On the Killing form of Lie Algebras
in Symmetric Ribbon Categories[⋆]

Igor BUCHBERGER and Jürgen FUCHS

Teoretisk fysik, Karlstads Universitet, Universitetsgatan 21, S–65188 Karlstad, Sweden
E-mail: *igor.buchberger@kau.se, juerfuch@kau.se*
URL: http://www.ingvet.kau.se/juerfuch/

Abstract. As a step towards the structure theory of Lie algebras in symmetric monoidal categories we establish results involving the Killing form. The proper categorical setting for discussing these issues are symmetric ribbon categories.

Key words: Lie algebra; monoidal category; ribbon category; Killing form; Lie superalgebra

2010 Mathematics Subject Classification: 17Bxx; 18D35; 18D10; 18E05

1   Introduction

At times, a mathematical notion reveals its full nature only after viewing it in a more general context than the one in which it had originally appeared, and often this is achieved by formulating it in the appropriate categorical framework. To mention a few examples, the octonions are really an instance of an *associative* commutative algebra, namely once they are regarded as an object in the category of $\mathbb{Z}_2^{\times 3}$-graded vector spaces with suitably twisted associator [1, 2]; chiral algebras in the sense of Beilinson and Drinfeld can be viewed as Lie algebras in some category of $\mathcal{D}$-modules [48]; and vertex algebras (which are Beilinson–Drinfeld chiral algebras on the formal disk) are singular commutative associative algebras in a certain functor category [5]. As another illustration, not only do Hopf algebras furnish a vast generalization of group algebras, but indeed a group *is* a Hopf algebra, namely a Hopf algebra in the category of sets.

In this note we study aspects of Lie algebras from a categorical viewpoint. This already has a long tradition, see e.g. [3, 9, 15, 22, 23]. For instance, it is well known that Lie superalgebras and, more generally, Lie color algebras, are ordinary Lie algebras in suitable categories, see Example 2.3 below. Our focus here is on the structure theory of Lie algebras in symmetric monoidal categories which, to the best of our knowledge, has not been investigated in purely categorical terms before. We restrict our attention to a few particular issues, the main goal being to identify for each of them an adequate categorical setting that both allows one to develop the respective aspect closely parallel with the classical situation and at the same time covers an as broad as possible class of cases. In particular we do not assume the underlying category to be semisimple, or Abelian, or linear (i.e., enriched over vector spaces), or that the tensor product functor is exact, albeit large classes of examples that are prominent in the literature share (some of) these properties.

On the other hand we insist on imposing relevant conditions directly on the underlying category. They are thereby not necessarily the weakest possible, and in fact they could often be considerably weakened to conditions on subclasses of objects. For instance, even an object in a non-braided category can be endowed with a Lie algebra structure, provided that it comes with

---

a self-invertible Yang–Baxter operator in the sense of [21, Definition 2.6] that takes over the role of the self-braiding. But this way one potentially loses the clear distinction between categorical and non-categorical aspects. Indeed, a benefit of the abstraction inherent in the categorical point of view is that it allows one to neatly separate features that apply only to a subclass of examples from those which are essential for the concepts and results in question and are thereby generic. We are specifically interested in the proper notion of Lie subalgebra and ideal, and in the relevance of the Killing form. Recall that non-degeneracy of the Killing form furnishes a criterion for semisimplicity of a Lie algebra over the complex numbers, but ceases to do so for e.g. Lie algebras over a field of non-zero characteristic or for Lie superalgebras.

The rest of this note is organized as follows. Section 2 is devoted to basic aspects of Lie algebras in monoidal categories. In Section 2.1 we present the definition of a Lie algebra in a symmetric additive category and illustrate it by examples. We then introduce the notions of nilpotent and solvable Lie algebras in Section 2.2. In Section 2.3 we briefly describe subalgebras and ideals of Lie algebras, paying attention to the differences that result from the specification of the relevant class of monics. In Section 3 we study aspects of the Killing form. To this end we first review in Section 3.1 some properties of the partial trace. In Section 3.2 we give the definition of the Killing form of a Lie algebra in a symmetric ribbon category and show that it is symmetric, while invariance is established in Section 3.3. Finally in Section 3.4 we explore the role of non-degeneracy of the Killing form.

2   Lie algebras

We freely use pertinent concepts from category theory, and in particular for algebras in monoidal categories. Some of these are recalled in the Appendix. The product of a Lie algebra is anti-symmetric and satisfies the Jacobi identity. This requires that morphisms can be added and that there is a notion of exchanging the factors in a tensor product. Moreover, it turns out that one can proceed in full analogy to the classical case only if the latter is a *symmetric braiding*. Thus we make the

Convention 2.1. *In the sequel, unless stated otherwise, by a category we mean an additive symmetric monoidal category, with symmetric braiding c.*

To our knowledge, Lie algebras in this setting were first considered in [22, 23]. To present the definition, we introduce the short-hand notation

$$c_U^{(n)} := c_{U^{\otimes(n-1)},U} \in \mathrm{End}_{\mathcal{C}}\big(U^{\otimes n}\big) \tag{2.1}$$

for multiple self-braidings, as well as $h_{\lfloor 2 \rfloor} := h$ and

$$h_{\lfloor n \rfloor} := h \circ (\mathrm{id}_U \otimes h_{\lfloor n-1 \rfloor}) \in \mathrm{Hom}_{\mathcal{C}}\big(U^{\otimes n}, U\big) \qquad \text{for} \quad n > 2 \tag{2.2}$$

for iterations of a morphism $h \in \mathrm{Hom}_{\mathcal{C}}(U \otimes U, U)$.

2.1   Lie algebras in additive symmetric monoidal categories

With the notations (2.1) and (2.2) we have

Definition 2.2. A *Lie algebra* in a category $\mathcal{C}$ is a pair (L, ℓ) consisting of an object $L \in \mathcal{C}$ and a morphism $\ell \in \mathrm{Hom}_{\mathcal{C}}(L \otimes L, L)$ that satisfies

$$\ell_{\lfloor 2 \rfloor} \circ \big(\mathrm{id}_L^{\otimes 2} + c_L^{(2)} \big) = 0 \tag{2.3}$$

(antisymmetry) and

$$\ell_{\lfloor 3 \rfloor} \circ \left[\mathrm{id}_L^{\otimes 3} + c_L^{(3)} + \left(c_L^{(3)} \right)^2 \right] = 0 \tag{2.4}$$

(Jacobi identity).

Here and below we assume monoidal categories to be strict. The additional occurrences of the associativity constraint in the non-strict case are easily restored (the explicit expression can e.g. be found in [21, Definition 2.1]). By abuse of terminology, also the object L is called a Lie algebra; the morphism ℓ is referred to as the *Lie bracket* of L.

Example 2.3.

(i) For $\Bbbk$ a field and $\mathcal{C} = \mathcal{V}ect_{\Bbbk}$ the category of $\Bbbk$-vector spaces, with the symmetric braiding given by the flip $v \otimes v' \mapsto v' \otimes v$, we recover ordinary Lie algebras.

(ii) A Lie superalgebra over $\Bbbk$ [17, 30] is a Lie algebra in the category $\mathcal{SV}ect_{\Bbbk}$ of $\mathbb{Z}_2$-graded $\Bbbk$-vector spaces with the braiding given by the superflip, acting as $v \otimes v' \mapsto (-1)^{\deg(v)\deg(v')}(v' \otimes v)$ on homogeneous elements (and extended by bilinearity).

(iii) A Lie color algebra (or color Lie algebra) [4, 43, 45] is a Lie algebra in the category $\Gamma\text{-}\mathcal{V}ect$ of Γ-graded vector spaces, where Γ is a finite Abelian group endowed with a skew bicharacter φ. The braiding acts on homogeneous elements analogously as for superalgebras, i.e. as a flip multiplied by a phase factor $\varphi(\deg(v), \deg(v'))$.

For $\Gamma = \mathbb{Z}_2$ with the unique cohomologically nontrivial skew bicharacter this yields Lie superalgebras, while for φ cohomologically trivial it yields ordinary Lie algebras for any Γ.

(iv) A so-called Hom-Lie algebra L in a symmetric monoidal category $\mathcal{C}$, for which the Jacobi identity is deformed with the help of an automorphism of L (first introduced in [26] for the case $\mathcal{C} = \mathcal{V}ect_{\Bbbk}$) is a Lie algebra in a category $\mathcal{HC}$ whose objects are pairs consisting of an object U of $\mathcal{C}$ and an automorphism of U [6].

Remark 2.4.

(i) When trying to directly generalize the notion of a Lie algebra to monoidal categories with a generic braiding, two independent Jacobi identities must be considered [53].

(ii) A more conceptual approach to the case of generic braiding has led to the notion of a quantum Lie bracket on an object L, satisfying a generalized Jacobi identity, which is related to the adjoint action for Hopf algebras in $\mathcal{C}$ that are compatible with an additional coalgebra structure on L [38]. In another approach one deals with a whole family of n-ary products on an object of $\Gamma\text{-}\mathcal{V}ect$, with Γ a finite Abelian group endowed with a (not necessarily skew) bicharacter ψ [41]. (If ψ is skew, this reduces to Lie color algebras.) This approach has the advantage of keeping important aspects of the classical case, e.g. the primitive elements of a Hopf algebra in $\Gamma\text{-}\mathcal{V}ect$ and the derivations of an associative algebra in $\Gamma\text{-}\mathcal{V}ect$ carry such a generalized Lie algebra structure. There is also a further generalization [42] to the case that $\mathcal{C}$ is the category of Yetter–Drinfeld modules over a Hopf algebra with bijective antipode.

Remark 2.5. Given an associative algebra (A, m) in $\mathcal{C}$, the morphism

$$\ell_m := m - m \circ c_{A,A} \tag{2.5}$$

clearly satisfies the defining relations (2.3) and (2.4) of a Lie bracket. Thus the pair (A, ℓ_m) is a Lie algebra in $\mathcal{C}$; we refer to such a Lie algebra as a *commutator Lie algebra* and to its Lie bracket ℓ_m as the commutator of the associative multiplication m.

