Long-Time Asymptotics for the Defocusing Integrable Discrete Nonlinear Schrödinger Equation II

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Abstract. We investigate the long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation. If $|n| < 2t$, we have decaying oscillation of order $O(t^{-1/2})$ as was proved in our previous paper. Near $|n| = 2t$, the behavior is decaying oscillation of order $O(t^{-1/3})$ and the coefficient of the leading term is expressed by the Painlevé II function. In $|n| > 2t$, the solution decays more rapidly than any negative power of n.

Key words: discrete nonlinear Schrödinger equation; nonlinear steepest descent; Painlevé equation

2010 Mathematics Subject Classification: 35Q55; 35Q15

1 Introduction

In our previous paper [6], we studied the long-time behavior of the defocusing integrable discrete nonlinear Schrödinger equation (IDNLS)

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2(R_{n+1} + R_{n-1}) = 0 \tag{1}$$

in the region $|n| \le (2 - V_0)t$, $0 < V_0 < 2$. (In the present paper we refer to it as Region A.) We have proved that there exist $C_j = C_j(n/t) \in \mathbb{C}$ and $p_j = p_j(n/t), q_j = q_j(n/t) \in \mathbb{R}$ $(j = 1, 2)$ depending only on the ratio n/t such that

$$R_n(t) = \sum_{j=1}^{2} C_j t^{-1/2} e^{-i(p_j t + q_j \log t)} + O\big(t^{-1}\log t\big) \qquad \text{as} \quad t \to \infty.$$

The behavior of each term in the sum is decaying oscillation of order $t^{-1/2}$. Here C_j and q_j are defined in terms of the reflection coefficient $r = r(z)$ ([1], [6]) corresponding to the initial potential $\{R_n(0)\}$.

In the present paper, we study (1) in other regions, namely one including the rays $n = \pm 2t$ and another with $|n| > 2t$.

Painlevé asymptotics has been observed in the cases of the MKdV equation ([2]) and the Toda lattice ([3]). The proofs are based on the nonlinear steepest descent method. Unlike the saddle point case, one has to deal with a phase function of degree 3. Following these results, especially [2], we obtain the long-time asymptotics of (1) in Region B, i.e. near $n = \pm 2t$.

Roughly speaking, up to a time shift $t \mapsto t - t_0$, our result is as follows (Theorem 1).

Consider a curve defined by

$$t^{2/3}\frac{2 - n/t}{(6 - n/t)^{1/3}} = \text{a real constant}. \tag{2}$$

It approaches $n/t = 2$ with an error of $O(t^{-2/3})$ as $t \to \infty$. The behavior of $R_n(t)$ on it is of the form

$$R_n(t) = \text{const}\, e^{i(-4t+\pi n)/2} t^{-1/3} + O\big(t^{-2/3}\big).$$

The constant in the above expression is written in terms of the Painlevé II function with parameters determined by the reflection coefficient corresponding to $\{R_n(0)\}$. A similar result was obtained in [5] at least formally. Notice that an analogous phenomenon can be found in a different context [4]. In the result of [2] about the MKdV equation, no oscillatory factor appears together with the Painlevé function.

Remark 1. The equation (1) is invariant under the reflection $n \mapsto -n$. In the later sections, we assume $n > 0$ without loss of generality.

In Section 2 we state our main results. Sections 3–6 are devoted to the study of the region $2t - Mt^{1/3} < n < 2t$. In Section 7 we study $2t \le n < 2t + M't^{1/3}$. In Section 8 we investigate $n > 2t$.

2 Main results

Let $r(z)$ be the reflection coefficient determined by the initial potential $\{R_n(0)\}$. See [6] for the precise definition. We assume that $\{R_n(0)\}$ decreases rapidly in the sense that for any $s > 0$ there exists a constant $C_s > 0$ such that $|R_n(0)| \le C_s/(1 + |n|)^s$.[1] Then $r(z)$ is smooth on $C\colon |z| = 1$.

Let Region B[2] be defined by

$$2t - Mt^{1/3} < n < 2t + M't^{1/3}, \tag{3}$$

where M and M' are arbitrary positive constants. The solution to an initial value problem for (1) has the following asymptotic behavior there:

Theorem 1. *Let t_0 be such that $\pi^{-1}(\arg r(e^{-\pi i/4}) - 2t_0) - 1/2$ is an integer. Set $t' = t - t_0$, $p' = i(-4t' + \pi n)/4$, $\alpha' = [12t'/(6t' - n)]^{1/3}$, $q' = -2^{-4/3} 3^{1/3} (6t' - n)^{-1/3}(2t' - n)$. Then we have*

$$R_n(t) = \frac{e^{2p' - \pi i/4}\alpha'}{(3t')^{1/3}} u\left(\frac{4q'}{3^{1/3}}\right) + O\big(t'^{-2/3}\big).$$

Here u is a solution of the Painlevé II equation $u''(s) - su(s) - 2u^3(s) = 0$ and is specified in (20).

Let Region C be defined by

$$n > (2 + V_0)t,$$

where V_0 is an arbitrary positive constant.

Theorem 2. *Let j be an arbitrary positive integer. Then in Region C, we have $R_n(t) = O(n^{-j})$. More precisely, there exists a constant $C = C(j, V_0) > 0$ such that $|R_n(t)| \le Cn^{-j}$ holds.*

The solution decays exponentially if $r(z)$ is analytic on the circle $|z| = 1$: there exists a positive constant $\rho = \rho(V_0)$ with $0 < \rho < 1$ and a positive constant $C = C(V_0)$ such that $|R_n(t)| \le C\rho^n$ holds.

Remark 2. A sufficient condition for the analyticity of $r(z)$ is that $\{R_n(0)\}$ is finitely supported.

[1] It is equivalent to saying that $\sum_s (1 + |n|)^s |R_n(0)|$ converges for any s, see [6].

[2] We only consider the case $n > 0$, see Remark 1 above.

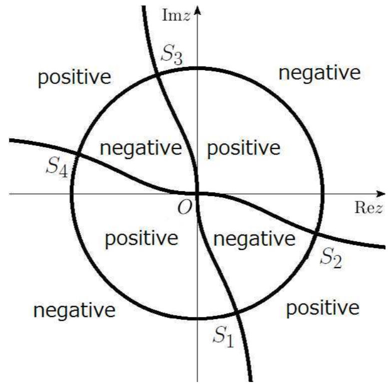

Figure 1. Real part of φ.

3 Decomposition and reduction

First we consider the long-time asymptotics in the 'left-hand half' of Region B defined in (3), namely

$$2t - Mt^{1/3} < n < 2t. \tag{4}$$

Notice that a curve like (2) is in this kind of region for M suitably chosen.

Set

$$\varphi = \varphi(z) = \varphi(z; n, t) = 2^{-1} it\big(z - z^{-1}\big)^2 - n \log z, \qquad \psi = \psi(z) = \varphi(z)/(it).$$

Choice of the branch of the logarithm is irrelevant because φ always appears in the form $e^{\pm\varphi}$.

We formulate a Riemann–Hilbert problem (RHP):

$$m_+(z) = m_-(z)v(z) \qquad \text{on} \quad C\colon |z| = 1, \tag{5}$$

$$m(z) \to I \qquad \text{as} \quad z \to \infty, \tag{6}$$

$$v(z) = e^{-\varphi \operatorname{ad} \sigma_3} \begin{bmatrix} 1 - |r(z)|^2 & -\bar{r}(z) \\ r(z) & 1 \end{bmatrix}. \tag{7}$$

Here m_+ and m_- are the boundary values from the *outside* and *inside* of C respectively of the unknown matrix-valued analytic function $m(z) = m(z; n, t)$ in $|z| \neq 1$. Namely C is endowed with clockwise orientation (a convention adopted by [1]). We employ the usual notation $\sigma_3 = \operatorname{diag}(1, -1)$, $a^{\operatorname{ad}\sigma_3} Q = a^{\sigma_3} Q a^{-\sigma_3}$ (a: a scalar, Q: a 2×2 matrix). In formulating other RHPs in the remaining part of the present article, we will always assume the normalization condition (6), which we often neglect to mention.

The sign of the real part of φ is as in Fig. 1. The function $\varphi(z)$ has four saddle points. They are

$$S_1 = e^{-\pi i/4} A, \qquad S_2 = e^{-\pi i/4} \bar{A}, \qquad S_3 = -S_1, \qquad S_4 = -S_2,$$

where $A = 2^{-1}(\sqrt{2 + n/t} - i\sqrt{2 - n/t})$. One can reconstruct $\{R_n(t)\}$ from $m(z)$ by

$$R_n(t) = -\lim_{z \to 0} \frac{1}{z} m(z)_{21} = -\frac{d}{dz} m(z)_{21}\bigg|_{z=0}. \tag{8}$$

We will frequently use the factorization

$$v = v(z) = e^{-\varphi \operatorname{ad} \sigma_3} \left\{ \begin{bmatrix} 1 & -\bar{r}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r(z) & 1 \end{bmatrix} \right\}$$

and its outcomes.

