic in z, and that the long-wave envelope moves to the right side.
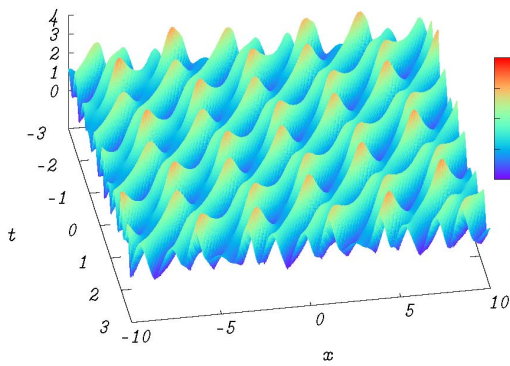
The three-phase solution of Hirota equation for $ab = 1$, $\sqrt{b/a} = 1.3$, $\varphi = 0.3\pi$, $\alpha = 0.1$ and for different values of λ_0 is presented in Figs. 14–17. It is easy to see all three phases of solution only in Fig. 15.
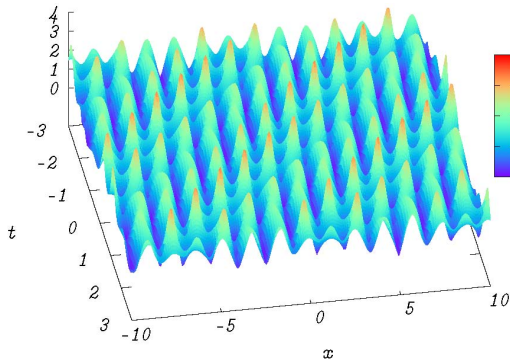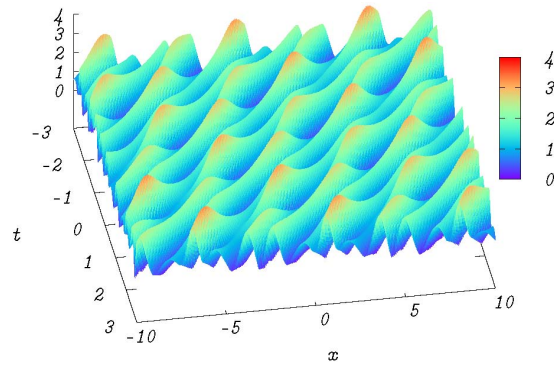
Figure 14. Amplitude of three-phase solution of Hirota equation for $\lambda_0 = 0$.
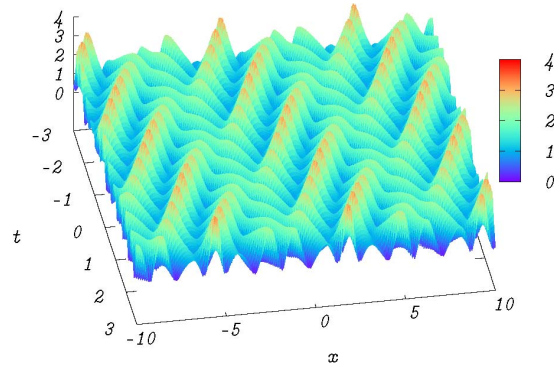
Figure 15. Amplitude of three-phase solution of Hirota equation for $\lambda_0 = 4$.

Figure 16. Amplitude of three-phase solution of Hirota equation for $\lambda_0 = k_2/(4k_1)$.

Figure 17. Amplitude of three-phase solution of Hirota equation for $\lambda_0 = k_2/(4k_3)$.

Acknowledgements

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