Remark 2.6. If the category $\mathcal{C}$ is in addition Abelian, then one can define the *universal enveloping algebra* of a Lie algebra $L \in \mathcal{C}$ as the quotient of the tensor algebra $T(L)$, as described in Example A.3(i), by the two-sided ideal $I(L)$ generated by the image of the morphism $\ell - \mathrm{id}_{L \otimes L} + c_{L,L}$ (regarded as an endomorphism of $T(L)$). In many cases of interest, such as for Lie color algebras [23, 33] or if $\mathcal{C}$ is the category of comodules over a coquasitriangular bialgebra in $\mathcal{V}\mathrm{ect}$ [15], the ideal $I(L)$ inherits a Hopf algebra structure from $T(L)$, so that the quotient is naturally a Hopf algebra as well. This still applies [42, Section 6] in the case of the Lie algebras in categories of Yetter–Drinfeld modules mentioned in Remark 2.4(ii).

Remark 2.7. The class of all Lie algebras in $\mathcal{C}$ forms the objects of a category, with a morphism f from a Lie algebra (L, ℓ) to a Lie algebra (L', ℓ') being a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(L, L')$ satisfying

$$\ell' \circ (f \otimes f) = f \circ \ell. \tag{2.6}$$

The category of associative algebras in $\mathcal{C}$ is defined analogously; supplementing the mapping from an associative algebra to its commutator Lie algebra by the identity on the sets of algebra morphisms defines a functor from the category of associative algebras in $\mathcal{C}$ to the one of Lie algebras in $\mathcal{C}$. Note that, as the condition (2.6) is nonlinear, these categories are not additive.

2.2 Nilpotent and solvable Lie algebras

Basic aspects of the structure theory of Lie algebras can be developed in the same way as in the classical case. We start with

Definition 2.8. An *Abelian* Lie algebra is a Lie algebra with vanishing Lie product, $\ell = 0$.

Remark 2.9.

(i) Trivially, $(U, \ell = 0)$ is an Abelian Lie algebra for any object $U \in \mathcal{C}$. A generic object cannot be endowed with any other Lie algebra structure.

(ii) The commutator Lie algebra (A, ℓ_m) built from a commutative associative algebra (A, m) is Abelian.

Next we introduce the concepts of nilpotent and solvable Lie algebras. Recall that according to the definition (2.2), for a Lie algebra (L, ℓ) the $(n-1)$-fold iteration of the Lie bracket is denoted by $\ell_{\lfloor n \rfloor} \in \mathrm{Hom}_{\mathcal{C}}(L^{\otimes n}, L)$. We now introduce in addition morphisms $\ell^{\lceil n \rceil} \in \mathrm{Hom}_{\mathcal{C}}(L^{\otimes 2^{(n-1)}}, L)$ by

$$\ell^{\lceil 2 \rceil} := \ell \qquad \text{and} \qquad \ell^{\lceil n \rceil} := \ell \circ \left(\ell^{\lceil n-1 \rceil} \otimes \ell^{\lceil n-1 \rceil} \right) \qquad \text{for } n > 2,$$

respectively. Then we can give

Definition 2.10. Let $L = (L, \ell)$ be a Lie algebra in $\mathcal{C}$.

(i) L is called *nilpotent* iff $\ell_{\lfloor n \rfloor} = 0$ for sufficiently large n.

(ii) L is called *solvable* iff $\ell^{\lceil n \rceil} = 0$ for sufficiently large n.

(iii) L is called *derived nilpotent* iff $\ell_{\lfloor n \rfloor} \circ \ell^{\otimes n} = 0$ for sufficiently large n.

It follows directly from the definitions that a nilpotent Lie algebra is solvable. Indeed,

$$\ell^{\lceil n \rceil} = \ell_{\lfloor n \rfloor} \circ \left(\ell^{\lceil n-1 \rceil} \otimes \ell^{\lceil n-2 \rceil} \otimes \cdots \otimes \ell \otimes \mathrm{id}_L \otimes \mathrm{id}_L \right).$$

But we even have

Proposition 2.11. *A derived nilpotent Lie algebra is solvable.*

Proof. We have directly $\ell^{\lceil 3 \rceil} = \ell_{\lfloor 2 \rfloor} \circ \ell^{\otimes 2}$, while for $n > 3$ one shows by induction that

$$\ell^{\lceil n+1 \rceil} = \ell_{\lfloor n \rfloor} \circ \ell^{\otimes n} \circ \left(\ell^{\lceil n-1 \rceil} \otimes \ell^{\lceil n-1 \rceil} \otimes \ell^{\lceil n-2 \rceil} \otimes \ell^{\lceil n-2 \rceil} \otimes \cdots \otimes \ell \otimes \ell \otimes \operatorname{id}_L^{\otimes 4} \right).$$

Thus $\ell_{\lfloor n \rfloor} \circ \ell^{\otimes n} = 0$ implies $\ell^{\lceil n+1 \rceil} = 0$, so that the claim follows directly from the definitions of derived nilpotency and of solvability. ∎

2.3   Subalgebras and ideals

A subalgebra of an algebra A is an isomorphism class of monics to A that are algebra morphisms. For spelling out this notion in detail, one must specify the class of monics one is considering. We present the cases that either all monics are admitted, corresponding to working with subobjects, or only split monics are admitted, corresponding to retracts. In the latter case we give

Definition 2.12.

(i) A *retract subalgebra*, or *subalgebra retract*, of an algebra (A, m) in $\mathcal{C}$ is a retract (B, e_B^A, r_A^B) of A for which

$$m \circ \left(e_B^A \otimes e_B^A \right) = p_A^B \circ m \circ \left(e_B^A \otimes e_B^A \right), \tag{2.7}$$

where $p_A^B = e_B^A \circ r_A^B$ is the idempotent associated with the retract.

(ii) A *retract Lie subalgebra*, or *Lie subalgebra retract*, of a Lie algebra (L, ℓ) in $\mathcal{C}$ is a retract (K, e_K^L, r_L^K) of L for which

$$\ell \circ \left(e_K^L \otimes e_K^L \right) = p_L^K \circ \ell \circ \left(e_K^L \otimes e_K^L \right). \tag{2.8}$$

Note that part (ii) of Definition 2.12 is redundant – it is merely a special case of part (i), as we do not specify any properties of the product m in (i). For clarity we present the Lie algebra version nevertheless separately; it is worth recalling that in the Lie algebra case the Convention 2.1 is in effect. Also note that owing to $p \circ e = e$, (2.7) is equivalent to

$$m \circ \left(p_A^B \otimes p_A^B \right) = p_A^B \circ m \circ \left(p_A^B \otimes p_A^B \right),$$

and analogously for (2.8), as well as for (2.9) below.

Definition 2.13. A *retract ideal*, or *ideal retract*, of a Lie algebra (L, ℓ) in $\mathcal{C}$ is a retract (K, e_K^L, r_L^K) of L for which

$$\ell \circ \left(e_K^L \otimes \operatorname{id}_L \right) = p_L^K \circ \ell \circ \left(e_K^L \otimes \operatorname{id}_L \right). \tag{2.9}$$

We call a Lie algebra L indecomposable iff it is non-Abelian and its only retract ideals are the zero object and L itself. Put differently, we have

Definition 2.14. An *indecomposable* Lie algebra is a Lie algebra which is not Abelian and which does not possess any non-trivial retract ideal.

By antisymmetry of ℓ, the equality (2.9) is equivalent to $\ell \circ (\operatorname{id}_L \otimes e_K^L) = p_L^K \circ \ell \circ (\operatorname{id}_L \otimes e_K^L)$. For an associative algebra, the equality analogous to (2.9) instead defines a *left* ideal, while exchanging the roles of the two morphisms id and e gives the notion of right ideal.

If (K, e, r) is a non-trivial retract of a Lie algebra (L, ℓ), then together with $p = r \circ e$ also $p' := \operatorname{id}_L - p$ is a non-zero idempotent, and hence if $\mathcal{C}$ is idempotent complete, then there is a non-trivial retract (K', e', r') of L such that $p' = r' \circ e'$ and $L \cong K \oplus K'$ as an object in $\mathcal{C}$. In the present context a particularly interesting case is that both K and K' are retract ideals of L. To account for this situation we give

Definition 2.15. Given a finite family of Lie algebras (K_i, ℓ_i), the *direct sum* Lie algebra of the family is the Lie algebra (L, ℓ) whose underlying object L is the direct sum $L := \bigoplus_i K_i$ of the objects underlying the Lie algebras (K_i, ℓ_i), and whose Lie bracket is given by

$$\ell := \sum_i e^L_{K_i} \circ \ell_i \circ \left(r^{K_i}_L \otimes r^{K_i}_L \right). \tag{2.10}$$

Note that since $r^{K_i}_L \circ e^L_{K_j} = \delta_{i,j} \, \mathrm{id}_{K_j}$, for a direct sum Lie algebra we have $\ell \circ (p^{K_i}_L \otimes p^{K_j}_L) = 0$ for $i \neq j$. Moreover, it follows that $\ell \circ (p^{K_i}_L \otimes \mathrm{id}_L) = p^{K_i}_L \circ \ell \circ (p^{K_i}_L \otimes \mathrm{id}_L)$ for every i, i.e. the Lie algebras (K_i, ℓ_i) are retract ideals of the direct sum Lie algebra L. Conversely, one has

Lemma 2.16. *If a Lie algebra (L, ℓ) is, as an object, the direct sum $L = \bigoplus_i K_i$, and each of the retracts K_i is a retract ideal of L, then (L, ℓ) is the direct sum of the K_i also as a Lie algebra.*

Proof. Because of $\mathrm{id}_L = \sum_i p^L_{K_i}$ the Lie bracket ℓ can be written as the triple sum

$$\ell = \sum_{i,j,k} p^{K_k}_L \circ \ell \circ \left(p^{K_i}_L \otimes p^{K_j}_L \right). \tag{2.11}$$