For $\psi = \psi(z) = \varphi(z)/(it)$, we have

$$\psi'(z) = z - z^{-3} - \frac{n}{it}z^{-1}, \qquad \psi''(z) = 1 + 3z^{-4} + \frac{n}{it}z^{-2}, \qquad \psi'''(z) = -12z^{-5} - \frac{2n}{it}z^{-3}.$$

Third-order approximation of ψ will be necessary, since we will deal with coalescence of saddle points.

We do not need the 'Δ-conjugation' as in [6, § 4], where a function called ρ was introduced. Here we decompose $\bar{r}$ and r on $\mathrm{arc}(S_2S_3) \cup \mathrm{arc}(S_4S_1)$ by using Taylor's theorem and Fourier analysis[3].

Set $\vartheta = \theta - \pi/4$, $z = e^{i\theta}$ and $\vartheta_0 = \pi/2 + \arg A = \pi/2 - \arctan\sqrt{(2t-n)/(2t+n)}$. (The definitions of ϑ and ϑ_0 are different from those in [6].) Then $\mathrm{arc}(S_2S_3)$ corresponds to $-\vartheta_0 \leq \vartheta \leq \vartheta_0$. We regard the function $\bar{r}$ on $\mathrm{arc}(S_2S_3)$ as a function in ϑ and denote it by $\bar{r}(\vartheta)$ by abuse of notation. We have

$$\bar{r}(\vartheta) = H_{\mathrm{e}}(\vartheta^2) + \vartheta H_{\mathrm{o}}(\vartheta^2), \qquad -\vartheta_0 \leq \vartheta \leq \vartheta_0,$$

for smooth functions H_{e} and H_{o}. By Taylor's theorem, they are expressed as follows:

$$H_{\mathrm{e}}(\vartheta^2) = \mu_0^{\mathrm{e}} + \cdots + \mu_k^{\mathrm{e}}(\vartheta^2 - \vartheta_0^2)^k + \frac{1}{k!}\int_{\vartheta_0^2}^{\vartheta^2} H_{\mathrm{e}}^{(k+1)}(\gamma)(\vartheta^2 - \gamma)^k d\gamma,$$
$$H_{\mathrm{o}}(\vartheta^2) = \mu_0^{\mathrm{o}} + \cdots + \mu_k^{\mathrm{o}}(\vartheta^2 - \vartheta_0^2)^k + \frac{1}{k!}\int_{\vartheta_0^2}^{\vartheta^2} H_{\mathrm{o}}^{(k+1)}(\gamma)(\vartheta^2 - \gamma)^k d\gamma.$$

Here $k = 4q + 1$ and q can be any positive integer.

We set

$$R(\vartheta) = R_k(\vartheta) = \sum_{i=0}^{k} \mu_i^{\mathrm{e}}(\vartheta^2 - \vartheta_0^2)^i + \vartheta \sum_{i=0}^{k} \mu_i^{\mathrm{o}}(\vartheta^2 - \vartheta_0^2)^i,$$
$$\alpha(z) = (z - S_2)^q (z - S_3)^q, \qquad h(\vartheta) = \bar{r}(\vartheta) - R(\vartheta)$$

and, by abuse of notation,

$$\alpha(\vartheta) = \alpha(e^{i(\vartheta+\pi/4)}) = \left[e^{i(\vartheta+\pi/4)} - e^{i(-\vartheta_0+\pi/4)}\right]^q \left[e^{i(\vartheta+\pi/4)} - e^{i(\vartheta_0+\pi/4)}\right]^q.$$

Notice that we have $R(\pm\vartheta_0) = \bar{r}(\pm\vartheta_0)$. The function R extends analytically from $\mathrm{arc}(S_2S_3)$ to a complex neighborhood. By abuse of notation, $R(z)$ denotes the analytic function thus obtained, so that $R(e^{i(\vartheta+\pi/4)}) = R(\vartheta)$ and $R(S_j) = \bar{r}(S_j)$.

On $\mathrm{arc}(S_2S_3)$ we have $d\psi/d\vartheta = -2\cos 2\vartheta - n/t$. Since $[-\vartheta_0, \vartheta_0] \ni \vartheta \mapsto \psi \in \mathbb{R}$ is strictly decreasing, we can consider its inverse $\vartheta = \vartheta(\psi)$, $\psi(\vartheta_0) \leq \psi \leq \psi(-\vartheta_0)$. We set

$$(h/\alpha)(\psi) = \begin{cases} h(\vartheta(\psi))/\alpha(\vartheta(\psi)) & \text{if } \ \psi(\vartheta_0) \leq \psi \leq \psi(-\vartheta_0), \\ 0 & \text{otherwise.} \end{cases}$$

Then $(h/\alpha)(\psi)$ is well-defined for $\psi \in \mathbb{R}$. It can be shown that $h/\alpha \in H^p(-\infty < \psi < \infty)$, where p can be any positive integer if we choose a sufficiently large value of k. Its norm is uniformly bounded with respect to (n, t). This argument is a 'curved' version of [2, equation (1.33)].

Notice that $d\psi/d\vartheta = -2\cos 2\vartheta - n/t$ has a zero of order *two* at $\vartheta = \pm\pi/2$ if $n/t = 2$. It may worsen the estimate of the Sobolev norm (cf. [2, equation (1.33)]) of h/α as a function of ψ in

[3] We sometimes denote $\mathrm{arc}(S_jS_k)$ by S_jS_k.

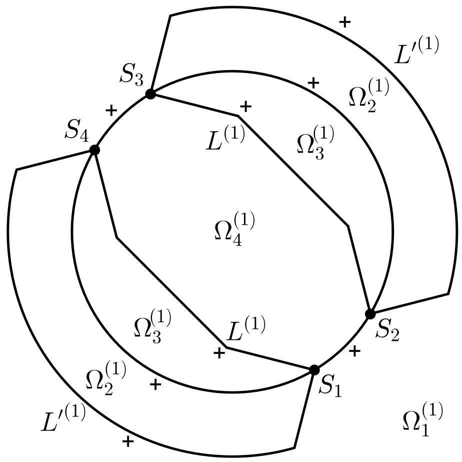

Figure 2. $\Sigma^{(1)}$.

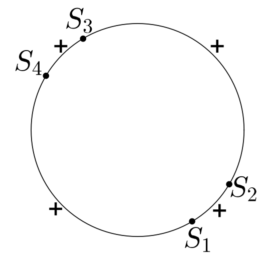

Figure 3. $\Sigma^{(2)}$.

contract to ϑ, especially in the case of Section 7, since it involves $d/d\psi = (d\psi/d\vartheta)^{-1}d/d\vartheta$. This kind of difficulty is overcome by choosing a sufficiently large value of k.

Set

$$\begin{aligned}
(\widehat{h/\alpha})(s) &= \int_{-\infty}^{\infty} e^{-is\psi}(h/\alpha)(\psi)\frac{d\psi}{\sqrt{2\pi}},\\
h_{\mathrm{I}}(\vartheta) &= \alpha(\vartheta)\int_t^{\infty} e^{is\psi(\vartheta)}(\widehat{h/\alpha})(s)\frac{ds}{\sqrt{2\pi}},\\
h_{\mathrm{II}}(\vartheta) &= \alpha(\vartheta)\int_{-\infty}^{t} e^{is\psi(\vartheta)}(\widehat{h/\alpha})(s)\frac{ds}{\sqrt{2\pi}},
\end{aligned}$$

then we have $h(\vartheta) = h_{\mathrm{I}}(\vartheta) + h_{\mathrm{II}}(\vartheta)$ and $\bar r(\vartheta) = R(\vartheta) + h_{\mathrm{I}}(\vartheta) + h_{\mathrm{II}}(\vartheta)$ on $|\vartheta| \le \vartheta_0$. We can apply the same process to r. We have

$$\begin{aligned}
&\bar r(z) = \bar r = h_{\mathrm{I}} + h_{\mathrm{II}} + R, &\qquad& r(z) = r = \bar h_{\mathrm{I}} + \bar h_{\mathrm{II}} + \bar R,\\
&\bar r(S_j) = R(S_j), && r(S_j) = \bar R(S_j).
\end{aligned} \tag{9}$$

The decomposition on $\mathrm{arc}(S_4S_1)$ immediately follows by symmetry. Notice that h_{II}, R, $\bar h_{\mathrm{II}}$ and $\bar R$ can be analytically continued to certain open sets. We still employ the same notation for the extended functions. For example, $\bar h_{\mathrm{II}} = \bar h_{\mathrm{II}}(z)$ is analytic, although the bar may seem a little strange.

