

# Harmonic Superfields in $\mathcal{N} = 4$ Supersymmetric Quantum Mechanics<sup>\*</sup>

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**Abstract.** This is a brief survey of applications of the harmonic superspace methods to the models of  $\mathcal{N} = 4$  supersymmetric quantum mechanics (SQM). The main focus is on a recent progress in constructing SQM models with couplings to the background non-Abelian gauge fields. Besides reviewing and systemizing the relevant results, we present some new examples and make clarifying comments.

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## 1 Introduction

Supersymmetric quantum mechanics (SQM) [1] is the simplest ( $d = 1$ ) supersymmetric theory. It has plenty of applications in various domains. Some salient features of the SQM models and their uses were already discussed during this Benasque meeting. Let us recall some other ones:

- SQM models are capable to catch characteristic properties of the higher-dimensional supersymmetric theories via the dimensional reduction [2];
- They provide superextensions of integrable models like Calogero–Moser systems [3, 4] and Landau-type models [5];
- An extended supersymmetry in  $d = 1$  is specific; it exhibits some features which are not shared by its  $d > 1$  counterparts. These are the so-called automorphic dualities between various supermultiplets with the different off-shell contents [6], the existence of nonlinear “cousins” of off-shell linear multiplets [7, 8], and some other ones.

An efficient tool to deal with extended supersymmetries in  $d > 1$  is known to be the *harmonic superspace* (HSS) [9, 10]. The HSS approach allowed to construct, for the first time, an off-shell formulation of hypermultiplets in  $\mathcal{N} = 2$ ,  $d = 4$  and  $\mathcal{N} = 1$ ,  $d = 6$  supersymmetry, as well as a formulation of  $\mathcal{N} = 4$ ,  $d = 4$  supersymmetric Yang–Mills theory with the maximal number  $\mathcal{N} = 3$  of off-shell supersymmetries (at cost of an infinite number of the auxiliary fields appearing in the harmonic expansions of the relevant superfields). Some further consequences of the HSS approach for the  $d > 1$  supersymmetric theories are listed, e.g., in [10, 11].

A natural extension of the HSS approach was applying it to  $d = 1$  supersymmetric theories, i.e. SQM models. An  $\mathcal{N} = 4$ ,  $d = 1$  version of the  $\mathcal{N} = 2$ ,  $d = 4$  HSS was worked out in [8]. It proved to be a powerful device of the  $\mathcal{N} \geq 4$  SQM model-building, as well as of getting new insights into the structure of  $d = 1$  supersymmetries and their representations. In particular, it allowed one to understand interrelations between various  $\mathcal{N} = 4$  SQM models via the manifestly

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$\mathcal{N} = 4$  covariant gauging procedure [12, 13, 14]. As one more important application, it helped to construct new  $\mathcal{N} = 4$  superextensions of the Calogero-type models [4].

The latest developments of the  $d = 1$  HSS approach concern applications in SQM models with the Lorentz-force type couplings to the external gauge fields, i.e. couplings of the form  $A_m(x(t))\dot{x}^m(t)$ . The major subject of this contribution is just a survey of these new applications from a common point of view, with some clarifying examples and further remarks.

Let us adduce some reasons why SQM models with external gauge fields are of interest.

One reason is that these models supply  $d = 1$  prototypes of the  $p$ -branes world-volume couplings. The other one is the close relation of such models to supersymmetric versions of the Wilson loops and Berry phase (see, e.g., [15]). Also, they provide superextensions of the Landau problem and of the quantum Hall effect (see, e.g., [5]) and give quantum-mechanical realizations of Hopf maps (see, e.g., [16]). At last, they yield, as the particular “extreme” case, superextensions of the Chern–Simons mechanics [17].

Our consideration will be limited to the  $\mathcal{N} = 4$  SQM models with the background gauge field<sup>1</sup>. Until recently, only  $\mathcal{N} = 4$  superextensions of the couplings to *Abelian* background gauge fields were known. Their off-shell formulation within the  $\mathcal{N} = 4$ ,  $d = 1$  HSS setting was given in the paper [8]. The coupling to *non-Abelian* gauge backgrounds was recently constructed in the papers [19, 20, 21] (see also [22, 23]). This construction essentially exploits the *semi-dynamical* (or *spin*, or *isospin*) supermultiplet  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  [4]. Bosonic fields of the spin multiplet are described by the U(1) gauged Wess–Zumino  $d = 1$  action and, after quantization, yield generators of the gauge group SU(2). A key role is also played by the manifestly  $\mathcal{N} = 4$  supersymmetric  $d = 1$  gauging procedure worked out in the papers [12, 13, 14]. It turns out that the requirement of off-shell  $\mathcal{N} = 4$  supersymmetry forces the external gauge potential to be (a) *self-dual* and (b) satisfying the 4D ’t Hooft ansatz or its 3D reduction. Moreover, in this case the gauge group should be SU(2). On the other hand, the *on-shell*  $\mathcal{N} = 4$  supersymmetry is compatible with the *general* self-dual background and an arbitrary gauge group [24, 19].

Most of the results reported here were obtained together with Francois Delduc, Sergey Fedoruk, Maxim Konyushikhin, Olaf Lechtenfeld, Jiri Niederle<sup>2</sup> and Andrei Smilga.