Now since K_i and K_j are ideals, one has $p^{K_k}_L \circ \ell \circ (p^{K_i}_L \otimes p^{K_j}_L) = \delta_{i,k} \delta_{j,k} p^{K_k}_L \circ \ell \circ (p^{K_i}_L \otimes p^{K_j}_L)$. As a consequence the expression (2.11) reduces to

$$\ell = \sum_i e^L_{K_i} \circ \ell_i \circ \left(r^{K_i}_L \otimes r^{K_i}_L \right) \qquad \text{with} \qquad \ell_i := r^{K_i}_L \circ \ell \circ \left(e^L_{K_i} \otimes e^L_{K_i} \right).$$

Hence there are brackets ℓ_i satisfying (2.10). Moreover, invoking the uniqueness of direct sum decomposition in the additive category $\mathcal{C}$, it follows (notwithstanding the non-additivity of the category of Lie algebras in $\mathcal{C}$) that these brackets coincide with the assumed Lie brackets on the retract ideals K_i up to isomorphism of Lie algebras. ∎

Next, consider a retract ideal (K, e, r) of a Lie algebra L in an idempotent-complete category. We call K a *primitive* retract ideal iff the idempotent $p = e \circ r$ cannot be written as a sum $p = p_1 + p_2$ of two non-zero idempotents such that at least one of the corresponding retracts (K_1, e_1, r_1) and (K_2, e_2, r_2) is a retract ideal of L. Now if the retracts associated with two idempotents in $\mathrm{End}_{\mathcal{C}}(L)$, say p and p', are retract ideals, then so is the retract associated with the idempotent $p \circ p' \in \mathrm{End}_{\mathcal{C}}(L)$. If p is in addition primitive, then either $p \circ p' = p$ or $p \circ p' = 0$. A primitive retract ideal is hence an indecomposable Lie algebra.

It is worth pointing out that in the Definitions 2.12 and 2.13 there is no need to assume the existence of an image of the Lie bracket ℓ. In contrast, if we spell out the corresponding definitions for the case of subobjects, we do have to make such an assumption:

Definition 2.17. Let $(L, \ell) \in \mathcal{C}$ be a Lie algebra for which the image $\mathrm{Im}(\ell)$ exists in $\mathcal{C}$.

(i) A *subobject Lie subalgebra*, or *Lie subalgebra subobject*, of L is a subobject (K, e^L_K) of L for which the image $\mathrm{Im}\left(\ell \circ (e^L_K \otimes e^L_K) \right)$ is a subobject of K.

(ii) A *subobject ideal*, or *ideal subobject*, of L is a subobject (K, e^L_K) of L for which the image $\mathrm{Im}\left(\ell \circ (e^L_K \otimes \mathrm{id}_L) \right)$ is a subobject of K.

Accordingly, Definition 2.14 then gets replaced by

Definition 2.18. A *simple* Lie algebra is a Lie algebra which is not Abelian and which does not possess any non-trivial subobject ideal.

Example 2.19.

(i) Given a unital associative algebra (A, m, η) in $\mathcal{C}$, the commutator ℓ_m (2.5) satisfies $\ell_m \circ (\eta \otimes \mathrm{id}_A) \equiv m \circ (\eta \otimes \mathrm{id}_A) - m \circ c_{A,A} \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A - \mathrm{id}_A = 0$. Thus, provided that $(\mathbf{1}, \eta)$ is a subobject of A – meaning that the unit morphism η is monic, which is true under rather weak conditions [37], e.g. if $\mathcal{C}$ is $\Bbbk$-linear with $\Bbbk$ a field [11, Section 7.7] – $(\mathbf{1}, \eta)$ is an Abelian subobject ideal of the commutator Lie algebra (A, ℓ_m).

(ii) If there exists in addition a morphism $\epsilon \in \mathrm{Hom}_{\mathcal{C}}(A, \mathbf{1})$ such that $\xi := \epsilon \circ \eta \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is invertible, then $(\mathbf{1}, \eta, \tilde{\epsilon})$ with $\tilde{\epsilon} := \xi^{-1} \circ \eta$ is a retract of A, and thus even an Abelian retract ideal of the Lie algebra (A, ℓ_m), and we have a direct sum decomposition

$$A = \mathbf{1} \oplus A' \tag{2.12}$$

of Lie algebras.

Example 2.20.

(i) If A carries the structure of a Frobenius algebra in $\mathcal{C}$, then its counit ε is a natural candidate for the morphism ϵ that is assumed in Example 2.19(ii). If, moreover, the Frobenius algebra A is strongly separable in the sense of Definition A.1(ii) as well as symmetric, then $\varepsilon \circ \eta = \dim_{\mathcal{C}}(A)$ equals the dimension of A, i.e. $(\mathbf{1}, \eta, \tilde{\varepsilon})$ being a retract ideal of the commutator Lie algebra (A, ℓ_m) is equivalent to A having invertible dimension.

(ii) As a special case, consider the tensor product $A_U := U \otimes U^\vee$ of an object U with its right dual in a (strictly) sovereign category $\mathcal{C}$. The object A_U carries a canonical structure of a symmetric Frobenius algebra $(A_U, m_U, \eta_U, \Delta_U, \varepsilon_U)$, for which all structural morphisms are simple combinations of the left and right evaluation and coevaluation morphisms for U, namely

$$m_U = \mathrm{id}_U \otimes d_U \otimes \mathrm{id}_{U^\vee}, \qquad \eta_U = b_U \tag{2.13}$$

and similarly for the coproduct Δ_U and counit ε_U. We refer to A_U with this algebra and coalgebra structure as a (full) *matrix algebra*. Any matrix algebra is Morita equivalent to the trivial Frobenius algebra $\mathbf{1}$, and we have $\varepsilon_U \circ \eta_U = \dim_{\mathcal{C}}(U)$, so $\mathbf{1}$ is a retract ideal of the commutator Lie algebra (A_U, ℓ_{m_U}) iff U has invertible dimension.

(iii) A simple realization of the situation described in (ii) is obtained by taking $\mathcal{C}$ to be the category of finite-dimensional complex representations of a classical finite-dimensional simple Lie algebra $\mathfrak{L}$ in $\mathcal{V}\mathrm{ect}_{\mathbb{C}}$ (or in $\mathcal{V}\mathrm{ect}_{\Bbbk}$, with $\Bbbk$ any algebraically closed field of characteristic zero) and $U = D \in \mathcal{C}$ to be the defining representation of $\mathfrak{L}$. The retract A'_D in the resulting direct sum decomposition $A_D = \mathbf{1} \oplus A'_D$ is then a simple Lie algebra iff $\mathfrak{L} = \mathfrak{sl}_n(\mathbb{C})$, in which case $A'_D = \mathrm{Ad}$ is (isomorphic to) the adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$ and $A_D = \mathbf{1} \oplus \mathrm{Ad}$ is just the familiar decomposition $\mathfrak{gl}_n(\mathbb{C}) \cong \mathbb{C} \oplus \mathfrak{sl}_n(\mathbb{C})$.

(iv) Another situation in which the retract A'_U is a simple Lie algebra is the case that $\mathcal{C}$ is the category of finite-dimensional complex representations of either the Mathieu group $\mathbb{M} = \mathbb{M}_{23}$ or of $\mathbb{M} = \mathbb{M}_{24}$, and U is the 45-dimensional irreducible representation of $\mathbb{M}$; A'_D is then the 2024-dimensional irreducible $\mathbb{M}$-representation. (The decomposition $A_D = \mathbf{1} \oplus A'_D$ is easily checked by hand from the character table of $\mathbb{M}$, which can e.g. be found in Tables V and VI of [27].)

Example 2.21. Let $B = (B, m, \eta, \Delta, \varepsilon)$ be a bialgebra and $(B, \ell = m - m \circ c)$ its commutator Lie algebra. Consider a subobject (P, e) of B satisfying

$$\Delta \circ e = e \otimes \eta + \eta \otimes e. \tag{2.14}$$

By the bialgebra axiom it follows directly that $\Delta \circ m \circ (e \otimes e) = (m \circ (e \otimes e)) \otimes \eta + 2e \otimes e + \eta \otimes (m \circ (e \otimes e))$, and similarly for $\Delta \circ m \circ c \circ (e \otimes e)$. Hence the morphism $\ell_P := \ell \circ (e \otimes e)$ obeys $\Delta \circ \ell_P = \ell_P \otimes \eta + \eta \otimes \ell_P$. In particular, if (P, e) is maximal with the property (2.14), then it is a subobject Lie subalgebra of (B, ℓ).

Remark 2.22. In applications, the relevance of a Lie algebra arises often through its linear representations. In the present note we do not dwell on representation theoretic issues, except for the following simple observation. For $A_U = U \otimes U^\vee$ the associative algebra with product m_U as in (2.13), the object U is naturally an A_U-module, with representation morphism $\rho := \mathrm{id}_U \otimes d_U \in \mathrm{Hom}_{\mathcal{C}}(A_U \otimes U, U)$. The representation property is verified as follows:

$$\rho_U \circ (\mathrm{id}_{A_U} \otimes \rho_U) \equiv (\mathrm{id}_U \otimes d_U) \circ (\mathrm{id}_{A_U} \otimes \mathrm{id}_U \otimes d_U) = \mathrm{id}_U \otimes d_U \otimes d_U$$
$$= (\mathrm{id}_U \otimes d_U) \circ (\mathrm{id}_U \otimes d_U \otimes \mathrm{id}_{U^\vee} \otimes \mathrm{id}_U) \equiv \rho_U \circ (m_U \otimes \mathrm{id}_U).$$

Being a module over the associative algebra (A_U, m_U), U is also a module over the commutator Lie algebra (A_U, ℓ_{m_U}) and thus, in case U has invertible dimension, also over the Lie algebra A_U' appearing in the direct sum decomposition (2.12). It then follows in particular that if every object of $\mathcal{C}$ is a retract of a suitable tensor power of U, then each object of $\mathcal{C}$ carries a natural structure of an A_U'-module, whereby $\mathcal{C}$ becomes equivalent to a subcategory of the category A_U'-mod$_{\mathcal{C}}$ of A_U'-modules in $\mathcal{C}$. In the special case of $U = D$ being the defining representation of $\mathfrak{sl}_n(\mathbb{C})$, $\mathcal{C}$ is in fact equivalent to A_U'-mod$_{\mathcal{C}}$ as a monoidal category.