We introduce a new contour $\Sigma^{(1)}$ as in Fig. 2. It is a variation of Σ in [6, Fig. 2]. The part $L^{(1)}$ is bent so that it stays away from the circle as $n \to 2t$ (except near the saddle points.) Some open sets, not necessarily connected, are defined in Fig. 2. Notice that $\Sigma^{(1)}$ remains finite even as $n \to 2t$.

We introduce a new unknown matrix $m^{(1)}$ by setting

$$m^{(1)} = \begin{cases} m & \text{in } \Omega_1^{(1)} \cup \Omega_4^{(1)},\\ m e^{-\varphi\,\mathrm{ad}\,\sigma_3}\begin{bmatrix} 1 & 0\\ -\bar h_{\mathrm{II}} & 1\end{bmatrix} & \text{in } \Omega_2^{(1)},\\ m e^{-\varphi\,\mathrm{ad}\,\sigma_3}\begin{bmatrix} 1 & -h_{\mathrm{II}}\\ 0 & 1\end{bmatrix} & \text{in } \Omega_3^{(1)}.\end{cases}$$

We define a new jump matrix $v^{(1)}$ by

$$v^{(1)} = v = \begin{cases} e^{-\varphi \,\mathrm{ad}\, \sigma_3} \left\{ \begin{bmatrix} 1 & -\bar{r} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} \right\} & \text{on} \quad S_1S_2 \cup S_3S_4, \\ e^{-\varphi \,\mathrm{ad}\, \sigma_3} \begin{bmatrix} 1 & -h_{\mathrm{II}} \\ 0 & 1 \end{bmatrix} & \text{on} \quad L^{(1)}, \\ e^{-\varphi \,\mathrm{ad}\, \sigma_3} \left\{ \begin{bmatrix} 1 & -h_{\mathrm{I}} - R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \bar{h}_{\mathrm{I}} + \bar{R} & 1 \end{bmatrix} \right\} & \text{on} \quad S_2S_3 \cup S_4S_1, \\ e^{-\varphi \,\mathrm{ad}\, \sigma_3} \begin{bmatrix} 1 & 0 \\ \bar{h}_{\mathrm{II}} & 1 \end{bmatrix} & \text{on} \quad L'^{(1)}. \end{cases}$$

Then we have

$$m_+^{(1)} = m_-^{(1)} v^{(1)} \quad \text{on} \quad \Sigma^{(1)}, \qquad m^{(1)} \to I \quad \text{as} \quad z \to \infty.$$

By (8), we have

$$R_n(t) = -\frac{d}{dz} m^{(1)}(z)_{21} \Big|_{z=0}. \tag{10}$$

We need some estimates. First, in the same way as [2, equation (1.36)] and [6, equation (43)],

$$|e^{-2\varphi} h_{\mathrm{I}}| \le C/t^{(3q+1)/2}, \qquad |e^{2\varphi} \bar{h}_{\mathrm{I}}| \le C/t^{(3q+1)/2}$$

holds for some $C > 0$ on $S_2S_3 \cup S_4S_1$.

The estimates of $|e^{-2\varphi} h_{\mathrm{II}}|$ on $L^{(1)}$ and of $|e^{2\varphi} \bar{h}_{\mathrm{II}}|$ on $L'^{(1)}$ must be handled with greater care. We have [6, § 4]

$$\psi''(S_j) = (-1)^j 2 S_j^{-2} (2 + n/t)^{1/2} (2 - n/t)^{1/2}.$$

It can be infinitely small and does not lead to a reasonably good estimate. We would rather rely on $\psi'''(S_j)$. The following lemma replaces [6, equation (44)] in our context.

Lemma 1. *Let $L^{(1)}(S_j)$ (resp. $L'^{(1)}(S_j)$) be the segment $\subset L^{(1)}$ (resp. $\subset L'^{(1)}$) emanating from S_j. Let d be the distance from S_j to $z \in L^{(1)}(S_j)$ (resp. to $z \in L'^{(1)}(S_j)$). Then there exists a positive constant C' such that*

$$\mathrm{Re}\, i\psi(z) \ge C' d^3, \quad z \in L^{(1)}(S_j), \qquad \mathrm{Re}\, i\psi(z) \le -C' d^3, \quad z \in L'^{(1)}(S_j). \tag{11}$$

Proof. First we assume $j = 2$. In view of Fig. 1, $L^{(1)}$ is in the region $\mathrm{Re}(i\psi) = \mathrm{Re}(t^{-1}\varphi) > 0$. Since $\psi'(S_2) = 0$, we have

$$i\psi(z) = i\psi(S_2) + \frac{i\psi''(S_2)}{2}(z - S_2)^2 + \frac{i\psi'''(S_2)}{6}(z - S_2)^3 + \text{higher order terms} \tag{12}$$

and $i\psi(S_2)$ is purely imaginary. It holds that

$$\psi''(S_2) = 2 S_2^{-2} (2 + n/t)^{1/2} (2 - n/t)^{1/2}, \qquad \psi'''(S_2) = -12 S_2^{-5} - \frac{2n}{it} S_2^{-3}.$$

The segment $L^{(1)}(S_2)$ is tangent to the steepest ascent path of $\varphi = it\psi$, hence also of $i\psi$. Assume $z \in L^{(1)}(S_2)$. If $n/t \approx 2$, then S_2 is close to $T_1 = e^{-\pi i/4}$ and $z - S_2$ is close to id. We have

$$i\psi''(S_2)(z - S_2)^2 \approx 2(2 + n/t)^{1/2} (2 - n/t)^{1/2} d^2, \tag{13}$$

$$i\psi'''(S_2)(z-S_2)^3 \approx \left(12e^{\pi i/4} + \frac{2n}{t}e^{-3\pi i/4}\right)d^3. \tag{14}$$

The right-hand side of (13) is positive. In estimating $\operatorname{Re} i\psi$ from below, we can neglect the term of degree 2 in (12). On the other hand, the quantity in the parentheses on the right-hand side of (14) has a positive real part (close to $4\sqrt{2}$). Hence we get the first inequality of (11). By symmetry, (11) also holds on $L^{(1)}(S_4)$.

In a similar way, we can show that the second inequality of (11) holds on $L'^{(1)}(S_2)$. Notice that $z - S_2 \approx d$ on it. The case $j = 2$ is now finished.

By symmetry, we get (11) for $j = 4$.

Since $S_1 \approx S_2$, the estimates on $L^{(1)}(S_1)$ and $L'^{(1)}(S_1)$ are similar to those on $L'^{(1)}(S_2)$ and $L^{(1)}(S_2)$ respectively. Notice that L and L' are exchanged. We get (11) for $j = 1$. The case $j = 3$ follows by symmetry. ∎

Assume $z \in L^{(1)}(S_j)$. We have $|\alpha(z)| \le \mathrm{const}\, d^q$. By modifying the argument of [6, equation (45)], we obtain

$$|e^{-2\varphi}h_{\mathrm{II}}| \le \mathrm{const}\, d^q e^{-2C'td^3} \le \mathrm{const}\, t^{-q/3} \sup_{\tau>0} \tau^{q/3} e^{-2C'\tau} \le \mathrm{const}\, t^{-q/3}. \tag{15}$$

This kind of estimate obviously holds on any compact subset of $\{\operatorname{Re}\varphi > 0\}$. Hence we get the following lemma.

Lemma 2.

$$|e^{-2\varphi}h_{\mathrm{II}}| \le \mathrm{const}\, t^{-q/3} \quad \text{on} \quad L^{(1)}, \qquad |e^{2\varphi}\bar{h}_{\mathrm{II}}| \le \mathrm{const}\, t^{-q/3} \quad \text{on} \quad L'^{(1)}.$$

The contribution to $R_n(t)$ by h_{II} and $\bar{h}_{\mathrm{II}}$ on $L^{(1)} \cup L'^{(1)}$, as well as by h_{I} and $\bar{h}_{\mathrm{I}}$ on $S_3S_2 \cup S_1S_4$, are of order t^{-l} as $t \to \infty$, where $l > 0$ is arbitrarily large. It is justified by choosing sufficiently large q. We are left with an RHP over $\Sigma^{(2)} = C$, the union of four arcs oriented clockwise. See Fig. 3.

We follow [2, equations (5.9) and (5.10)]. The new jump matrix $v^{(2)} = v^{(2)}(z)$ is given by

$$v^{(2)} = \begin{cases} e^{-\varphi\,\mathrm{ad}\,\sigma_3}\left\{\begin{bmatrix}1 & -\bar{r}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ r & 1\end{bmatrix}\right\} & \text{on} \quad S_1S_2 \cup S_3S_4,\\[2ex] e^{-\varphi\,\mathrm{ad}\,\sigma_3}\left\{\begin{bmatrix}1 & -R\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \bar{R} & 1\end{bmatrix}\right\} & \text{on} \quad S_2S_3 \cup S_4S_1.\end{cases}$$

Here S_jS_k denotes the minor arc joining S_j and S_k. Let $m^{(2)}$ be the solution to the RHP corresponding to $v^{(2)}$. Then by (10), for any $l > 0$,

$$R_n(t) = -\frac{d}{dz}m^{(2)}(z)_{21}\bigg|_{z=0} + O(t^{-l}).$$

See Section 4 for a more precise (routine) argument based on the Beals–Coifman formula.