## 2 Harmonic $\mathcal{N} = 4$ , $d = 1$ superspace

As a prerequisite to the main subject, let us recall the salient features of the  $d = 1$  version of HSS.

### 2.1 From the ordinary $\mathcal{N} = 4$ , $d = 1$ superspace to the harmonic one

The ordinary  $\mathcal{N} = 4$ ,  $d = 1$  superspace is parametrized by the co-ordinates:

$$(t, \theta^\alpha, \bar{\theta}_\alpha), \quad \alpha = 1, 2.$$

Its harmonic extension is defined as:

$$(t, \theta^\alpha, \bar{\theta}_\alpha) \Rightarrow (t, \theta^\alpha, \bar{\theta}_\alpha, u_\alpha^\pm), \quad u^{+\alpha}u_\alpha^- = 1, \quad u_\alpha^\pm \in \text{SU}(2)_{\text{Aut}}. \quad (1)$$

The main property of this  $d = 1$  HSS is the existence of the so-called analytic basis in it<sup>3</sup>

$$(t_A, \theta^+, \bar{\theta}^+, u_\alpha^\pm, \theta^-, \bar{\theta}^-) \equiv (\zeta, u^\pm, \theta^-, \bar{\theta}^-), \\ \theta^\pm = \theta^\alpha u_\alpha^\pm, \bar{\theta}^\pm = \bar{\theta}^\alpha u_\alpha^\pm, \quad t_A = t + i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+).$$

<sup>1</sup>The on-shell  $\mathcal{N} = 2$  SQM models with couplings to a non-Abelian monopole background were considered, e.g., in [18].

<sup>2</sup>Deceased.

<sup>3</sup>The original parametrization (1) will be referred to as the “central basis”.

Passing to the analytic basis makes manifest the presence of the analytic subspace in the  $d = 1$  HSS:

$$(t_A, \theta^+, \bar{\theta}^+, u_\alpha^\pm) \equiv (\zeta, u^\pm) \subset (\zeta, u^\pm, \theta^-, \bar{\theta}^-).$$

It is closed by itself under the  $\mathcal{N} = 4$  supersymmetry, but has twice as less Grassmann coordinates compared to the full HSS. The superfields given on this subspace are called *analytic superfields*. They can be defined by the constraints which resemble the well known chirality condition:

$$D^+\Phi = \bar{D}^+\Phi = 0 \quad \Rightarrow \quad \Phi = \Phi(\zeta, u^\pm), \quad D^+ = \frac{\partial}{\partial\theta^-}, \quad \bar{D}^+ = -\frac{\partial}{\partial\bar{\theta}^-}. \quad (2)$$

An important ingredient of the HSS formalism is the harmonic derivatives, i.e. the derivatives with respect to the harmonic variables:

$$D^{\pm\pm} = u_\alpha^\pm \frac{\partial}{\partial u_\alpha^\mp} + \theta^\pm \frac{\partial}{\partial\theta^\mp} + \bar{\theta}^\pm \frac{\partial}{\partial\bar{\theta}^\mp} + 2i\theta^\pm\bar{\theta}^\pm \frac{\partial}{\partial t_A}.$$

The derivative  $D^{++}$  is distinguished in that it commutes with the spinor derivatives  $D^+$ ,  $\bar{D}^+$ . Then, if the superfield  $\Phi$  is analytic, the superfield  $D^{++}\Phi$  is analytic as well:

$$[D^+, D^{++}] = [\bar{D}^+, D^{++}] = 0 \quad \Rightarrow \quad D^{++}\Phi(\zeta, u^\pm) \text{ is analytic,} \\ D^+(D^{++}\Phi) = \bar{D}^+(D^{++}\Phi) = 0.$$

The harmonic derivatives form an  $SU(2)$  algebra:

$$[D^{++}, D^{--}] = D^0, \quad D^0 = u_\alpha^+ \frac{\partial}{\partial u_\alpha^+} - u_\alpha^- \frac{\partial}{\partial u_\alpha^-} + \theta^+ \frac{\partial}{\partial\theta^+} + \bar{\theta}^+ \frac{\partial}{\partial\bar{\theta}^+} - \theta^- \frac{\partial}{\partial\theta^-} - \bar{\theta}^- \frac{\partial}{\partial\bar{\theta}^-}.$$

The operator  $D^0$  counts the harmonic  $U(1)$  charge of the superfields given on the  $d = 1$  HSS. It preserves the analyticity and is reduced to its pure harmonic part in the central basis.

## 2.2 Basic $\mathcal{N} = 4$ , $d = 1$ multiplets

It turns out that the basic off-shell multiplets of  $\mathcal{N} = 4$ ,  $d = 1$  supersymmetry are represented by analytic harmonic superfields subjected to the proper additional constraints. Below we briefly characterize these multiplets and quote their free superfield actions. We shall use for them the abbreviation  $(\mathbf{b}, \mathbf{4}, \mathbf{4} - \mathbf{b})$ , with  $\mathbf{b}$  standing for the physical bosonic fields and  $\mathbf{4} - \mathbf{b}$  for the auxiliary bosonic fields<sup>4</sup>. Depending on the choice of the action, some of the fields having a ‘‘physical’’ engineering dimension can become auxiliary, i.e. appear in the component action without time derivative on them.