3 The Killing form

3.1 Partial traces

The Killing form of a Lie algebra in $\mathcal{V}\mathrm{ect}_{\Bbbk}$ is defined as a trace, actually as a partial trace. Canonical traces, and thus partial traces, of morphisms exist in a category $\mathcal{C}$ iff $\mathcal{C}$ is a full monoidal subcategory of a ribbon category [29]. Accordingly from now on we restrict our attention to ribbon categories (but for now do not yet impose our Convention 2.1).

One may think of the basic data of a (strictly monoidal) ribbon category to be the right and left dualities, b, d and $\tilde{b}$, $\tilde{d}$, the braiding c, and the sovereign structure σ, rather than the twist θ. Then instead of expressing the sovereign structure with the help of the twist as in (A.2), conversely the twist is expressed as

$$\theta_U = (\mathrm{id}_U \otimes \tilde{d}_U) \circ (c_{U,U} \otimes \sigma_U) \circ (\mathrm{id}_U \otimes b_U). \tag{3.1}$$

It will prove to be convenient to exploit the graphical calculus for (strict) monoidal categories and for algebras therein that has been developed repeatedly in different guises, see e.g. [7, 28, 32, 38, 47, 51] or Appendix A of [16]. For instance, in this graphical description the defining property (A.1) of the sovereign structure takes the form

while the expression (3.1) for the twist becomes

$$\theta_U = \quad \sigma_U \quad .$$

The (left and right) *trace* of an endomorphism $f \in \mathrm{End}_{\mathcal{C}}(V)$ in a ribbon category is given by $\mathrm{tr}(f) = \tilde{d}_V \circ (f \otimes \sigma_V) \circ b_V = d_V \circ (\sigma_V^{-1} \otimes f) \circ \tilde{b}_V \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$; pictorially,

$$\mathrm{tr}(f) \;=\; \boxed{f} \;\; \sigma_V \;=\; \sigma_V^{-1} \; \boxed{f} \quad .$$

For defining the Killing form we also need a generalization, the notion of a *partial trace*: for a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(U_1 \otimes \cdots \otimes U_n \otimes V, V)$ we introduce the partial trace with respect to V by

$$\mathrm{tr}_{n+1}(f) := \tilde{d}_V \circ (f \otimes \sigma_V) \circ (\mathrm{id}_{U_1 \otimes \cdots \otimes U_n} \otimes b_V) \in \mathrm{Hom}_{\mathcal{C}}(U_1 \otimes \cdots \otimes U_n, \mathbf{1}).$$

For $n = 1$ this looks graphically as follows:

$$\mathrm{tr}_2(f) \;=\; \boxed{f} \;\; \sigma_V \quad \in \; \mathrm{Hom}_{\mathcal{C}}(U, \mathbf{1}) \tag{3.2}$$

for $f \in \mathrm{Hom}_{\mathcal{C}}(U \otimes V, V)$. A fundamental property of the partial trace (3.2) is the following:

Lemma 3.1. *For any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(U \otimes V, V)$ in a ribbon category one has*

$$\mathrm{tr}_2(f) \circ \theta_U = \mathrm{tr}_2(f). \tag{3.3}$$

Proof. This follows directly by invoking functoriality of the twist:

$$\mathrm{tr}_2(f) \circ \theta_U = \mathrm{tr}_2\big(\theta_V \circ f \circ \big(\mathrm{id}_U \otimes \theta_V^{-1}\big)\big) = \mathrm{tr}_2\big(f \circ \big[\mathrm{id}_U \otimes \big(\theta_V^{-1} \circ \theta_V\big)\big]\big) = \mathrm{tr}_2(f).$$

Alternatively one may 'drag the morphism f along the V-loop'; graphically:

$$\mathrm{tr}_2(f) \equiv \boxed{f} = \boxed{^\vee f} = \boxed{f^\vee} = \boxed{f} = \boxed{f} \tag{3.4}$$

Here the first and third equalities hold by definition of the left and right dual morphisms $^\vee f$ and $f^\vee$, respectively (and by monoidality of σ), the second equality is sovereignty, and the final equality follows by the functoriality of the braiding. ∎

By the same sequence of manipulations one obtains analogous identities for other partial traces. In particular we have the following generalization of the cyclicity property of the ordinary trace:

Lemma 3.2. *Partial traces are cyclic up to braidings and twists: For any pair of morphisms* $f \in \mathrm{Hom}_{\mathcal{C}}(U \otimes X, Y)$ *and* $g \in \mathrm{Hom}_{\mathcal{C}}(V \otimes Y, X)$ *in a ribbon category one has*

$$\mathrm{tr}_3\big(f \circ [\mathrm{id}_U \otimes g]\big) = \mathrm{tr}_3\big(g \circ [\mathrm{id}_V \otimes f] \circ [(c_{U,V} \circ (\theta_U \otimes \mathrm{id}_V)) \otimes \mathrm{id}_X]\big). \tag{3.5}$$

Remark 3.3. Instead of dragging, as in (3.4), the morphism f along the V-loop in clockwise direction, we could equally well drag it in counter-clockwise direction. In the situation considered in Lemma 3.2, this amounts to the identity

$$\mathrm{tr}_3\big(f \circ [\mathrm{id}_U \otimes g]\big) = \mathrm{tr}_3\big(g \circ [\mathrm{id}_V \otimes f] \circ \big[\big(c_{V,U}^{-1} \circ \big(\mathrm{id}_U \otimes \theta_V^{-1}\big)\big) \otimes \mathrm{id}_X\,\big]\big)$$

instead of (3.5). This is, however, equivalent to (3.5), as is seen by combining it with the result (3.3), as applied to the morphism $f \circ (\mathrm{id}_U \otimes g)$, and using that $\theta_{U \otimes V} = (\theta_U \otimes \theta_V) \circ c_{U,V} \circ c_{V,U}$.

Example 3.4. The category $\mathcal{SV}\mathrm{ect}$ of super vector spaces over a field $\Bbbk$ is $\Bbbk$-linear rigid monoidal and is endowed with a braiding given by the super-flip map. $\mathcal{SV}\mathrm{ect}$ is semisimple with, up to isomorphims, two simple objects, namely the tensor unit $\mathbf{1}$ which is the $\mathbf{1}$-dimensional vector space $\Bbbk$ in degree zero, and an object S given by $\Bbbk$ in degree one. There are two structures of ribbon category on this braided rigid monoidal category. For the first, the twist obeys (besides $\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$, which is true in any ribbon category) $\theta_S = -\mathrm{id}_S$, which after identifying $^{\vee}S = S^{\vee}$ results in a strictly sovereign structure and in $\dim(S) = 1$. For the second ribbon structure instead the twist is trivial, $\theta_S = \mathrm{id}_S$, while the sovereign structure is non-trivial, namely $\sigma_S = -\mathrm{id}_{S^{\vee}}$ (upon still identifying $^{\vee}S = S^{\vee}$), and the trace is a *supertrace*, in particular now $\dim(S) = -1$. This second ribbon structure arises from the conventions used in the bulk of the literature on associative and Lie superalgebras (see e.g. [30]), while in many applications of super vector spaces in supersymmetric quantum field theory the first ribbon structure is implicit.

3.2   The Killing form of a Lie algebra in a symmetric ribbon category

We now turn to morphisms involving Lie brackets. Accordingly in the sequel our Convention 2.1 will again be in effect. In particular, as the braiding of $\mathcal{C}$ is now symmetric, the twist of $\mathcal{C}$ squares to the identity, and $\theta \equiv \mathrm{id}$ is an allowed twist.

In a symmetric ribbon category, we give

Definition 3.5. The *Killing form* of a Lie algebra (L, ℓ) is the morphism $\kappa \in \mathrm{Hom}_{\mathcal{C}}(L \otimes L, \mathbf{1})$ given by the partial trace

$$\kappa := \mathrm{tr}_3(\theta_L \circ \ell_{\lfloor 3 \rfloor}).$$

Writing out the partial trace, this expression reads more explicitly

$$\kappa = \tilde{d}_L \circ \big([\theta_L \circ \ell_{\lfloor 3 \rfloor}] \otimes \sigma_L\big) \circ \big(\mathrm{id}_L^{\otimes 2} \otimes b_L\big).$$

In the sequel we will use the graphical notation

for the Lie bracket and the twist of L; then the Killing form is depicted as

$$\kappa \;=\; \vcenter{\hbox{[diagram]}} \quad.$$

The presence of the twist in Definition 3.5, which might not have been expected, is needed for the Killing form to play an analogous role in the structure theory as in the classical case of Lie algebras in $\mathcal{V}\mathrm{ect}_{\Bbbk}$. For comparison we will below also briefly comment on the morphism

$$\kappa_0 := \mathrm{tr}_3(\ell_{\lfloor 3 \rfloor}). \tag{3.6}$$

which of course coincides with κ if, as in the classical case, $\theta_L = \mathrm{id}_L$.

As illustrated by Example 3.4, there are important classes of ribbon categories for which the sovereign structure is not strict. Nevertheless, for convenience in the sequel we will suppress the sovereign structure or, in other words, take it to be strict. We are allowed to do so because it is entirely straightforward to restore the sovereign structure in all relevant expressions.