Let $\Sigma^{(3)}$ be the contour in Fig. 4. The parts inside and outside the circle are denoted by $L^{(3)}$ and $L'^{(3)}$ respectively. The latter consists of four half-lines. Following [2, equations (5.13)–(5.15)], we set

$$v^{(3)} = v^{(2)} = \begin{cases} e^{-\varphi\,\mathrm{ad}\,\sigma_3}\left\{\begin{bmatrix}1 & -\bar{r}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ r & 1\end{bmatrix}\right\} & \text{on} \quad S_1S_2 \cup S_4S_3,\\[2ex] e^{-\varphi\,\mathrm{ad}\,\sigma_3}\begin{bmatrix}1 & -R\\ 0 & 1\end{bmatrix} & \text{on} \quad L^{(3)},\\[2ex] e^{-\varphi\,\mathrm{ad}\,\sigma_3}\begin{bmatrix}1 & 0\\ \bar{R} & 1\end{bmatrix} & \text{on} \quad L'^{(3)}.\end{cases}$$

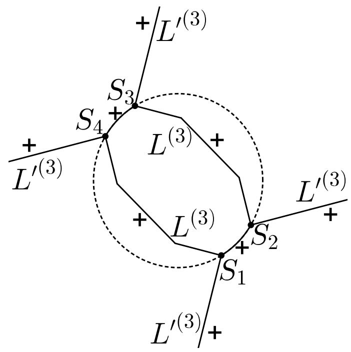

Figure 4. $\Sigma^{(3)}$.

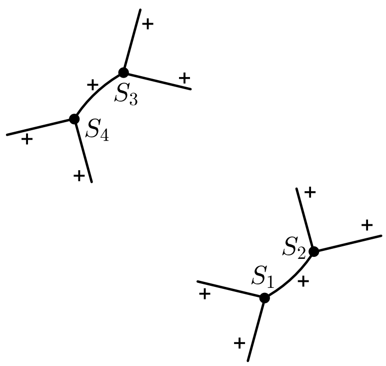

Figure 5. $\Sigma^{(4)}$.

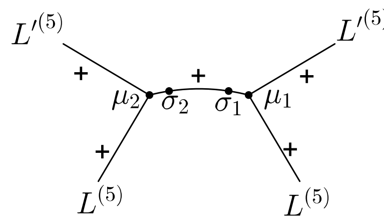

Figure 6. $\Sigma^{(5)}$.

Notice that $v^{(3)} = v^{(3)}(z) \to I$ as $L'^{(3)} \ni z \to \infty$. The new unknown function $m^{(3)}(z)$ is defined in the usual way: $m^{(3)}(z) \to I$. It implies that $m^{(3)}(z) = m^{(2)}(z)$ near $z = 0$.

By using the method of [6, § 7.2, § 9] (originally of [2, § 2, § 3]), we can replace $\Sigma^{(3)}$ by the *bounded* contour $\Sigma^{(4)}$ in Fig. 5 up to an error of order $O(t^{-1})$, hence without changing the leading part in the asymptotics. We can assume that the lengths of the 'branches' emanating from the saddle points are independent of n and t. The new jump matrix $v^{(4)}$ equals $v^{(3)}$ on $\Sigma^{(4)}$ and is the identity matrix elsewhere.

Owing to the technique of [2, Proposition 3.66] and [6, Proposition 9.2], the contribution from the two connected components of $\Sigma^{(4)}$ can be separated out, with an error of $O(t^{-1})$. Notice that the two terms arising from the two components are actually the same because $r(-z) = -r(z)$ for $z \in C$ (cf. [6, Proposition 12.4]). It is enough to investigate the lower part (containing $T_1 = e^{-\pi i/4}$), which is referred to as $\Sigma^{(4)}_{\text{lower}}$.

4 Scaling and rotation

We have

$$\varphi(T_1) = \frac{i}{4}(-4t + \pi n), \qquad \varphi'(T_1) = (2t - n)e^{\pi i/4},$$
$$\varphi''(T_1) = i(-2t + n), \qquad \varphi'''(T_1) = (-12t + 2n)e^{-\pi i/4},$$
$$\varphi(z) = \sum_{k=0}^{3} \frac{\varphi^{(k)}(T_1)}{k!}(z - T_1)^k + \varphi_4(z), \qquad \varphi_4(z) = O\big((z - T_1)^4\big)$$

near $z = T_1$. Let $\varepsilon > 0$ be such that $\Sigma^{(4)}_{\text{lower}}$ is within the circle $|z - T_1| = \varepsilon/2$.

Now we define an operator sc by

$$z \mapsto \operatorname{sc}(z) = t^{-1/3}e^{-3\pi i/4}z + T_1, \qquad T_1 = T_2 = e^{-\pi i/4}.$$

Set $\sigma_j = \operatorname{sc}^{-1}(S_j) = t^{1/3}e^{3\pi i/4}(S_j - T_1)$. We have $S_j - T_1 = O\big(\sqrt{2-n/t}\big) = O(t^{-1/3})$, the latter equality being a consequence of (4). It follows that σ_j is bounded in spite of the magnifying factor $t^{1/3}$. There is a constant $\tilde{M} > 0$ such that $|\operatorname{Re}\sigma_j| < \tilde{M}$. Let μ_j be such that $\operatorname{Re}\mu_j = (-1)^{j-1}\tilde{M}$ and that $|\operatorname{sc}(\mu_j)| = 1$, $\Im\operatorname{sc}(\mu_j) < 0$. We modify $\operatorname{sc}^{-1}\big(\Sigma^{(4)}_{\text{lower}}\big)$ without moving the endpoints to get the contour $\Sigma^{(5)}$ in Fig. 6. The arc $\mu_1\mu_2$ is a part of a circle of radius $t^{1/3}$ and looks like a segment of length $2\tilde{M}$ if t is large. The lengths of $L^{(5)}$ and $L'^{(5)}$ are of order $t^{1/3}$ and their directions approach $\pm\pi/4$ or $\pm 3\pi/4$ as $t \to \infty$. We choose $\Sigma^{(5)}$ so that $\operatorname{sc}\big(\Sigma^{(5)}\big)$ is within the circle $|z - T_1| = \varepsilon$.

We want to approximate $\varphi(\operatorname{sc}(z))$ by a cubic polynomial which is related to the Painlevé II function (up to a constant term). We introduce

$$\phi = \phi(z) = \frac{i}{4}(-4t + \pi n) + i(-2t+n)t^{-1/3}z + \frac{i(6t-n)t^{-1}}{3}z^3.$$

Then we have $\varphi(\operatorname{sc}(z)) = \phi(z) + 2^{-1}(2t-n)t^{-2/3}z^2 + \varphi_4(\operatorname{sc}(z))$. The following proposition is an analogue of [6, Proposition 10.1].

Proposition 1. *Fix a constant γ with $0 < \gamma < 1 < (6t-n)t^{-1}/3$. Then on $L^{(5)}$, we have*

$$\big|e^{-2\varphi(\operatorname{sc}(z))}R(\operatorname{sc}(z)) - e^{-2\phi(z)}\bar{r}(T_1)\big| \le Ct^{-1/3}\big|e^{-i\gamma z^3}\big|,$$
$$\big|\operatorname{sc}(z)^{-2}e^{-2\varphi(\operatorname{sc}(z))}R(\operatorname{sc}(z)) - T_1^{-2}e^{-2\phi(z)}\bar{r}(T_1)\big| \le Ct^{-1/3}\big|e^{-i\gamma z^3}\big|$$

for some constant $C > 0$.

Proof. We show only the latter inequality; the former is easier. We have

$$\begin{aligned} &e^{i\gamma z^3}\big[\operatorname{sc}(z)^{-2}e^{-2\varphi(\operatorname{sc}(z))}R(\operatorname{sc}(z)) - T_1^{-2}e^{-2\phi(z)}\bar{r}(T_1)\big] \\ &\quad = e^{-i\gamma z^3}\big[\operatorname{sc}(z)^{-2}ER(\operatorname{sc}(z)) - T_1^{-2}e^{-2\phi(z)+2i\gamma z^3}\bar{r}(T_1)\big], \end{aligned}$$

where $E = \exp(-2\varphi(\operatorname{sc}(z)) + 2i\gamma z^3)$. Each factor is uniformly bounded. Notice that $\operatorname{sc}(z)$ remains in the ε-neighborhood of T_1.