**1.  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  multiplet.** The multiplet  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  is described by the superfield  $q^{+a}(\zeta, u) \propto (x^{\alpha a}, \chi^a, \bar{\chi}^a)$ ,  $a = 1, 2$ , with the harmonic constraint

$$D^{++}q^{+a} = 0, \quad q^{+a} = x^{\alpha a}u_\alpha^+ - 2\theta^+\chi^a - 2\bar{\theta}^+\bar{\chi}^a - 2i\theta^+\bar{\theta}^+\dot{x}^{\alpha a}u_\alpha^-. \quad (3)$$

The free action of this multiplet reads

$$S_{\text{free}}(q) \sim \int dt d^4\theta du q^{+a} D^{--}q_a^+ \sim \int dt (\dot{x}^{\alpha a}\dot{x}_{\alpha a} + i\bar{\chi}^a\dot{\chi}_a). \quad (4)$$

<sup>4</sup>An additional set of admissible multiplets can be gained by changing the overall Grassmann parity of the relevant superfields.

It also admits a Wess–Zumino (WZ) term which, in its simplest variant, is given by the analytic subspace action

$$S_{\text{WZ}}(q) \sim \int dt_A du d\theta^+ d\bar{\theta}^+ C_{(ab)} q^{+a} q^{+b} \sim \int dt C_{(ab)} (ix^{\alpha a} \dot{x}_\alpha^b + 4\chi^a \bar{\chi}^b). \quad (5)$$

**2. (3, 4, 1) multiplet.** The multiplet **(3, 4, 1)** is described by the superfield  $L^{++}(\zeta, u) \propto (\ell^{(\alpha\beta)}, \psi^\alpha, \bar{\psi}^\alpha, F)$  which is subjected to the constraint

$$D^{++}L^{++} = 0, L^{++} = \ell^{\alpha\beta} u_\alpha^+ u_\beta^+ + i(\theta^+ \chi^\alpha + \bar{\theta}^+ \bar{\chi}^\alpha) u_\alpha^+ + \theta^+ \bar{\theta}^+ (F - 2i\dot{\ell}^{\alpha\beta} u_\alpha^+ u_\beta^-).$$

The free action is:

$$S_{\text{free}}(\ell) \sim \int dt d^4\theta du L^{++} (D^{--})^2 L^{++} \sim \int dt \left[ \left( \dot{\ell}^{\alpha\beta} \dot{\ell}_{\alpha\beta} - \frac{1}{2} F^2 \right) + i\bar{\psi}^\alpha \dot{\psi}_\alpha \right].$$

This multiplet also admits WZ-type  $\mathcal{N} = 4$  superfield invariants.

**3. Gauge multiplet.** An important multiplet is the gauge one described by an unconstrained analytic superfield  $V^{++}(\zeta, u)$ , which exhibits the gauge freedom with an analytic superfield parameter:

$$V^{++'} = V^{++} + D^{++}\Lambda, \quad \Lambda = \Lambda(\zeta, u).$$

This gauge freedom can be fixed so as to bring  $V^{++}$  into the Wess–Zumino gauge with only one component  $B(t)$  ( $d = 1$  “gauge field”):

$$V_{\text{WZ}}^{++} = 2i\theta^+ \bar{\theta}^+ B(t), \quad \delta B = \dot{\lambda}(t).$$

**4. Gauged (4, 4, 0) multiplet.** Using the superfield  $V^{++}$ , one can define a gauged version of the multiplet **(4, 4, 0)**. It is represented by the superfields  $(v^+, \bar{v}^+)$ ,  $v^{+'} = e^{i\Lambda} v^+$ ,  $\bar{v}^{+'} = e^{-i\Lambda} \bar{v}^+$ , obeying a gauge-covariant version of the constraint in **(3)**:

$$(D^{++} + iV^{++})v^+ = 0 \quad \Rightarrow \quad v^+ = \phi^\alpha u_\alpha^+ + \theta^+ \omega_1 + \bar{\theta}^+ \bar{\omega}_2 - 2i\theta^+ \bar{\theta}^+ (\dot{\phi}^\alpha + iB\phi^\alpha) u_\alpha^-.$$

This multiplet (“spin multiplet”) is an important ingredient of SQM models with non-Abelian background gauge fields (Sections 4 and 5).

**5. Some other multiplets.** Of use in the  $\mathcal{N} = 4$  SQM model-building are also a fermionic counterpart **(0, 4, 4)** of the multiplet **(4, 4, 0)**, as well as the multiplet **(1, 4, 3)** and the chiral multiplet **(2, 4, 2)**. The first multiplet is described by a fermionic analog  $\Psi^{+A}(\zeta, u)$  of the superfield  $q^{+a}$ , with the harmonic constraint

$$D^{++}\Psi^{+A} = 0 \quad \Rightarrow \quad \Psi^{+A} = \psi^{\alpha A} u_\alpha^+ + \theta^+ d^A + \bar{\theta}^+ \bar{d}^A - 2i\theta^+ \bar{\theta}^+ \dot{\psi}^{\alpha A} u_\alpha^-, \quad (6)$$

and the free action

$$S_{\text{free}}(\Psi) \sim \int dt_A du d\theta^+ d\bar{\theta}^+ \Psi^{+A} \Psi_A^+ \sim \int dt (i\psi^{\alpha A} \dot{\psi}_{\alpha A} - d^A \bar{d}_A).$$

The multiplets **(1, 4, 3)** and **(2, 4, 2)** can be also described within the harmonic superspace setting, though in a rather indirect manner [13, 14].