Example 3.6. For the matrix Lie algebra (A_U, ℓ_{m_U}), introduced in Example 2.20(ii) as the commutator Lie algebra of the matrix algebra $A_U = U \otimes U^\vee$, for any $U \in \mathcal{C}$, the Killing form evaluates to

$$\kappa_{A_U} = 2 \dim_\theta(U)\big(\tilde{d}_U \circ [\mathrm{id}_U \otimes d_U \otimes \theta_{U^\vee}]\big) - 2\big(\tilde{d}_U \circ [\mathrm{id}_U \otimes \theta_{U^\vee}]\big)^{\otimes 2}$$

$$= 2 \dim_\theta(U) \;\vcenter{\hbox{[diagram]}}\; - \; 2 \;\vcenter{\hbox{[diagram]}}\;,$$

where

$$\dim_\theta(U) := \tilde{d}_U \circ (\theta \otimes \mathrm{id}_U) \circ b_U.$$

Further, recall that if the dimension $\dim(U)$ is invertible, then we have a direct sum decomposition $A_U = \mathbf{1} \oplus A_U'$. Using the graphical description

$$\mathrm{id}_{U \otimes U^\vee} = e_{\neq} \circ r^{\neq} + e_{A_U'} \circ r^{A_U'} = \frac{b_U \circ \tilde{d}_U}{\dim(U)} + e_{A_U'} \circ r^{A_U'} = \frac{1}{\dim(U)} \;\vcenter{\hbox{[diagram]}}\; + \;\vcenter{\hbox{[diagram]}}$$

of the corresponding idempotents, the Killing form on the ideal A_U' reads

$$\kappa_{A_U'} = 2 \dim_\theta(U)\big(\tilde{d}_U \circ [\mathrm{id}_U \otimes d_U \otimes \theta_{U^\vee}] \circ [e_{A_U'} \otimes e_{A_U'}]\big) = 2 \dim_\theta(U) \;\vcenter{\hbox{[diagram]}}\;.$$

Proposition 3.7. *The Killing form is* symmetric, *i.e. satisfies*

$$\kappa = \kappa \circ c_{L,L}. \tag{3.7}$$

Proof. By functoriality of the twist we can rewrite the Killing form as

$$\kappa = \mathrm{tr}_3(\theta_L \circ \ell_{\lfloor 3 \rfloor}) \equiv \;\;\;\;\;\; = \;\;\;\;\;\; \equiv \mathrm{tr}_3(\ell_{\lfloor 3 \rfloor} \circ (\mathrm{id}_L^{\otimes 2} \otimes \theta_L)). \tag{3.8}$$

Using the cyclicity (3.5) of the partial trace we thus get

$$\kappa = \;\;\;\;\;\; = \;\;\;\;\;\; \equiv \mathrm{tr}_3([\ell \circ (\mathrm{id}_L \otimes \theta_L)] \circ [\mathrm{id}_L \otimes \ell] \circ [(c_{L,L} \circ (\theta_L \otimes \mathrm{id}_L)) \otimes \mathrm{id}_L]). \tag{3.9}$$
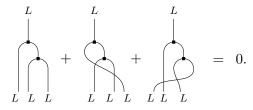
The claim now follows by noticing that, via functoriality of the twist, the right hand side of (3.9) equals $\mathrm{tr}_3(\ell_{\lfloor 3 \rfloor} \circ (\mathrm{id}_L^{\otimes 2} \otimes \theta_L)) \circ c_{L,L} = \kappa \circ c_{L,L}$. ∎

The morphism κ_0 defined in (3.6) satisfies $\kappa_0 = \kappa_0 \circ c_{L,L} \circ (\theta_L \otimes \mathrm{id}_L)$ instead of (3.7).

Example 3.8. Recall from Example 3.4 that there are two ribbon structures on the category $\mathcal{SV}\mathrm{ect}$. For both of them the Killing form κ of a Lie algebra L in $\mathcal{SV}\mathrm{ect}$, i.e. a Lie superalgebra, is supersymmetric, $\kappa(y, x) = (-1)^{|x||y|} \kappa(x, y)$ for homogeneous elements $x, y \in L$. For the ribbon structure that has trivial twist, this trivially holds for κ_0 as well, while for the one with non-trivial twist (and strict sovereign structure), κ_0 is instead symmetric, $\kappa_0(y, x) = \kappa_0(x, y)$.

3.3 Invariance of the Killing form

We are now going to show that the Killing form is invariant. The proof relies on the use of the Jacobi identity. In the graphical description, the Jacobi identity (slightly rewritten using that the braiding is symmetric) reads

$$\;\;\;\;\;\; + \;\;\;\;\;\; + \;\;\;\;\;\; = \; 0.$$

An alternative version, obtained by further rewriting with the help of antisymmetry, is

$$\;\;\;\;\;\; - \;\;\;\;\;\; - \;\;\;\;\;\; = \; 0 \tag{3.10}$$

(here, as well as in the pictures below, we omit the obvious L-labels).

Proposition 3.9. *The Killing form is invariant, i.e.*

$$\kappa \circ (\ell \otimes \mathrm{id}_L) = \kappa \circ (\mathrm{id}_L \otimes \ell). \tag{3.11}$$

Graphically,

$$
\text{(diagram)} \quad = \quad \text{(diagram)} \quad . \tag{3.12}
$$

Proof. We write the Killing form as in (3.8) and invoke the Jacobi identity in the form of (3.10) to express the left hand side of (3.12) as

$$
\text{(diagram)} \quad = \quad \text{(diagram)} \quad - \quad \text{(diagram)} \quad = \quad \text{(diagram)} \quad - \quad \text{(diagram)} \quad , \tag{3.13}
$$

where the second equality holds because of the cyclicity property (3.5). On the other hand, applying the Jacobi identity to the right hand side of (3.11) yields

$$
\text{(diagram)} \quad = \quad \text{(diagram)} \quad - \quad \text{(diagram)} \quad . \tag{3.14}
$$

Analogously as in the proof of symmetry, the claim then follows noticing that the expressions on the right hand sides of (3.13) and (3.14) are equal, owing to the functoriality of the twist. ∎

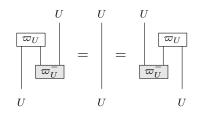## 3.4 Implications of non-degeneracy of the Killing form

Non-degeneracy of the Killing form plays an important role for Lie algebras in categories of vector spaces. It still does so for Lie algebras in generic additive symmetric ribbon categories. We start by recalling the appropriate notion of non-degeneracy of a pairing on an object U, i.e. for a morphism in $\mathrm{Hom}_{\mathcal{C}}(U \otimes U, \mathbf{1})$:

Definition 3.10. A pairing $\varpi_U \in \mathrm{Hom}_{\mathcal{C}}(U \otimes U, \mathbf{1})$ in a monoidal category $\mathcal{C}$ is called *non-dege-nerate* iff there exists a copairing $\varpi_U^- \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, U \otimes U)$ that is side-inverse to ϖ_U, i.e. iff the adjointness relations

$$
(\varpi_U \otimes \mathrm{id}_U) \circ (\mathrm{id}_U \otimes \varpi_U^-) = \mathrm{id}_U = (\mathrm{id}_U \otimes \varpi_U) \circ (\varpi_U^- \otimes \mathrm{id}_U) \tag{3.15}
$$

hold.

We describe the equalities (3.15) graphically as follows:

$$
\text{(diagram)}
$$

If the pairing is symmetric, then each of the two equalities in (3.15) implies the other. If the monoidal category $\mathcal{C}$ has a right (say) duality, an equivalent definition of non-degeneracy of ϖ_U is that the morphism $\varpi_U^\wedge := (\varpi_U \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_U \otimes b_U)$ in $\mathrm{Hom}_{\mathcal{C}}(U, U^\vee)$ is an isomorphism, implying in particular that $U^\vee$ is isomorphic, albeit not necessarily equal, to U.

We have immediately the

Lemma 3.11. *If a Lie algebra has a non-degenerate Killing form, then its only Abelian retract ideal is* 0.

Proof. Assume that (K, e, r) is an Abelian retract ideal of a Lie algebra L. Then, with $p = e \circ r$,

$$\kappa \circ (e \otimes \mathrm{id}_L) = \mathrm{tr}_3(p \circ \ell \circ (e \otimes \ell)) = \mathrm{tr}_3(\ell \circ (e \otimes \ell) \circ (\mathrm{id}_K \otimes \mathrm{id}_L \otimes p))$$
$$= \mathrm{tr}_3(\ell \circ (e \otimes p) \circ (\mathrm{id}_K \otimes \ell) \circ (\mathrm{id}_K \otimes \mathrm{id}_L \otimes p)) = 0, \tag{3.16}$$

where abelianness enters in the last step. Graphically, with $e = \downarrow$ and $r = \uparrow$, equation (3.16) reads

Using that κ is non-degenerate, (3.16) implies immediately that $0 = (\kappa \circ \mathrm{id}_L) \circ (e \otimes \kappa^-) = e$. ∎

We are now in a position to establish

Proposition 3.12. *Let (L, ℓ) be a Lie algebra in an idempotent complete symmetric ribbon category and let $\mathrm{End}_{\mathcal{C}}(L)$ have finitely many idempotents. If the Killing form κ of L is non-degenerate, then L is a finite direct sum of indecomposable Lie algebras.*

Proof. Assume that L is decomposable as a Lie algebra, and let (M, e, r) be a non-trivial retract ideal of L, with corresponding idempotent $p = e \circ r$. The kernel of the idempotent $\widehat{p} := (\kappa \otimes \mathrm{id}_L) \circ p \circ (\mathrm{id}_L \otimes \kappa^-)$ exists and is obtained by splitting the idempotent $p' = \mathrm{id}_L - \widehat{p}$; we denote this kernel retract by (M', e', r'), so that
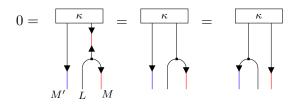

Since κ is non-degenerate, we have

$$\kappa \circ (e' \otimes e) = 0. \tag{3.17}$$

It follows in particular that

$$0 = \kappa \circ (e' \otimes [p \circ \ell \circ (\mathrm{id}_L \otimes e)]) = \kappa \circ (e' \otimes [\ell \circ (\mathrm{id}_L \otimes e)]) = \kappa \circ ([\ell \circ (e' \otimes \mathrm{id}_L)] \otimes e) \tag{3.18}$$

or, graphically,

where in the second equality it is used that M is an ideal, and in the third that κ is invariant. By the universal property of M' as the kernel of $\widehat{p}$, (3.18) implies

$$\ell \circ (e' \otimes \mathrm{id}_L) = p' \circ \ell \circ (e' \otimes \mathrm{id}_L)$$

showing that (M', e', r') is in fact a retract *ideal* of L.