Set $f(w) = w^{-2}$. For any fixed z, $\operatorname{sc}(z)^{-2} = f(\operatorname{sc}(z))$ and $R(\operatorname{sc}(z))$ tend to T_1^{-2} and $\bar{r}(T_1)$ respectively as $t \to \infty$. This convergence is uniform on $L^{(5)}$ in the following sense:

$$\big|e^{-i\gamma z^3}\big[\operatorname{sc}(z)^{-2} - T_1^{-2}\big]\big| \le \big|e^{-i\gamma z^3}\big|\big|t^{-1/3}e^{-3\pi i/4}z\big| \sup_{|w-T_1|\le\varepsilon}|f'(w)| \le \operatorname{const} t^{-1/3},$$
$$\big|e^{-i\gamma z^3}\big[R(\operatorname{sc}(z)) - \bar{r}(T_1)\big]\big| \le \big|e^{-i\gamma z^3}\big|\big|t^{-1/3}e^{-3\pi i/4}z\big| \sup_{|w-T_1|\le\varepsilon}|R'(w)| \le \operatorname{const} t^{-1/3}.$$

We have used the fact that $e^{-i\gamma z^3}z$ is bounded on either branch of $L^{(5)}$.

Since $e^{-i\gamma z^3}z^j$ $(j = 2, 4)$ is bounded and $2t - n = O(t^{1/3})$, we have

$$\begin{aligned} &\big|e^{-i\gamma z^3}\big(E - e^{-2\phi(z)+2i\gamma z^3}\big)\big| \\ &\quad\le \big|e^{-i\gamma z^3}\big| \sup_{0\le s\le 1}\left|\frac{d}{ds}\exp\big(-2\phi(z) + 2i\gamma z^3 + s\big[(2t-n)t^{-2/3}z^2 - 2\varphi_4(\operatorname{sc}(z))\big]\big)\right| \\ &\quad\le C\big|e^{-i\gamma z^3}\big|\big[(2t-n)t^{-2/3}|z|^2 + 2|t^{-1/3}z|^4\big] \le Ct^{-1/3}. \end{aligned}$$

Combining the three estimates above, we can derive the desired inequality. ■

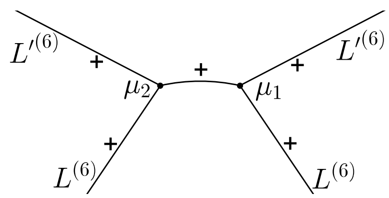

Figure 7. $\Sigma^{(6)}$.

The factorization problem on $\Sigma^{(4)}_{\text{lower}}$ is equivalent, up to the change of variables $z \mapsto \mathrm{sc}(z)$, to one on $\Sigma^{(5)}$, where the jump matrix $v^{(5)} = v^{(5)}(z)$ is

$$v^{(5)}(z) = \begin{cases} e^{-\varphi(\mathrm{sc}(z))\,\mathrm{ad}\,\sigma_3}\left\{\begin{bmatrix}1 & -\bar{r}(\mathrm{sc}(z))\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ r(\mathrm{sc}(z)) & 1\end{bmatrix}\right\} & \text{on } \sigma_2\sigma_1,\\ e^{-\varphi(\mathrm{sc}(z))\,\mathrm{ad}\,\sigma_3}\left\{\begin{bmatrix}1 & -R(\mathrm{sc}(z))\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \bar{R}(\mathrm{sc}(z)) & 1\end{bmatrix}\right\} & \text{on } \sigma_1\mu_1 \cup \mu_2\sigma_2,\\ e^{-\varphi(\mathrm{sc}(z))\,\mathrm{ad}\,\sigma_3}\begin{bmatrix}1 & -R(\mathrm{sc}(z))\\ 0 & 1\end{bmatrix} & \text{on } L^{(5)},\\ e^{-\varphi(\mathrm{sc}(z))\,\mathrm{ad}\,\sigma_3}\begin{bmatrix}1 & 0\\ \bar{R}(\mathrm{sc}(z)) & 1\end{bmatrix} & \text{on } L'^{(5)}.\end{cases}$$

Notice that $v^{(5)}$ is smooth across σ_1 and σ_2 to any desired order (choose k sufficiently large).

Let $\Sigma^{(6)}$ be the contour in Fig. 7 obtained by extending $L^{(5)}$ and $L'^{(5)}$ infinitely. Then we can regard $v^{(5)}(z)$ as a jump matrix on $\Sigma^{(6)}$: set $v^{(5)} = I$ on $\Sigma^{(6)} \setminus \Sigma^{(5)}$. Because of Proposition 1 (and its variants about $L'^{(5)}$ and $\mu_1\mu_2$), the jump matrix $v^{(5)}(z)$ is approximated by $v^{(6)}(z)$ up to an error of order $O(t^{-1/3})$, where

$$v^{(6)}(z) = \begin{cases} e^{-\phi(z)\,\mathrm{ad}\,\sigma_3}\left\{\begin{bmatrix}1 & -\bar{r}(T_1)\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ r(T_1) & 1\end{bmatrix}\right\} & \text{on } \mu_1\mu_2,\\ e^{-\phi(z)\,\mathrm{ad}\,\sigma_3}\begin{bmatrix}1 & -\bar{r}(T_1)\\ 0 & 1\end{bmatrix} & \text{on } L^{(6)},\\ e^{-\phi(z)\,\mathrm{ad}\,\sigma_3}\begin{bmatrix}1 & 0\\ r(T_1) & 1\end{bmatrix} & \text{on } L'^{(6)}.\end{cases}$$

We rescale by the factor $\alpha = [12t/(6t-n)]^{1/3} > 0$, which satisfies $\alpha^3 t^{-1}(6t-n)/3 = 4$ and tends to $3^{1/3}$ as $t \to \infty$. We have

$$\phi(\alpha z) = \frac{i}{4}(-4t + \pi n) + 4i\left\{z^3 + \frac{\alpha t^{-1/3}}{4}(-2t+n)z\right\}.$$

Set $p = i(-4t+\pi n)/4, q = \alpha t^{-1/3}(-2t+n)/4 = 2^{-4/3}3^{1/3}(6t-n)^{-1/3}(-2t+n)$, then we have

$$\phi(\alpha z) = p + 4i(z^3 + qz), \qquad p \in i\mathbb{R}.$$

We have normalized the coefficient of z^3. The term $4i(z^3+qz)$ will play an important role in Section 6.

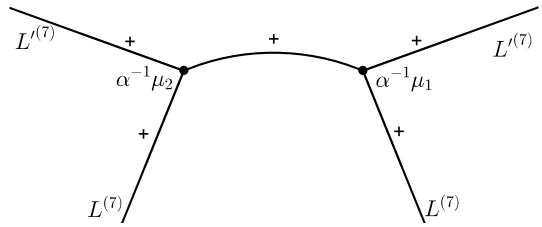

Figure 8. $\Sigma^{(7)} = \Sigma^{(8)}$.

The jump matrix $v^{(7)}(z) = v^{(6)}(\alpha z)$ on $\Sigma^{(7)} = \alpha^{-1}\Sigma^{(6)}$ is given by

$$v^{(7)}(z) = \begin{cases} e^{-[p+4i(z^3+qz)]\,\mathrm{ad}\,\sigma_3} \left\{ \begin{bmatrix} 1 & -\bar{r}(T_1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r(T_1) & 1 \end{bmatrix} \right\} & \text{on } \ (\alpha^{-1}\mu_1)(\alpha^{-1}\mu_2), \\ e^{-[p+4i(z^3+qz)]\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & -\bar{r}(T_1) \\ 0 & 1 \end{bmatrix} & \text{on } \ L^{(7)}, \\ e^{-[p+4i(z^3+qz)]\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & 0 \\ r(T_1) & 1 \end{bmatrix} & \text{on } \ L'^{(7)}, \end{cases}$$

where $(\alpha^{-1}\mu_1)(\alpha^{-1}\mu_2)$ is the arc in $\Sigma^{(7)}$. We have an RHP $m_+^{(7)}(z) = m_-^{(7)}(z)v^{(7)}(z)$ on $\Sigma^{(7)}$.