Most of the analytic  $\mathcal{N} = 4$  multiplets listed here have their *nonlinear* counterparts, with the nonlinearly modified harmonic constraints. Their implications in the  $\mathcal{N} = 4$  SQM models have not yet been fully explored so far. Also, in accordance with the fact that the full automorphism group of  $\mathcal{N} = 4$ ,  $d = 1$  superalgebra is  $\text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2)$ , each  $\mathcal{N} = 4$  supermultiplet from the above list has its “mirror” (or “twisted”) counterpart, with the two  $\text{SU}(2)$  automorphism groups switching their roles.

The free actions of all these  $\mathcal{N} = 4$  multiplets can be generalized to involve a non-trivial self-interaction. The corresponding bosonic manifolds exhibit interesting target space geometries.

### 2.3 Bi-harmonic superfields

A unified description of  $\mathcal{N} = 4$  supermultiplets and their mirror cousins is achieved in the framework of *bi-harmonic*  $\mathcal{N} = 4$ ,  $d = 1$  HSS [25], with the two independent sets of harmonic variables  $u^{\pm 1\alpha}$ ,  $v^{\pm 1i}$ ,  $u^{1\alpha}u_{\alpha}^{-1} = 1$ ,  $v^{1i}v_i^{-1} = 1$ , for either two mutually commuting  $SU(2)$  automorphism groups of  $\mathcal{N} = 4$ ,  $d = 1$  supersymmetry. In this approach, the  $\mathcal{N} = 4$ ,  $d = 1$  spinor derivatives are combined into the  $SU(2) \times SU(2)$  quartet,

$$D^{\alpha i} = (D^{\alpha}, \bar{D}^{\alpha}),$$

and then are split into a set of bi-harmonic projections

$$D^{\alpha i} \Rightarrow (D^{1,1}, D^{1,-1}, D^{-1,1}, D^{-1,-1}), \quad \text{where } D^{\pm 1\pm 1} = D^{\alpha i}u_{\alpha}^{\pm 1}v_i^{\pm 1}, \quad \text{etc.}$$

In this language, the standard harmonic analytic superfields discussed in the previous subsections are defined by the constraints

$$(a) \quad D^{1,1}\Phi^{(I,0)} = D^{1,-1}\Phi^{(I,0)} = 0, \quad (b) \quad D^{0,2}\Phi^{(I,0)} = 0, \quad (7)$$

where  $I$  is the harmonic charge with respect to the  $u$ -harmonics and  $D^{0,2}$  is the analyticity-preserving covariant derivative with respect to the  $v$ -harmonics (the harmonic constraint (7b) just eliminates the  $v$ -dependence in the central basis). The mirror multiplets are represented by the alternative analytic superfields  $\Phi^{(0,J)}$ :

$$(a) \quad D^{1,1}\Phi^{(0,J)} = D^{-1,1}\Phi^{(0,J)} = 0, \quad (b) \quad D^{2,0}\Phi^{(0,J)} = 0,$$

with  $D^{2,0}$  being the same as  $D^{++}$  defined above and  $J$  the harmonic charge associated with the  $v$ -harmonics. These two types of  $\mathcal{N} = 4$ ,  $d = 1$  harmonic analyticity conditions cannot be imposed on the bi-harmonic superfields simultaneously, since  $\{D^{1,-1}, D^{-1,1}\} \sim \partial_t$ .

One of the advantages of the bi-harmonic approach is that it makes manifest both  $SU(2)$  automorphism groups in their realization on the component fields. For instance, two mutually mirror  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  multiplets amount to the following sets of  $d = 1$  fields:  $(x^{\alpha\alpha}, \chi^{ia})$  and  $(x^{ia'}, \chi^{\alpha\alpha'})$ , the first set being another form of the multiplet (3). Analogously, there are two versions of the fermionic off-shell multiplet  $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ ,  $(\psi^{\alpha A}, d^{iA})$  and  $(\psi^{iA'}, d^{\alpha A'})$ .

The bi-harmonic formalism also helps to establish direct relations between various  $\mathcal{N} = 4$ ,  $d = 1$  multiplets and their twisted cousins. Consider, e.g., a twisted version  $\Psi^{(0,1)A'}$  of the fermionic multiplet (6):

$$\begin{aligned} D^{1,1}\Psi^{(0,1)A'} &= D^{-1,1}\Psi^{(0,1)A'} = 0, & D^{2,0}\Psi^{(0,1)A'} &= D^{0,2}\Psi^{(0,1)A'} = 0 \\ \Rightarrow \Psi^{(0,1)A'} &\propto (\psi^{iA'}, d^{\alpha A'}). \end{aligned}$$

Then the bosonic superfield

$$Q^{(1,0)A'} \equiv D^{1,-1}\Psi^{(0,1)A'}$$

satisfies the standard  $u$ -type harmonic analyticity constraints (2), (3)

$$D^{1,1}Q^{(1,0)A'} = D^{1,-1}Q^{(1,0)A'} = 0, \quad D^{2,0}Q^{(1,0)A'} = D^{0,2}Q^{(1,0)A'} = 0,$$

and so it is a ‘‘composite’’ version of the  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  multiplet,  $Q^{(1,0)A'} \propto (d^{\alpha A'}, \chi^{iA'} \sim \dot{\psi}^{iA'})$ . The invariants (4) and (5), upon substitution  $q^{+a} \equiv q^{(1,0)a} \Rightarrow Q^{(1,0)A'}$ , would produce some non-minimal actions for the multiplet  $\Psi^{(0,1)A'}$  with non-canonical numbers of time derivatives on the fermionic field  $\psi^{iA'}$ ; in particular, the WZ-type invariant (5) would contain the term with two derivatives  $\sim \int dt \dot{\psi}^{iA'} \dot{\psi}_i^{B'} C_{(A'B')}$ .