Now take M to be a *primitive* retract ideal. Then either $p \circ p' = p$ or $p \circ p' = 0$. In case that $p \circ p' = p$, it follows from (3.17) that $\kappa \circ (e \otimes e) = 0$, which in turn implies

$$
\begin{aligned}
\kappa \circ \big([\ell \circ (p \otimes p)] \otimes \mathrm{id}_L \big) &= \kappa \circ \big(p \otimes [\ell \circ (p \otimes \mathrm{id}_L)] \big) \\
&= \kappa \circ (p \otimes p) \circ \big(\mathrm{id}_L \otimes [\ell \circ (p \otimes \mathrm{id}_L)] \big) = 0
\end{aligned}
$$

and therefore, when combined with Lemma 3.11, contradicts the non-degeneracy of κ. We thus conclude that $p \circ p' = 0$, and hence, again by non-degeneracy, that the Lie algebra L is the direct sum of the retract ideals M and M'. Furthermore, $\kappa \circ (e \otimes e)$ and $\kappa \circ (e' \otimes e')$ are just the Killing form of M and M', respectively.

The claim now follows by iteration, which terminates after a finite number of steps because $\mathrm{End}_{\mathcal{C}}(L)$ only has finitely many idempotents. ∎

Remark 3.13. A large class of categories for which the assumptions of Proposition 3.12 are satisfied for any Lie algebra are Krull–Schmidt categories, for which every object is a finite direct sum of indecomposables.

Remark 3.14. For $\mathcal{C}$ the category of finite-dimensional vector spaces over a field (of arbitrary characteristic), the proof of Proposition 3.12 reduces to the one given in [13].

As is well known, for Lie algebras in $\mathcal{V}\mathrm{ect}_{\Bbbk}$, semisimplicity does not imply non-degeneracy of the Killing form unless $\Bbbk$ is a field of characteristic zero. Still, the following weaker statement holds.

Proposition 3.15. *The Killing form of an indecomposable Lie algebra in a symmetric ribbon category is either zero or non-degenerate.*

Proof. Let (M, e, r) be a maximal retract of the Lie algebra (L, ℓ) such that for the Killing form κ of L one has

$$\kappa \circ (\mathrm{id}_L \otimes e) = 0. \tag{3.19}$$

Then we have

$$0 = \kappa \circ (\ell \otimes e) = \kappa \circ (\mathrm{id}_L \otimes \ell) \circ (\mathrm{id}_L \otimes \mathrm{id}_L \otimes e),$$

which implies that M is a retract ideal of L. Since L is indecomposable, this means that M either equals L or is zero. In the former case, (3.19) says that κ is zero, while the latter case amounts to κ being non-degenerate. ∎

A   Appendix

A.1   Categorical background

For the sake of fixing terminology and notation, we recall a few pertinent concepts from category theory. Besides in standard textbooks, details can e.g. be found on the nLab web site[1]; for a condensed exposition of several of the relevant notions see [35].

A *subobject* (U', e) of an object U is a pair consisting of an object U' and a monomorphism $e \in \mathrm{Hom}_\mathcal{C}(U', U)$ or, to be precise, an equivalence class of such pairs, with equivalence defined via factorization of morphisms with the same codomain (but it is common to use the term subobject also for individual representatives.) A *simple* object is an object that does not have any non-trivial subobject. A *retract* of an object U is a subobject of U for which e has a left-inverse, i.e. is a triple (U', e, r) such that (U', e) is a subobject of U and $r \in \mathrm{Hom}_\mathcal{C}(U', U)$ is an epimorphism satisfying $r \circ e = \mathrm{id}_{U'}$. A retract (U', e, r) for which the idempotent $p = e \circ r \in \mathrm{End}_\mathcal{C}(U)$ cannot be written as the sum of two non-zero idempotents is called *primitive*.

The *image* of a morphism $f \in \mathrm{Hom}_\mathcal{C}(U, V)$ is a pair (V', h) consisting of an object V' and a monic $h \in \mathrm{Hom}_\mathcal{C}(V', V)$ such that there exists a morphism $g \in \mathrm{Hom}_\mathcal{C}(U, V')$ satifying $h \circ g = f$ and such that (V', h) is universal with this property, i.e. for any object W with a monic $k \in \mathrm{Hom}_\mathcal{C}(W, V)$ and morphism $j \in \mathrm{Hom}_\mathcal{C}(U, W)$ satifying $k \circ j = f$ there exists a unique morphism $l \in \mathrm{Hom}_\mathcal{C}(V', W)$ such that $k \circ l = h$. In short, in case it exists, the image is the smallest subobject (V', h) of V through which the morphism f factors.

A *preadditive* category is a category $\mathcal{C}$ which is enriched over the monoidal category of Abelian groups, i.e. for which the morphism set $\mathrm{Hom}_\mathcal{C}(U, V)$ for any pair of objects $U, V \in \mathcal{C}$ is an Abelian group and composition of morphisms is bilinear. An *additive* category is a preadditive category admitting all finitary biproducts, or direct sums. A direct sum is unique up to isomorphism, and the empty biproduct is a zero object 0 satisfying $\mathrm{Hom}_\mathcal{C}(U, 0) = 0 = \mathrm{Hom}_\mathcal{C}(0, U)$ for any object $U \in \mathcal{C}$. An object U of an additive category is called *indecomposable* iff it is non-zero and has only trivial direct sum decompositions, i.e. $U = U_1 \oplus U_2$ implies that $U_1 = 0$ or $U_2 = 0$. Each summand U_i in a direct sum $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ forms a retract (U_i, e_i, r_i), i.e. there are monics $e_i \in \mathrm{Hom}_\mathcal{C}(U_i, U)$ and epis $r_i \in \mathrm{Hom}_\mathcal{C}(U, U_i)$ such that

$$r_i \circ e_j = \delta_{i,j}\, \mathrm{id}_{U_j} \qquad \text{for all} \quad i, j = 1, 2, \ldots, n \qquad \text{and} \qquad \sum_{i=1}^{n} e_i \circ r_i = \mathrm{id}_U \,.$$

An indecomposable object thus only has trivial retracts, the zero object and itself. A primitive retract is an indecomposable direct summand.

An *idempotent* p is an endomorphism satisfying $p^2 = p$; an idempotent $p \in \mathrm{End}_\mathcal{C}(U)$ is called *split* iff there exists a retract (V, e, r) of U such that $e \circ r = p$. A category $\mathcal{C}$ is said to be *idempotent complete* iff every idempotent in $\mathcal{C}$ is split. In an idempotent complete category, any retract is a direct sum of primitive retracts. A *Krull–Schmidt category* is an additive category for which every object decomposes into a finite direct sum of objects whose endomorphism sets are local rings. The objects with local endomorphism rings are then precisely the indecomposable objects. Krull–Schmidt categories are idempotent complete. The class of Krull–Schmidt categories contains in particular every Abelian category in which each object has finite length, and thus e.g. all finite tensor categories in the sense of [14], as well as [25] every triangulated category whose morphism sets are finite-dimensional vector spaces and for which the endomorphism ring of any indecomposable object is local.

The data of a *monoidal* category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ consist of a category $\mathcal{C}$, the tensor product functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the monoidal unit $\mathbf{1} \in \mathcal{C}$, the associativity constraint $a = (a_{U,V,W})$, and

---

[1]http://ncatlab.org.

left and right unit constraints $l = (l_U)$ and $r = (r_U)$ (see e.g. [39] for a review). If $\mathcal{C}$ is additive, then the tensor product is required to be additive in each argument. Every monoidal category is equivalent to a *strict* one, i.e. one for which the associativity and unit constraints are identities. We denote by $U^{\otimes n}$ the tensor product of n copies of U (which can be defined unambiguously even if $\mathcal{C}$ isn't strict). A *braided* monoidal category has in addition a commutativity constraint, called the braiding and denoted by $c = (c_{U,V})$, i.e. a natural family of isomorphisms $c_{U,V} \in \mathrm{Hom}_{\mathcal{C}}(U \otimes V, V \otimes U)$ satisfying two coherence conditions known as hexagon equations. A *symmetric* monoidal category is a braided monoidal category with symmetric braiding, i.e. for which $c_{V,U} \circ c_{U,V} = \mathrm{id}_{U \otimes V}$. A *rigid* (or autonomous) monoidal category has right and left dualities, with natural families $d = (d_U)$ and $\tilde{d} = (\tilde{d}_U)$ of evaluation and $b = (b_U)$ and $\tilde{b} = (\tilde{b}_U)$ of coevaluation morphisms, respectively. The objects that are right and left dual to U are denoted by $U^{\vee}$ and $^{\vee}U$, respectively, so that $d_U \in \mathrm{Hom}_{\mathcal{C}}(U^{\vee} \otimes U, \mathbf{1})$, $b_U \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, U \otimes U^{\vee})$ and $\tilde{d}_U \in \mathrm{Hom}_{\mathcal{C}}(U \otimes {}^{\vee}U, \mathbf{1})$, $\tilde{b}_U \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, {}^{\vee}U \otimes U)$. (With this convention, the functor $U^{\vee} \otimes -$ is left adjoint to $U \otimes -$, while $^{\vee}U \otimes -$ is right adjoint to $U \otimes -$.) Dualities are also defined naturally on morphisms, namely for $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$ by

$$f^{\vee} := (d_V \otimes \mathrm{id}_{U^{\vee}}) \circ (\mathrm{id}_{V^{\vee}} \otimes f \otimes \mathrm{id}_{U^{\vee}}) \circ (\mathrm{id}_{V^{\vee}} \otimes b_U) =$$

and

$$^{\vee}f := (\mathrm{id}_{^{\vee}U} \otimes \tilde{d}_V) \circ (\mathrm{id}_{^{\vee}U} \otimes f \otimes \mathrm{id}_{^{\vee}V}) \circ (\tilde{b}_U \otimes \mathrm{id}_{^{\vee}V}) = \qquad ,$$

respectively, whereby they furnish (contravariant) endofunctors of $\mathcal{C}$. For a monoidal category with a right (say) duality, (U, e, r) is a retract of V iff $(U^{\vee}, r^{\vee}, e^{\vee})$ is a retract of $V^{\vee}$.