We want to remove p in $v^{(7)}(z)$. (Notice that p contributes to the oscillatory factor in Theorem 1.) Set $m^{(8)}(z) = e^{p\,\mathrm{ad}\,\sigma_3} m^{(7)}(z)$. Then $m^{(8)}(z)$ is the solution to

$$\begin{aligned} & m_+^{(8)}(z) = m_-^{(8)}(z)v^{(8)}(z) && \text{on} \quad \Sigma^{(8)} = \Sigma^{(7)}, \\ & m^{(8)}(z) \to I && \text{as} \quad z \to \infty, \end{aligned}$$

where $v^{(8)}(z) = e^{p\,\mathrm{ad}\,\sigma_3} v^{(7)}(z)$. We have

$$v^{(8)}(z) = \begin{cases} e^{-[4i(z^3+qz)]\,\mathrm{ad}\,\sigma_3} \left\{ \begin{bmatrix} 1 & -\bar{r}(T_1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r(T_1) & 1 \end{bmatrix} \right\} & \text{on } \ (\alpha^{-1}\mu_1)(\alpha^{-1}\mu_2), \\ e^{-[4i(z^3+qz)]\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & -\bar{r}(T_1) \\ 0 & 1 \end{bmatrix} & \text{on } \ L^{(8)} = L^{(7)}, \\ e^{-[4i(z^3+qz)]\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & 0 \\ r(T_1) & 1 \end{bmatrix} & \text{on } \ L'^{(8)} = L'^{(7)}. \end{cases}$$

We have explained steps of reduction in terms of contours and jump matrices. It should be supplemented with reconstruction formulas (up to some errors) involving integrals. Recall that each $v^{(j)}$ has a factorization of the form $v^{(j)} = (I + w_-^{(j)})(I + w_+^{(j)})$, where the diagonal components of $w_\pm^{(j)}$ is zero. We have $(I + w_-^{(j)})^{-1} = I - w_-^{(j)}$.

Let

$$\big(C_\pm^{(j)} f\big)(z) = \int_{\Sigma^{(j)}} \frac{f(\zeta)}{\zeta - z_\pm} \frac{d\zeta}{2\pi i} = \lim_{\substack{y \to z \\ y \in \{\pm\text{-side of}\Sigma^{(j)}\}}} \int_{\Sigma^{(j)}} \frac{f(\zeta)}{\zeta - y} \frac{d\zeta}{2\pi i}, \qquad z \in \Sigma^{(j)},$$

be the Cauchy operators on $\Sigma^{(j)}$. Define $C_{w^{(j)}} \colon L^2(\Sigma^{(j)}) \to L^2(\Sigma^{(j)})$ by

$$C_{w^{(j)}} f = C_+^{(j)}\big(f w_-^{(j)}\big) + C_-^{(j)}\big(f w_+^{(j)}\big)$$

for a 2×2 matrix-valued function f (cf. [2, § 2], [6, § 7]). The Cauchy differential form is invariant under an affine change of variables: $z = az'+b$ and $\zeta = a\zeta'+b$ imply $(\zeta-z)^{-1}d\zeta = (\zeta'-z')^{-1}d\zeta'$. The operator $C_{w^{(j)}}$ commutes with an affine change of variables in the sense that

$$(C_{w^{(j)}(z)}f(\bullet))(az'+b) = (C_{w^{(j)}(az'+b)}f(a\bullet+b))(z').$$

We have

$$m^{(j)}(z) = I + \int_{\Sigma^{(j)}} \frac{\left((1-C_{w^{(j)}})^{-1}I\right)(\zeta)w^{(j)}(\zeta)}{\zeta - z}\frac{d\zeta}{2\pi i},$$

where $w^{(j)}(\zeta) = w_+^{(j)}(\zeta) + w_-^{(j)}(\zeta)$. By (10), we obtain

$$R_n(t) = -\int_{\Sigma^{(1)}} z^{-2}\big[\big((1-C_{w^{(1)}})^{-1}I\big)w^{(1)}\big]_{21}(z)\frac{dz}{2\pi i}.$$

Repeated replacement of contours and integrands leads to (cf. [6])

$$\begin{aligned} R_n(t) &= -\int_{\Sigma^{(4)}} z^{-2}\big[\big((1-C_{w^{(4)}})^{-1}I\big)w^{(4)}\big]_{21}(z)\frac{dz}{2\pi i} + O\big(t^{-1}\big) \\ &= -2\int_{\Sigma^{(4)}_{\text{lower}}} z^{-2}\big[\big((1-C_{w^{(4)}})^{-1}I\big)w^{(4)}\big]_{21}(z)\frac{dz}{2\pi i} + O\big(t^{-1}\big). \end{aligned}$$

By repeated affine changes of variables and Proposition 1, we get

$$\begin{aligned} R_n(t) &= -\frac{2e^{-3\pi i/4}}{t^{1/3}}\int_{\Sigma^{(5)}} \mathrm{sc}(z')^{-2}\big[\big((1-C_{w^{(5)}})^{-1}I\big)w^{(5)}\big]_{21}(z')\frac{dz'}{2\pi i} + O\big(t^{-1}\big) \\ &= -\frac{2e^{-3\pi i/4}}{t^{1/3}}\int_{\Sigma^{(6)}} T_1^{-2}\big[\big((1-C_{w^{(6)}})^{-1}I\big)w^{(6)}\big]_{21}(z')\frac{dz'}{2\pi i} + O\big(t^{-2/3}\big) \\ &= -\frac{2e^{-\pi i/4}\alpha}{t^{1/3}}\int_{\Sigma^{(7)}} \big[\big((1-C_{w^{(7)}})^{-1}I\big)w^{(7)}\big]_{21}(z)\frac{dz}{2\pi i} + O\big(t^{-2/3}\big), \end{aligned} \tag{16}$$

where $\alpha = (12t)^{1/3}(6t-n)^{-1/3} > 0$. We have used the fact that $\mathrm{sc}(z') - T_1 = O(t^{-1/3})$ and the second resolvent identity. See [6, Remark 7.4].

Let us calculate the integral in (16). As $z \to \infty$,

$$z\big[\sigma_3, m^{(j)}(z)\big]_{21} \to 2\left[\int_{\Sigma^{(j)}} \big((1-C_{w^{(j)}})^{-1}I\big)w^{(j)}\right]_{21}(\zeta)\frac{d\zeta}{2\pi i}. \tag{17}$$

On the other hand, we have $[\sigma_3, m^{(8)}(z)] = e^{p\,\mathrm{ad}\,\sigma_3}[\sigma_3, m^{(7)}(z)]$. These two formulas imply

$$\begin{aligned} \left[\int_{\Sigma^{(8)}} \big((1-C_{w^{(8)}})^{-1}I\big)w^{(8)}\right]_{21}(\zeta)\frac{d\zeta}{2\pi i} &= \left[e^{p\,\mathrm{ad}\,\sigma_3}\int_{\Sigma^{(7)}} \big((1-C_{w^{(7)}})^{-1}I\big)w^{(7)}\right]_{21}(\zeta)\frac{d\zeta}{2\pi i} \\ &= e^{-2p}\left[\int_{\Sigma^{(7)}} \big((1-C_{w^{(7)}})^{-1}I\big)w^{(7)}\right]_{21}(\zeta)\frac{d\zeta}{2\pi i}. \end{aligned} \tag{18}$$

By using (16), (17) and (18), we obtain

$$\begin{aligned} R_n(t) &= -\frac{2\alpha e^{2p-\pi i/4}}{t^{1/3}}\int_{\Sigma^{(8)}} \big[\big((1-C_{w^{(8)}})^{-1}I\big)w^{(8)}\big]_{21}(z)\frac{dz}{2\pi i} + O\big(t^{-2/3}\big) \\ &= \frac{\alpha e^{2p-\pi i/4}}{t^{1/3}}\lim_{z\to\infty}\big\{-z\big[\sigma_3, m^{(8)}(z)\big]_{21}\big\} + O\big(t^{-2/3}\big). \end{aligned} \tag{19}$$

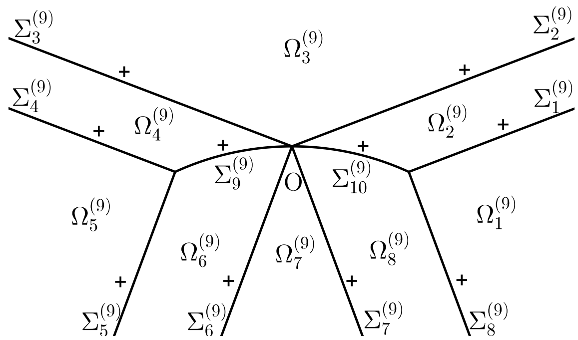

Figure 9. $\Sigma^{(9)}$.

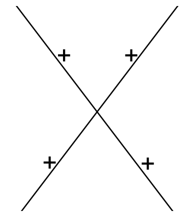

Figure 10. $\Sigma^{(10)}$.

5 Time shift

If $r(T_j)$ is purely imaginary, it is easy to apply the argument of [2, p. 359] to our case. Otherwise, we perform the following reduction. As is proved in [1], the time evolution of the reflection coefficient is given by

$$r(T_1, t) = r(T_1)\exp\bigl(it(T_1 - \bar{T}_1)^2\bigr) = r(T_1)\exp(-2it), \qquad r(T_1) = r(T_1, 0).$$

Therefore $r(T_1, t_0)$ is purely imaginary for some t_0. The condition to be satisfied is

$$\arg r(T_1) - 2t_0 - \pi/2 \in \pi\mathbb{Z}.$$

Notice that (3) is preserved if t is replaced by $t - t_0$.