For simplicity, in the subsequent consideration we shall stick to the standard  $\mathcal{N} = 4$ ,  $d = 1$  HSS with one set of harmonic variables  $u^{\pm\alpha}$ .

### 3 Gauging in $\mathcal{N} = 4$ , $d = 1$ HSS

The  $\mathcal{N} = 4$ ,  $d = 1$  superfield gauging procedure has been worked out in [12, 13, 14]. It allows one to relate various  $\mathcal{N} = 4$  multiplets and their invariant superfield actions with preserving, at each step, the manifest  $\mathcal{N} = 4$ ,  $d = 1$  supersymmetry.

#### 3.1 A simple example of $d = 1$ gauging in bosonic system

Let us start with a simple clarifying bosonic example. Consider a complex  $d = 1$  field  $z(t)$ ,  $\bar{z}(t)$  with the following Lagrangian:

$$L_0 = \dot{z}\dot{\bar{z}} + i\kappa(\dot{z}\bar{z} - z\dot{\bar{z}}). \quad (8)$$

The first term is the kinetic energy, the second one is the simplest  $d = 1$  WZ term. One of the symmetries of this system is the invariance under U(1) transformations:

$$z' = e^{-i\lambda}z, \quad \bar{z}' = e^{i\lambda}\bar{z}.$$

Now we gauge this symmetry by promoting  $\lambda \rightarrow \lambda(t)$ . The gauge invariant action involves the  $d = 1$  gauge field  $A(t)$

$$L_{\text{gauge}} = (\dot{z} + iAz)(\dot{\bar{z}} - iA\bar{z}) + i\kappa(\dot{z}\bar{z} - z\dot{\bar{z}} + 2iAz\bar{z}) + 2cA, \quad A' = A + \dot{\lambda},$$

where a ‘‘Fayet–Iliopoulos term’’  $\sim c$  has been also added. This term is gauge invariant (up to a total derivative) by itself.

The next step is to choose the appropriate gauge in  $L_{\text{gauge}}$ :

$$z = \bar{z} \equiv \rho(t).$$

We substitute it into  $L_{\text{gauge}}$  and obtain:

$$L_{\text{gauge}} = (\dot{\rho} + iA\rho)(\dot{\rho} - iA\rho) + 2i\kappa A\rho^2 + 2cA = (\dot{\rho})^2 + A^2\rho^2 - 2\kappa A\rho^2 + 2cA.$$

The field  $A(t)$  is the typical example of auxiliary field: it can be eliminated by its algebraic equation of motion:

$$\delta A : \quad A = \kappa - \frac{c}{\rho^2}.$$

The final form of the gauge-fixed action is as follows

$$L_{\text{gauge}} \Rightarrow (\dot{\rho})^2 - \left( \kappa\rho - \frac{c}{\rho} \right)^2. \quad (9)$$

This is a one-particle prototype of the renowned Calogero–Moser system. At  $\kappa = 0$ , one recovers the standard conformal mechanics:

$$L_{\text{gauge}}^{(\kappa=0)} = (\dot{\rho})^2 - \frac{c^2}{\rho^2}.$$

This gauging procedure can be interpreted as an off-shell Lagrangian analog of the well known Hamiltonian reduction. In the present case, in the parametrization  $z = \rho e^{i\varphi}$ , the Hamiltonian reduction consists in imposing the constraints  $p_\varphi - 2c \approx 0$ ,  $\varphi \approx 0$ , upon which the Hamiltonian of the system (8) is reduced to the Hamiltonian of (9).

### 3.2 An example of supersymmetric gauging in $\mathcal{N} = 4$ , $d = 1$ HSS

Now we start from the free action of the multiplet  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ ,

$$S = \int dt d^4\theta du q^{+a} D^{--} q_a^+. \quad (10)$$

It is invariant under the shifts  $q^{+a} \rightarrow q^{+a} + \lambda u^{+a}$ ,  $a = 1, 2$ . The gauging of this Abelian symmetry is accomplished by replacing  $\lambda \rightarrow \Lambda(\zeta, u)$  and properly covariantizing (3)

$$\begin{aligned} D^{++} q^{+a} = 0 &\Rightarrow \nabla^{++} q^{+a} = D^{++} q^{+a} - V^{++} u^{+a} = 0, \\ S \Rightarrow S_g = \int dt d^4\theta du q^{+a} \nabla^{--} q_a^+, \quad \nabla^{--} q_a^+ &= D^{--} q^{+a} - V^{--} u^{+a}, \\ [\nabla^{++}, \nabla^{--}] = D^0 &\Rightarrow D^{++} V^{--} - D^{--} V^{++} = 0, \quad V^{--} = V^{--}(V^{++}, u). \end{aligned}$$