A *sovereign* structure on a rigid monoidal category is a (choice of) monoidal natural isomorphism between the right and left duality functors, i.e. a natural family of isomorphisms $\sigma_U \in \mathrm{Hom}_{\mathcal{C}}(U^{\vee}, {}^{\vee}U)$ such that

$$^{\vee}f \circ \sigma_V = \sigma_U \circ f^{\vee} \tag{A.1}$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$. A sovereign structure is equivalent [52, Proposition 2.11] to a pivotal (or balanced) structure, i.e. to a monoidal natural isomorphism between the (left or right) double dual functor and the identity functor. If the natural isomorphism defining a sovereign structure is the identity, the category is called *strictly sovereign*. In a sovereign category, an endomorphism $f \in \mathrm{End}_{\mathcal{C}}(U)$ has a right trace $\mathrm{tr}(f) \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$, given (for strictly sovereign $\mathcal{C}$) by $\mathrm{tr}(f) = \tilde{d}_U \circ (f \otimes \mathrm{id}_{U^{\vee}}) \circ b_U$, as well as an analogous left trace, and thus in particular every object U has a right (and analogously, left) dimension $\dim_{\mathcal{C}}(U) \equiv \mathrm{tr}(\mathrm{id}_U) \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$. When the left and right traces coincide, the category is called *spherical*.

A *twist* (or balancing) for a braided monoidal category with a (right, say) duality is a natural family $\theta = (\theta_U)$ of isomorphisms $\theta_U \in \mathrm{End}_{\mathcal{C}}(U)$ satisfying $\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}$, $\theta_{U \otimes V} = (\theta_U \otimes \theta_V) \circ c_{V,U} \circ c_{U,V}$ and $\theta_{U^{\vee}} = (\theta_U)^{\vee}$. A braided rigid monoidal category equipped with a compatible twist is called a *ribbon* (or tortile) category. A ribbon category is sovereign, with sovereign structure

$$\sigma_U = (\mathrm{id}_{^{\vee}U} \otimes d_U) \circ (c_{U,U}^{-1} \otimes \theta_U) \circ (\mathrm{id}_{U^{\vee}} \otimes \tilde{b}_U) \in \mathrm{Hom}_{\mathcal{C}}(U^{\vee}, {}^{\vee}U), \tag{A.2}$$

as well as spherical. In a symmetric ribbon category the twist θ_U squares to id_U (see e.g. [47, Remark 4.32]).

A.2   Algebras and coalgebras

We also collect a few elementary concepts concerning algebras in monoidal categories. An *algebra* (or monoid, or monoid object) in a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is just a pair (A, m) consisting of an object $A \in \mathcal{C}$ and a morphism $m \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, A)$; m is called a product or a multiplication, and by the usual abuse of terminology also the object A is referred to as an algebra.

   This concept is rather empty unless an interesting property of the product and/or additional structure on A is imposed, which in turn may require also the category $\mathcal{C}$ to have additional structure beyond being monoidal. Monoidal categories without any further structure furnish the proper setting for *associative* algebras, for which the associativity property

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \circ a_{A,A,A} \tag{A.3}$$

is imposed, and for *unital* algebras, for which there exists a morphism $\eta \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A)$ satisfying

$$m \circ (\eta \otimes \ \mathrm{id}_A) \circ l_A^{-1} = \mathrm{id}_A = m \circ (\mathrm{id}_A \otimes \eta) \circ r_A^{-1}. \tag{A.4}$$

Likewise, in any monoidal category there is the notion of a coassociative and co-unital coalgebra (C, Δ, ε), with morphisms $\Delta \in \mathrm{Hom}_{\mathcal{C}}(C, C \otimes C)$ and $\varepsilon \in \mathrm{Hom}_{\mathcal{C}}(C, \mathbf{1})$ that satisfy relations obtained by reversing the arrows in (A.3) and (A.4), i.e. $(\Delta \otimes \mathrm{id}_C) \circ \Delta \circ a_{C,C,C} = (\mathrm{id}_C \otimes \Delta) \circ \Delta$ and $l_C \circ (\varepsilon \otimes \mathrm{id}_C) \circ \Delta = \mathrm{id}_C = r_C \circ (\mathrm{id}_C \otimes \varepsilon) \circ \Delta$. In the sequel we will simplify such equalities by taking the monoidal category $\mathcal{C}$ to be strict.

   Still working merely in a (strict) monoidal category $\mathcal{C}$ there is an interesting way to combine algebras and coalgebras:

Definition A.1.

 (i) A *Frobenius algebra* A in $\mathcal{C}$ is a quintuple $(A, m, \eta, \Delta, \varepsilon)$ such that (A, m, η) is a unital associative algebra, (A, Δ, ε) is co-unital coassociative coalgebra, and such that the coproduct is a morphism of A-bimodules, i.e.

$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = \Delta \circ m = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta).$$

 (ii) A *strongly separable*[2] Frobenius algebra is a Frobenius algebra for which the product is a left inverse of the coproduct, i.e. $m \circ \Delta = \mathrm{id}_A$.

Example A.2.

 (i) Coalgebras, and in particular Frobenius algebras, in symmetric monoidal categories which admit the notion of an adjoint (or dagger [46]) of a morphism arise naturally in a categorical formalization of fundamental issues of quantum mechanics, like complementarity of observables or the no-cloning theorem; see e.g. [7, 8, 34].

 (ii) To a pair of functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$, such that G is right adjoint to F, there is naturally associated a monad in $\mathcal{C}$, i.e. an algebra in the category of endofunctors of $\mathcal{C}$. If G is also left adjoint to F, then this monad has a natural Frobenius structure [49, Proposition 1.5].

   If the monoidal category $\mathcal{C}$ is in addition braided, one can also define (co-)commutativity as well as bi- and Hopf algebras. An algebra (A, m) in a braided (strict) monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, c)$ is said to be *commutative* iff $m \circ c_{A,A} = m$; dually, a coalgebra (C, Δ) in $\mathcal{C}$ is *cocommutative* iff $c_{A,A} \circ \Delta = \Delta$. A *bialgebra* B is both an (associative unital) algebra and

---

[2]In the literature also variants of this definition are in use.

a (coassociative co-unital) coalgebra, with the coproduct and counit being algebra morphisms, i.e. $\Delta \circ m = (m \otimes m) \circ (\mathrm{id}_B \otimes c_{B,B} \otimes \mathrm{id}_B) \circ (\Delta \otimes \Delta)$ and $\varepsilon \circ m = \varepsilon \otimes \varepsilon$. A *Hopf* algebra H is a bialgebra with an antipode $s \in \mathrm{End}_{\mathcal{C}}(H)$ that is a two-sided inverse of id_H for the convolution product.

Example A.3.

(i) If $\mathcal{C}$ is braided monoidal and countable direct sums are defined in $\mathcal{C}$,[3] then for any object $U \in \mathcal{C}$ the object $T(U) := \bigoplus_{n=0}^{\infty} U^{\otimes n}$ carries a natural structure of a unital associative algebra, with the product defined via the tensor product of the retracts $U^{\otimes n}$ and the unit morphism given by the retract embedding of $U^{\otimes 0} = \mathbf{1}$, see e.g. [12, 44]; $T(U)$ is called the *tensor algebra* of U. In fact [44, Corollary 2.4], $T(U)$ has in fact a natural *bi*algebra structure, and by extending the endomorphism $-\mathrm{id}_U$ of U to an anti-algebra morphism of $T(U)$ defines an antipode, so that $T(U)$ is even a Hopf algebra. $T(U)$ can also be defined by a universal property, it is an associative algebra together with a morphism $i_U \in \mathrm{Hom}_{\mathcal{C}}(U, T(U))$ such that for any algebra morphism f from U to some algebra A in $\mathcal{C}$ there exists a unique morphism g from $T(U)$ to A such that $f = g \circ i_U$.

(ii) If in addition $\mathcal{C}$ is idempotent complete, is $\Bbbk$-linear with $\Bbbk$ a field of characteristic zero, and has symmetric braiding, then the direct sum $S(U) := \bigoplus_{n=0}^{\infty} S_n(U)$ of all symmetric tensor powers of U carries a natural structure of a commutative unital associative algebra.

(iii) Hopf algebras appear naturally in constructions within categories of three-dimensional cobordisms. Specifically, the manifold obtained from a 2-torus by excising an open disk is a Hopf algebra 1-morphism in the bicategory of three-dimensional cobordisms with corners [10].

(iv) For a braided finite tensor category $\mathcal{C}$ the coend $\int^{U \in \mathcal{C}} U^{\vee} \otimes U$ exists and can be used to construct quantum invariants of three-manifolds and representations of mapping class groups [36]. It also plays a role (for $\mathcal{C}$ semisimple, i.e. a fusion category) in establishing the relationship between Reshetikhin–Turaev and Turaev–Viro invariants [50]. For these results it is crucial that this coend carries a natural structure of a Hopf algebra (with some additional properties) in $\mathcal{C}$.

Remark A.4. A Hopf algebra H in an additive rigid braided category which has invertible antipode and which possesses a left integral $\Lambda \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, H)$ and right cointegral $\lambda \in \mathrm{Hom}_{\mathcal{C}}(H, \mathbf{1})$ such that $\lambda \circ \Lambda \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is invertible, naturally also has the structure of a Frobenius algebra with the same algebra structure, and with the Frobenius counit given by λ [19, 31, 40].