6 Painlevé function

We assume that $r(T_1)$ is purely imaginary. See the previous section for justification.

Augment $\Sigma^{(8)} \to \Sigma^{(9)}$ (cf. [2, Fig. 5.5]) as in Fig. 9. The contour $\Sigma^{(9)}$ contains four pairs of parallel half-lines.

Define the new unknown function $m^{(9)}(z)$ by

$$m^{(9)}(z) = \begin{cases} m^{(8)}(z), & z \in \Omega_1^{(9)} \cup \Omega_3^{(9)} \cup \Omega_5^{(9)} \cup \Omega_7^{(9)}, \\ m^{(8)}(z) e^{-\{4i(z^3+qz)\}\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & 0 \\ -r(T_1) & 1 \end{bmatrix}, & z \in \Omega_2^{(9)} \cup \Omega_4^{(9)}, \\ m^{(8)}(z) e^{-\{4i(z^3+qz)\}\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & -\bar{r}(T_1) \\ 0 & 1 \end{bmatrix}, & z \in \Omega_6^{(9)} \cup \Omega_8^{(9)}. \end{cases}$$

Direct computation shows that $m^{(9)}(z)$ has no jump across $\Sigma_j^{(9)}$, $j = 1, 4, 5, 8, 9, 10$. Its jump is given by J_{23} across $\Sigma_2^{(9)} \cup \Sigma_3^{(9)}$ and by J_{67} across $\Sigma_6^{(9)} \cup \Sigma_7^{(9)}$, where

$$J_{23} = e^{-\{4i(z^3+qz)\}\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & 0 \\ r(T_1) & 1 \end{bmatrix}, \qquad J_{67} = e^{-\{4i(z^3+qz)\}\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & -\bar{r}(T_1) \\ 0 & 1 \end{bmatrix}.$$

Thus the RHP is reduced to one along $\Sigma_2^{(9)} \cup \Sigma_3^{(9)} \cup \Sigma_6^{(9)} \cup \Sigma_7^{(9)}$. It is not exactly a cross, but a simple deformation enables us to replace it by $\Sigma^{(10)}$ as in Fig. 10, the counterpart of the contour in [2, Fig. 5.6]. In the upper and lower halves, the jump matrix coincides with J_{23} and J_{67} respectively. We apply the argument in [2, pp. 357–360]. In particular, we employ the parameters p, q, r (roman font) in it. See Appendix for explanation. We use $|r(T_j)| < 1$ and $r(T_j) + \bar{r}(T_j) = 0$, the latter being true if the time variable t is replaced by $t - t_0$ for some t_0 (see the previous section).

We employ the notation explained in Appendix. We set $\mathrm{p} = r(T_1)$, $\mathrm{q} = -r(T_1) = \overline{r}(T_1)$, $\mathrm{r} = (\mathrm{p}+\mathrm{q})/(1-\mathrm{pq}) = 0$ and consider the solution $u(s; r(T_1), -r(T_1), 0)$ to the Painlevé II equation $u'' - su - 2u^3 = 0$. Since

$$4i\big(z^3 + qz\big) = \frac{4i}{3}\big(3^{1/3}z\big)^3 + i\frac{4q}{3^{1/3}}\big(3^{1/3}z\big),$$

we have

$$\lim_{z\to\infty}\big(-z\big[\sigma_3, m^{(8)}(z)\big]_{21}\big) = \frac{1}{3^{1/3}}u\left(\frac{4q}{3^{1/3}}; r(T_1), -r(T_1), 0\right). \tag{20}$$

We combine (20) with (19). The result is

$$R_n(t) = \frac{e^{2p-\pi i/4}\alpha}{(3t)^{1/3}}u\left(\frac{4q}{3^{1/3}}; r(T_1), -r(T_1), 0\right) + O\big(t^{-2/3}\big).$$

Theorem 1 holds at least in the region (4).

7 Asymptotics in the remaining part of Region B

We consider the long-time asymptotics in the region

$$2t \le n < 2t + Mt^{1/3}, \tag{21}$$

where M' is an arbitrary positive constant. It is the 'right-hand half' of the Region B defined by (3).

If $2t = n$, then the function $\varphi(z)$ has no saddle points. Indeed, S_j and S_{j+1} ($j = 1,\ 3$) coalesce. If $2t < n$, then $\varphi(z)$ has four saddle points on the line $\mathrm{Re}\, z + \mathrm{Im} z = 0$. Set $A = 2^{-1}(\sqrt{2+n/t} + \sqrt{-2+n/t})$, $A' = 2^{-1}(\sqrt{2+n/t} - \sqrt{-2+n/t})$, then the four saddle points are $\pm e^{-\pi i/4}A$ and $\pm e^{-\pi i/4}A'$. Notice that $A > 1$, $AA' = 1$, $0 < A' < 1$.

For $z = re^{i\theta}$ (here r is not the reflection coefficient), we have $\mathrm{Re}\,\varphi = \frac{-1}{2}t(r^2 - r^{-2})\sin 2\theta - n\log r$. It vanishes for any θ if $r = 1$. If $r \ne 1$, the equation $\mathrm{Re}\,\varphi = 0$ is equivalent to saying that

$$\sin 2\theta = -\frac{2n}{t}\frac{\log r}{r^2 - r^{-2}}.$$

The function $\log r/(r^2 - r^{-2})$ can be continuously extended to $0 < r < \infty$. It is strictly increasing in $0 < r < 1$ and is strictly decreasing in $r > 1$. It attains its maximum $1/4$ at $r = 1$. We can calculate the number of solutions θ (modulo 2π) for each fixed value of r. Figs. 1, 11 and 12 show the curve $\mathrm{Re}\,\varphi = 0$ in the cases $n < 2t$, $n = 2t$ and $2t < n$ respectively.

Set $\psi_0 = \psi_0(z) = 2^{-1}(z - z^{-1})^2 + 2i\log z$. It is nothing but what ψ is if $n = 2t$. We employ it as the Fourier variable in the region (21), not only on the ray $n = 2t$. Then we get a decomposition like (9) on $\mathrm{arc}(T_2T_3)$ and on $\mathrm{arc}(T_4T_1)$, where $T_1 = T_2 = e^{-\pi i/4}$ and $T_3 = T_4 = e^{3\pi i/4}$. In the formulas below, h_{I}, h_{II} etc. denote the terms obtained by this decomposition.

Set $\varphi_0 = it\psi_0$. We have $\mathrm{Re}\,\varphi > \mathrm{Re}\,\varphi_0$ in $|z| < 1$, and $\mathrm{Re}\,\varphi < \mathrm{Re}\,\varphi_0$ in $|z| > 1$. We introduce a new contour $\Sigma^{(11)}$ as in Fig. 13. Notice that $L^{(11)}$ and $L'^{(11)}$ are in $\{\mathrm{Re}\,\varphi_0 > 0, |z| < 1\}$ and $\{\mathrm{Re}\,\varphi_0 < 0, |z| > 1\}$ respectively.

In the same way as (15), we can derive estimates of $|e^{-2\varphi_0}h_{\mathrm{II}}|$ on $L^{(11)}$. It is good enough even in the case $n > 2t$, because we have $|e^{-2\varphi}h_{\mathrm{II}}| \le |e^{-2\varphi_0}h_{\mathrm{II}}|$ on $L^{(11)}$. By this observation, we can perform a simplified version of the argument in the preceding section. We conclude that Theorem 1 holds in the whole region (3).

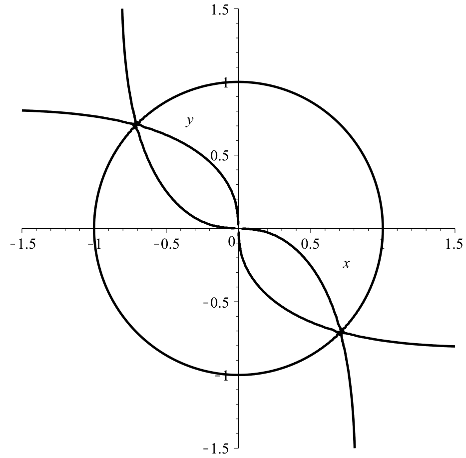

Figure 11. $n = 2t$.