As the next step, we choose the gauge  $u^{-a} q_a^+ = 0 \Rightarrow q^{+a} = u^{-a} L^{++}$ . Then

$$\begin{aligned} D^{++} q^{+a} - V^{++} u^{+a} = 0 &\Rightarrow V^{++} = L^{++}, \quad D^{++} L^{++} = 0, \\ D^{++} V^{--} - D^{--} L^{++} = 0 &\Rightarrow V^{--} = \frac{1}{2} (D^{--})^2 L^{++}, \\ S_g = \int dt d^4\theta du V^{--} L^{++} &= \frac{1}{2} \int dt d^4\theta du L^{++} (D^{--})^2 L^{++}. \end{aligned}$$

Thus, starting from the free action of the  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  multiplet and gauging a symmetry of this action, we have eventually arrived at the free action of the multiplet  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ ! As distinct from the previous example, in the present case the gauging procedure does not produce any interaction of the multiplet  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ . Such an interaction could be induced [12] if we would gauge another Abelian symmetry of the action (10), that with respect to the U(1) transformations  $\delta q^{+a} = \lambda C_b^a q^{+b}$ , where  $C_b^a$  is a constant traceless matrix,  $C_a^a = 0$ .

### 3.3 Further gaugings

The superfield gauging procedure just described can be equally applied to other  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  Lagrangians exhibiting some isometries and involving an interaction, equally as to other  $\mathcal{N} = 4$ ,  $d = 1$  multiplets. These multiplets and their superfield actions can be reproduced as the appropriate gaugings of the multiplet  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  or of some nonlinear generalizations of the latter. Below we give a list of such gaugings:

- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow$  the linear  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$  multiplet – via gauging of shifting or rotational U(1) symmetry of  $q^{+a}$ ;
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow$  the non-linear  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$  multiplet – via gauging target space scaling symmetry,  $q^{+a'} = \lambda q^{+a}$ ;
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow$  the chiral  $(\mathbf{2}, \mathbf{4}, \mathbf{2})$  multiplets – via gauging some two-generator solvable symmetries realized on  $q^{+a}$ ;
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow$  the  $(\mathbf{1}, \mathbf{4}, \mathbf{3})$  multiplet – via gauging  $SU(2)_{PG}$  symmetry,  $q^{+a'} = \lambda_b^a q^{+b}$ ;
- $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \Rightarrow$  the fermionic  $(\mathbf{0}, \mathbf{4}, \mathbf{4})$  multiplet – via gauging the semi-direct product of  $SU(2)_{PG}$  and the shift symmetry  $\delta q^{+a} = \lambda u^{+a}$ .

It is worth noting that the  $d = 1$  gauging procedure outlined here resembles the gauging of isometries by non-propagating gauge fields in  $d = 2$  sigma models, which provides a field-theoretical realization of  $T$ -duality [26]. There is an essential difference between the  $d = 1$  and  $d = 2$  cases, however. An important part of the  $d = 2$  procedure is the insertion into the

action, with a Lagrange multiplier, the condition that the corresponding gauge field strength is vanishing. No gauge field strength can be defined in  $d = 1$ , so no analogous constraint is possible. The gauge field finally becomes just the auxiliary field of another  $\mathcal{N} = 4$ ,  $d = 1$  multiplet, and its actual role is to produce some new potential terms in the on-shell action of the latter.

## 4 $\mathcal{N} = 4$ , $4D$ SQM models in the gauge field backgrounds

### 4.1 $\mathcal{N} = 4$ , $4D$ SQM with Abelian external gauge field

$\mathcal{N} = 4$  SQM model with coupling of  $(4, 4, 0)$  multiplet to the background Abelian gauge field is described by the action [8]:

$$S = \int dt d^4\theta du R_{\text{kin}}(q^{+a}, D^{--}q^{+b}, u) + \int dt_A dud\theta^+ d\bar{\theta}^+ \mathcal{L}^{+2}(q^{+a}, u) \equiv S_1 + S_2. \quad (11)$$

The second term in (11) involves only one time derivative on the bosonic field  $x^{\alpha a}$ , and so it is an example of  $d = 1$  WZ term

$$S_2 \sim \int dt (\mathcal{A}_{\alpha b}(x) \dot{x}^{\alpha b} + \text{fermions}), \quad \mathcal{A}_{\alpha b}(x) = \int du u_{\alpha}^{-} \frac{\partial \mathcal{L}^{+2}}{\partial q^{+b}} \Big|_{\theta=0},$$

$$\mathcal{F}_{\alpha b \beta d} = \partial_{\alpha b} \mathcal{A}_{\beta d} - \partial_{\beta d} \mathcal{A}_{\alpha b} = \epsilon_{\alpha \beta} \mathcal{F}_{(bd)}, \quad \mathcal{F}_{(\alpha \beta)} = 0 \quad (\text{self-duality condition}).$$

Thus  $\mathcal{N} = 4$  supersymmetry requires the external gauge field to be *self-dual*<sup>5</sup>. No such a requirement is implied, e.g., by  $\mathcal{N} = 2$ ,  $d = 1$  supersymmetry.

How to extend this to the most interesting non-Abelian case?