Frobenius algebras F can, just like in the classical case, also be defined with the help of a non-degenerate invariant pairing $\varpi \in \mathrm{Hom}_{\mathcal{C}}(F \otimes F, \mathbf{1})$; in terms of the description chosen above, $\varpi = \varepsilon \circ m$, see e.g. [20, 51]. A *symmetric* Frobenius algebra is a Frobenius algebra F for which the pairing ϖ is symmetric.

Example A.5. In two-dimensional rational conformal field theory (RCFT), full local RCFTs that come from one and the same chiral RCFT are in bijection with Morita classes of strongly separable symmetric Frobenius algebras of non-zero dimension in the modular tensor category that underlies the chiral RCFT [18].

Acknowledgments

---

[3] If $\mathcal{C}$ is Abelian, then the condition that countable direct sums exist is dispensable: instead of $\mathcal{C}$ one may then work with the category of ind-objects in $\mathcal{C}$, into which $\mathcal{C}$ is fully embedded [24, Section 3.1].

References

[1] Albuquerque H., Majid S., Quasialgebra structure of the octonions, *J. Algebra* **220** (1999), 188–224, math.QA/9802116.

[2] Baez J.C., The octonions, *Bull. Amer. Math. Soc.* **39** (2002), 145–205, math.RA/0105155.

[3] Bahturin Y., Fischman D., Montgomery S., On the generalized Lie structure of associative algebras, *Israel J. Math.* **96** (1996), 27–48.

[4] Bahturin Y.A., Mikhalev A.A., Petrogradsky V.M., Zaicev M.V., Infinite-dimensional Lie superalgebras, *de Gruyter Expositions in Mathematics*, Vol. 7, Walter de Gruyter & Co., Berlin, 1992.

[5] Borcherds R.E., Quantum vertex algebras, in Taniguchi Conference on Mathematics Nara '98, *Adv. Stud. Pure Math.*, Vol. 31, Math. Soc. Japan, Tokyo, 2001, 51–74, math.QA/9903038.

[6] Caenepeel S., Goyvaerts I., Monoidal Hom–Hopf algebras, *Comm. Algebra* **39** (2011), 2216–2240, arXiv:0907.0187.

[7] Coecke B., Duncan R., Interacting quantum observables: categorical algebra and diagrammatics, *New J. Phys.* **13** (2011), 043016, 85 pages, arXiv:0906.4725.

[8] Coecke B., Pavlovic D., Vicary J., A new description of orthogonal bases, *Math. Structures Comput. Sci.* **23** (2013), 555–567, arXiv:0810.0812.

[9] Cohen M., Fischman D., Westreich S., Schur's double centralizer theorem for triangular Hopf algebras, *Proc. Amer. Math. Soc.* **122** (1994), 19–29.

[10] Crane L., Yetter D., On algebraic structures implicit in topological quantum field theories, *J. Knot Theory Ramifications* **8** (1999), 125–163, hep-th/9412025.

[11] Deligne P., Catégories tannakiennes, in The Grothendieck Festschrift, Vol. II, *Progr. Math.*, Vol. 87, Birkhäuser Boston, Boston, MA, 1990, 111–195.

[12] Deligne P., Morgan J.W., Notes on supersymmetry (following Joseph Bernstein), in Quantum Fields and Strings: a Course for Mathematicians, Vols. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, 41–97.

[13] Dieudonné J., On semi-simple Lie algebras, *Proc. Amer. Math. Soc.* **4** (1953), 931–932.

[14] Etingof P., Ostrik V., Finite tensor categories, *Mosc. Math. J.* **4** (2004), 627–654, math.QA/0301027.

[15] Fischman D., Montgomery S., A Schur double centralizer theorem for cotriangular Hopf algebras and generalized Lie algebras, *J. Algebra* **168** (1994), 594–614.

[16] Fjelstad J., Fuchs J., Runkel I., Schweigert C., TFT construction of RCFT correlators. V. Proof of modular invariance and factorisation, *Theory Appl. Categ.* **16** (2006), 342–433, hep-th/0503194.

[17] Freund P.G.O., Kaplansky I., Simple supersymmetries, *J. Math. Phys.* **17** (1976), 228–231.

[18] Fuchs J., Runkel I., Schweigert C., TFT construction of RCFT correlators. I. Partition functions, *Nuclear Phys. B* **646** (2002), 353–497, hep-th/0204148.

[19] Fuchs J., Schweigert C., Hopf algebras and finite tensor categories in conformal field theory, *Rev. Un. Mat. Argentina* **51** (2010), 43–90, arXiv:1004.3405.

[20] Fuchs J., Stigner C., On Frobenius algebras in rigid monoidal categories, *Arab. J. Sci. Eng. Sect. C Theme Issues* **33** (2008), 175–191, arXiv:0901.4886.

[21] Goyvaerts I., Vercruysse J., A note on the categorification of Lie algebras, in Lie Theory and its Applications in Physics, *Springer Proceedings in Mathematics & Statistics*, Vol. 36, Editor V. Dobrev, Springer-Verlag, Tokyo, 2013, 541–550, arXiv:1202.3599.

[22] Gurevich D.I., Generalized translation operators on Lie groups, *Soviet J. Contemporary Math. Anal.* **18** (1983), 57–70.

[23] Gurevich D.I., Elements of the formal Lie theory and the Poincaré–Birkhoff–Witt theorem for generalized shift operators, *Funct. Anal. Appl.* **20** (1986), 315–317.

[24] Hái P.H., An embedding theorem for abelian monoidal categories, *Compositio Math.* **132** (2002), 27–48, math.CT/0004160.

[25] Happel D., On the derived category of a finite-dimensional algebra, *Comment. Math. Helv.* **62** (1987), 339–389.

[26] Hartwig J.T., Larsson D., Silvestrov S.D., Deformations of Lie algebras using σ-derivations, *J. Algebra* **295** (2006), 314–361, math.QA/0408064.

[27] James G.D., The modular characters of the Mathieu groups, *J. Algebra* **27** (1973), 57–111.

[28] Joyal A., Street R., The geometry of tensor calculus. I, *Adv. Math.* **88** (1991), 55–112.

[29] Joyal A., Street R., Verity D., Traced monoidal categories, *Math. Proc. Cambridge Philos. Soc.* **119** (1996), 447–468.

[30] Kac V.G., Lie superalgebras, *Adv. Math.* **26** (1977), 8–96.

[31] Kadison L., Stolin A.A., Separability and Hopf algebras, in Algebra and its Applications (Athens, OH, 1999), *Contemp. Math.*, Vol. 259, Amer. Math. Soc., Providence, RI, 2000, 279–298.

[32] Kassel C., Quantum groups, *Graduate Texts in Mathematics*, Vol. 155, Springer-Verlag, New York, 1995.

[33] Kharchenko V.K., Connected braided Hopf algebras, *J. Algebra* **307** (2007), 24–48.

[34] Kissinger A., Pictures of processes: automated graph rewriting for monoidal categories and applications to quantum computing, Ph.D. Thesis, University of Oxford, 2011, arXiv:1203.0202.

[35] Krause H., Krull–Schmidt categories and projective covers, arXiv:1410.2822.

[36] Lyubashenko V., Modular transformations for tensor categories, *J. Pure Appl. Algebra* **98** (1995), 279–327.

[37] MacDonald J.L., Conditions for a universal mapping of algebras to be a monomorphism, *Bull. Amer. Math. Soc.* **80** (1974), 888–892.

[38] Majid S., Quantum and braided-Lie algebras, *J. Geom. Phys.* **13** (1994), 307–356, hep-th/9303148.

[39] Müger M., Tensor categories: a selective guided tour, *Rev. Un. Mat. Argentina* **51** (2010), 95–163, arXiv:0804.3587.

[40] Pareigis B., When Hopf algebras are Frobenius algebras, *J. Algebra* **18** (1971), 588–596.

[41] Pareigis B., On Lie algebras in braided categories, in Quantum Groups and Quantum Spaces (Warsaw, 1995), *Banach Center Publ.*, Vol. 40, Polish Acad. Sci., Warsaw, 1997, 139–158, q-alg/9612002.

[42] Pareigis B., On Lie algebras in the category of Yetter–Drinfeld modules, *Appl. Categ. Structures* **6** (1998), 151–175, q-alg/9612023.

[43] Rittenberg V., Wyler D., Generalized superalgebras, *Nuclear Phys. B* **139** (1978), 189–202.

[44] Schauenburg P., A characterization of the Borel-like subalgebras of quantum enveloping algebras, *Comm. Algebra* **24** (1996), 2811–2823.

[45] Scheunert M., Generalized Lie algebras, *J. Math. Phys.* **20** (1979), 712–720.

[46] Selinger P., Dagger compact closed categories and completely positive maps: (extended abstract), *Electron. Notes Theor. Computer Sci.* **170** (2007), 139–163.

[47] Selinger P., A survey of graphical languages for monoidal categories, in New Structures for Physics, *Lecture Notes in Phys.*, Vol. 813, Springer, Heidelberg, 2011, 289–355, arXiv:0908.3347.

[48] Soibelman Ya.S., Meromorphic tensor categories, q-alg/9709030.

[49] Street R., Frobenius monads and pseudomonoids, *J. Math. Phys.* **45** (2004), 3930–3948.

[50] Turaev V.G., Virelizier A., On two approaches to 3-dimensional TQFTs, arXiv:1006.3501.

[51] Yamagami S., Frobenius algebras in tensor categories and bimodule extensions, in Galois Theory, Hopf Algebras, and Semiabelian Categories, *Fields Inst. Commun.*, Vol. 43, Amer. Math. Soc., Providence, RI, 2004, 551–570.

[52] Yetter D.N., Framed tangles and a theorem of Deligne on braided deformations of Tannakian categories, in Deformation Theory and Quantum Groups with Applications to Mathematical Physics (Amherst, MA, 1990), *Contemp. Math.*, Vol. 134, Amer. Math. Soc., Providence, RI, 1992, 325–349.

[53] Zhang S., Zhang Y.-Z., Braided m-Lie algebras, *Lett. Math. Phys.* **70** (2004), 155–167, math.RA/0308095.