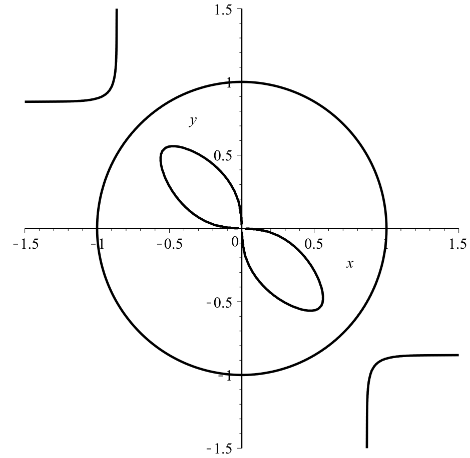

Figure 12. $n > 2t$.

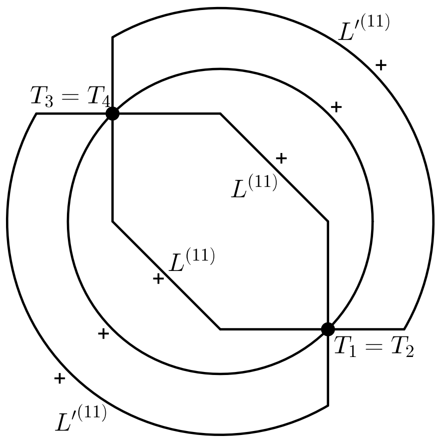

Figure 13. $\Sigma^{(11)}$.

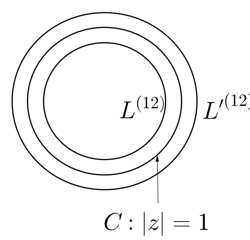

Figure 14. $\Sigma^{(12)}$.

8 Region C

We consider the case $2t < n \to \infty$. The four saddle points of φ are not on the circle $C : |z| = 1$. Two of them are inside and the other two are outside. For $z = re^{i\theta}$, we have

$$\mathrm{Re}\left[\frac{2\varphi}{n}\right] = -\frac{t}{n}\left(r^2 - r^{-2}\right)\sin 2\theta - 2\log r.$$

Set $f(r) = n^{-1}t(r^2 - r^{-2}) - 2\log r$. If $r > 1$, then $\mathrm{Re}[2\varphi/n] \leq f(r)$ and if $r < 1$, then $\mathrm{Re}[-2\varphi/n] \leq -f(r)$. Notice that $f(1) = 0$, $f'(1) = 2(2t - n)/n < 0$. If $r > 1$ is sufficiently close to 1, then we have $\mathrm{Re}[2\varphi/n] < 0$. On the other hand, if $r < 1$ is sufficiently close to 1, then we have $\mathrm{Re}[-2\varphi/n] < 0$.

We introduce a contour as in Fig. 14 consisting of three concentric circles $L^{(12)}$, $L'^{(12)}$ and $C : |z| = 1$. Their radii are sufficiently close. There exists a positive number $p = p(V_0) < 1$ such that $|e^{-2\varphi}| \leq p^n$ on $L^{(12)}$ and $|e^{2\varphi}| \leq p^n$ on $L'^{(12)}$.

Since $r(z)$ is smooth on $|z| = 1$, its complex conjugate can be written in terms of a Fourier series:

$$\bar{r}(z) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} = \sum_{k=-\infty}^{\infty} a_k z^k.$$

For any $\alpha \in \mathbb{N}$, there exists a constant $A_\alpha > 0$ such that $|a_k| \le A_\alpha/|k|^{\alpha+1}$ holds for any $k \in \mathbb{Z}$. If $r(z)$ is analytic, then a contour deformation leads to $a_k = (2\pi i)^{-1} \int_{|z|=1\pm\varepsilon} z^{-k-1}\bar{r}(z)dz$. So a_k is exponentially decreasing: $|a_k| \le \mathrm{const}(1\pm\varepsilon)^k$.

Set $h_{\mathrm{I}}(z) = \sum_{k<-n} a_k z^k$, $h_{\mathrm{II}}(z) = \sum_{k\ge -n} a_k z^k$, $\bar{h}_{\mathrm{I}}(z) = \sum_{k<-n} \bar{a}_k z^{-k}$ and $\bar{h}_{\mathrm{II}}(z) = \sum_{k\ge -n} \bar{a}_k z^{-k}$. We have $r(z) = \bar{h}_{\mathrm{I}}(z) + \bar{h}_{\mathrm{II}}(z)$. We employ z^{-k} rather than $\bar{z}^k$ with analytic continuation in mind. Indeed, h_{II} and $\bar{h}_{\mathrm{II}}$ can be analytically continued up to $L^{(12)}$ and $L'^{(12)}$ respectively. It is easy to see that h_{I} and $\bar{h}_{\mathrm{I}}$ decay faster than any negative power as $n \to \infty$ on the circle, since they would have fewer terms. On the other hand, we can show that $e^{-2\varphi}h_{\mathrm{II}}$ and $e^{2\varphi}\bar{h}_{\mathrm{II}}$ decay exponentially on $L^{(12)}$ and $L'^{(12)}$ respectively.

We define a new jump matrix $v^{(12)}$ by

$$v^{(12)} = \begin{cases} e^{-\varphi\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & -h_{\mathrm{II}} \\ 0 & 1 \end{bmatrix} & \text{on } \ L^{(12)}, \\ e^{-\varphi\,\mathrm{ad}\,\sigma_3} \left\{ \begin{bmatrix} 1 & -h_{\mathrm{I}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \bar{h}_{\mathrm{I}} & 1 \end{bmatrix} \right\} & \text{on } \ |z| = 1, \\ e^{-\varphi\,\mathrm{ad}\,\sigma_3} \begin{bmatrix} 1 & 0 \\ \bar{h}_{\mathrm{II}} & 1 \end{bmatrix} & \text{on } \ L'^{(12)}. \end{cases}$$

The factorization problem (5)–(7) is equivalent to the one involving $v^{(12)}$. We can show that $v^{(12)}$ tends to the identity matrix as $n \to \infty$. The error is smaller than any negative power of n. Indeed, we have exponential decay on $L^{(12)}$ and $L'^{(12)}$ due to φ. The decay on the circle $|z| = 1$ is not so good in general. If $r(z)$ is analytic, however, h_{I} and $\bar{h}_{\mathrm{I}}$ decay exponentially as $n \to \infty$. This completes the proof of Theorem 2.

A Parametrization of the Painlevé functions

For readers' convenience, we collect some useful facts employed in [2].

Let p, q and r be constants satisfying the constraint $\mathrm{r} = \mathrm{p} + \mathrm{q} + \mathrm{pqr}$. We define six matrices S_i by

$$S_1 = \begin{bmatrix} 1 & 0 \\ \mathrm{p} & 1 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 1 & \mathrm{r} \\ 0 & 1 \end{bmatrix}, \qquad S_3 = \begin{bmatrix} 1 & 0 \\ \mathrm{q} & 1 \end{bmatrix},$$
$$S_4 = \begin{bmatrix} 1 & -\mathrm{p} \\ 0 & 1 \end{bmatrix}, \qquad S_5 = \begin{bmatrix} 1 & 0 \\ -\mathrm{r} & 1 \end{bmatrix}, \qquad S_6 = \begin{bmatrix} 1 & -\mathrm{q} \\ 0 & 1 \end{bmatrix}.$$

We introduce the contour $\Sigma^{(13)}$ (the intersection is the origin and all the rays are oriented outward) and the regions $\Omega_i^{(13)}$ in Fig. 15. Then we consider the Riemann–Hilbert problem

$$\Psi_{i+1}(s,z) = \Psi_i(s,z)S_j \qquad \text{on} \quad \Sigma_i^{(13)} (1 \le i \le 6),$$

where Ψ_i ($\Psi_7 = \Psi_1$) is holomorphic in $\Omega_i^{(13)}$. It has a unique solution with the asymptotics

$$\Psi(s,z) = \left(I + \frac{(\hat{Y}_i)_1}{z} + \frac{(\hat{Y}_i)_2}{z^2} + \cdots \right) e^{-([4i/3]z^3 + isz)\sigma_3}$$

as $z \to \infty$ in $\Omega_i^{(13)}$. The function u defined by

$$u = u(s; \mathrm{p}, \mathrm{q}, \mathrm{r}) = -\lim_{z\to\infty} z\big[\sigma_3, \hat{Y}_i(z)\big]_{21},$$

where the limit is taken with respect to $z \in \Omega_i^{(13)}$ for any $i \in \{1, \ldots, 6\}$, satisfies the Painlevé II equation $u''(s) - su(s) - 2u^3(s) = 0$.

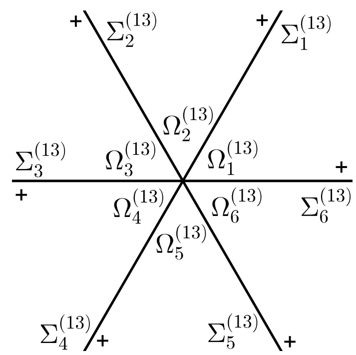

Figure 15. $\Sigma^{(13)}$.

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