### 4.2 Non-Abelian self-dual background

The coupling to non-Abelian backgrounds can be accomplished by adding the ‘‘spin’’ multiplet  $(4, 4, 0)$  [20]. The relevant superfield action consists of the three pieces:

$$S = \int dt d^4\theta du R_{\text{kin}}(q^{+a}, D^{--}q^{+b}, u) - i \frac{k}{2} \int dt_A dud\theta^+ d\bar{\theta}^+ V^{++}$$

$$- \frac{1}{2} \int dt_A dud\theta^+ d\bar{\theta}^+ K(q^{+a}, u) v^+ \bar{v}^+ \equiv S_1 + S_2 + S_3,$$

where the new spin multiplet superfields obey the constraints:

$$(D^{++} + iV^{++})v^+ = (D^{++} - iV^{++})\bar{v}^+ = 0.$$

In the total action, the piece  $S_1$  describes a sigma-model type interaction of  $x^{\alpha a}$ :

$$S_1 \sim \int dt (f^{-2}(x) \dot{x}^{\alpha a} \dot{x}_{\alpha a} + \text{fermions}), \quad f^{-2}(x) \sim \int du \square R_{\text{kin}}|_{\theta=0} - \text{conformal factor}.$$

The term  $S_2$  is the one-dimensional ‘‘Fayet–Iliopoulos’’ term:

$$S_2 = k \int dt B.$$

<sup>5</sup>The analytic function  $\mathcal{L}^{+2}(x^{\alpha a} u_{\alpha}^+, u^{\pm})$  (prepotential) yields the *most general* solution of the  $\mathbb{R}^4$  self-duality constraint in the Abelian case [27, 10].

The term  $S_3$  is most important for our purpose. It is a generalized WZ term:

$$S_3 \sim \int dt \left[ i\bar{\varphi}^\alpha (\dot{\varphi}_\alpha + iB\varphi_\alpha) - \frac{1}{2}\bar{\varphi}^\beta \varphi_\gamma (\mathcal{A}_{\alpha\beta})_\beta^\gamma \dot{x}^{\alpha\beta} + \text{fermions without } \partial_t \right],$$

where

$$(\mathcal{A}_{\alpha\beta})_\beta^\gamma = \frac{i}{h} \left( \varepsilon_{\alpha\beta} \partial_b^\gamma h - \frac{1}{2} \delta_\beta^\gamma \partial_{\alpha b} h \right), \quad h(x) = \int du K(x^{+a}, u_\beta^\pm), \quad \square h(x) = 0.$$

The background gauge field  $\mathcal{A}_{\alpha\beta}$  is self-dual,  $\mathcal{F}_{\alpha\beta} = 0$ . It precisely matches with the general 't Hooft ansatz for 4D self-dual SU(2) gauge fields:

$$\begin{aligned} (\mathcal{A}_{\alpha\beta})_\gamma^\beta &\Rightarrow (\mathcal{A}_\mu)_\gamma^\beta = \frac{1}{2} \mathcal{A}_\mu^i (\sigma_i)_\gamma^\beta, \\ \mathcal{A}_\mu^i &= -\bar{\eta}_{\mu\nu}^i \partial_\nu \ln h(x), \quad \bar{\eta}_{ij}^k = \varepsilon_{kij}, \quad \bar{\eta}_{0i}^k = -\bar{\eta}_{i0}^k = \delta_{ki}, \quad i, j, k = 1, 2, 3. \end{aligned}$$

An instructive example is the one-instanton configuration on  $S^4$ :

$$ds^2 = \frac{4R^4 dx_\mu^2}{(x^2 + R^2)^2}, \quad \mathcal{A}_\mu^i = \frac{2R^2 \bar{\eta}_{\mu\nu}^i x_\nu}{x^2(x^2 + R^2)}.$$

It corresponds to the following choice of the functions  $K(x^{+a}, u)$  and  $h(x)$ :

$$K(x^{+a}, u_\beta^\pm) = 1 + \frac{1}{(c_a^- x^{+a})^2}, \quad h(x) = 1 + \frac{R^2}{x_\mu^2}, \quad c^{-a} = c^{\alpha a} u_\alpha^-, \quad R^2 = |c|^{-2},$$

and can be brought in the BPST form,  $\hat{\mathcal{A}}_\mu^i = \frac{2\eta_{\mu\nu}^i x_\nu}{x^2 + R^2}$ ,  $\hat{\mathcal{F}}_{\mu\nu}^i = -\frac{4R^2 \eta_{\mu\nu}^i}{(x^2 + R^2)^2}$ , by the singular gauge transformation

$$\mathcal{A}_\mu \rightarrow \hat{\mathcal{A}}_\mu = U^\dagger \mathcal{A}_\mu U + iU^\dagger \partial_\mu U, \quad U(x) = -i\sigma_\mu x_\mu / \sqrt{x^2}. \quad (12)$$

### 4.3 $\mathcal{N} = 4$ SQM with Yang monopole

As a by-product, our non-Abelian SQM construction solves the long-lasting problem of setting up  $\mathcal{N} = 4$  SQM model with *Yang* monopole as a background.

Let us consider the following bosonic Lagrangian:

$$L_{\mathbb{R}^5} = \frac{1}{2} (\dot{y}_5 \dot{y}_5 + \dot{y}_\mu \dot{y}_\mu) + \mathcal{B}_\mu^i(y) \frac{1}{2} (\bar{\varphi} \sigma^i \varphi) \dot{y}_\mu, \quad \mu = 1, 2, 3, 4, \quad (13)$$

where

$$\mathcal{B}_\mu^i = \frac{\eta_{\mu\nu}^i y_\nu}{r(r + y_5)}, \quad r = \sqrt{y_5^2 + y_\mu^2},$$

is the standard form of the Yang monopole potential [28]. Thus (13) describes a coupling of the non-relativistic particle  $(y_5, y_\mu)$  in the 5-dimensional Euclidean space  $\mathbb{R}^5$  to the external Yang monopole field.

After the polar decomposition of  $\mathbb{R}^5$  into the angular  $\mathbb{S}^4 \sim \{\tilde{y}_\mu\}$  and the radial  $r$  parts as

$$(y_5, y_\mu) \Rightarrow \left( r, \sqrt{1 - \tilde{y}_\mu^2}, \tilde{y}_\mu \right),$$

and passing to the stereographic-projection coordinates as

$$\tilde{y}_\mu = 2 \frac{x_\mu}{1 + x^2},$$

we obtain

$$L_{\mathbb{R}^5} = \frac{1}{2} \left\{ \dot{r}^2 + 4r^2 \frac{\dot{x}_\mu \dot{x}_\mu}{(1+x^2)^2} \right\} + \frac{2\eta_{\mu\nu}^i x_\nu \dot{x}_\mu}{1+x^2} \frac{1}{2} (\bar{\varphi} \sigma^i \varphi). \quad (14)$$

The external gauge field in this Lagrangian is just BPST instanton on  $S^4$ . Hence, if we set the radial coordinate  $r$  in (14) equal to a constant, this Lagrangian can be extended to a particular form of the Lagrangian of  $\mathcal{N} = 4$  SQM with the self-dual SU(2) gauge field.

Thus the 5D mechanics with the gauge coupling to Yang monopole and “frozen” radial coordinate  $r$  admits an extension to  $\mathcal{N} = 4$  SQM model. The radial coordinate can presumably be described by the  $\mathcal{N} = 4$  supermultiplet  $(\mathbf{1}, \mathbf{4}, \mathbf{3})$  which also admits a description in the  $\mathcal{N} = 4$  HSS [13] and so can be properly coupled to the set of the basic analytic superfields  $q^{+a}$ ,  $v^+$  and  $\bar{v}^+$ .

#### 4.4 Quantization of spin variables

The (iso)spin variables  $\varphi_\alpha$ ,  $\bar{\varphi}^\alpha$  play the pivotal role for attaining the  $\mathcal{N} = 4$  coupling to the external non-Abelian gauge fields. Let us dwell in some detail on their role in the quantum theory.

The relevant part of the total action reads:

$$S = \int dt [i\bar{\varphi}^\alpha (\dot{\varphi}_\alpha + iB\varphi_\alpha) + kB + \mathcal{A}_\mu^i T^i \dot{x}_\mu], \quad T^i = \frac{1}{2} \bar{\varphi}^\alpha (\sigma^i)_\alpha^\beta \varphi_\beta, \quad (15)$$

where

$$k = \text{integer}. \quad (16)$$

The condition (16) can be deduced from the requirement of invariance of the Euclidean path integral under topologically non-trivial gauge transformations [29]:

$$B(t) \rightarrow B(t) + \dot{\alpha}(t), \quad \varphi(t) \rightarrow e^{-i\alpha(t)} \varphi(t).$$

By varying with respect to the “gauge field”  $B(t)$ , one obtains the constraint on  $\varphi$ ,  $\bar{\varphi}$ :

$$\bar{\varphi}^\alpha \varphi_\alpha = k. \quad (17)$$

Applying the standard Dirac quantization procedure, one is left with the commutation relations:

$$[\varphi_\alpha, \bar{\varphi}^\beta] = \delta_\alpha^\beta, \quad [\varphi_\alpha, \varphi_\beta] = [\bar{\varphi}^\alpha, \bar{\varphi}^\beta] = 0, \quad \varphi_\alpha \rightarrow \partial / \partial \bar{\varphi}^\alpha.$$

After quantization, the constraint (17) becomes the condition on the wave function

$$\bar{\varphi}^\alpha \varphi_\alpha \Psi = \bar{\varphi}^\alpha \frac{\partial}{\partial \bar{\varphi}^\alpha} \Psi = k\Psi. \quad (18)$$

It restricts the wave functions to be homogeneous polynomials of  $\bar{\varphi}^\alpha$  of degree  $k$ .

The bilinear combinations of the spin variables  $T^i$  appearing in (15), after quantization are identified as SU(2) generators:

$$T^i \rightarrow T^i = \frac{1}{2} \bar{\varphi}^\alpha (\sigma^i)_\alpha^\beta \frac{\partial}{\partial \varphi^\beta}, \quad [T^i, T^k] = i\varepsilon^{ikl} T^l.$$

Taking into account the constraint (18), one derives

$$T^i T^i = \frac{1}{4} [(\bar{\varphi}^\alpha \varphi_\alpha)^2 + 2(\bar{\varphi}^\alpha \varphi_\alpha)] = \frac{k}{2} \left( \frac{k}{2} + 1 \right).$$

Thus  $T^i$  are generators of SU(2) in the irrep of spin  $k/2$ . An interesting feature is that this gauge SU(2) group is at the same time the R-symmetry group of  $\mathcal{N} = 4$  supersymmetry<sup>6</sup>.

<sup>6</sup>The gauge transformation (12) converts this SU(2) into another SU(2) which acts on the extra indices  $a$  of  $x^{aa}$  and commutes with  $\mathcal{N} = 4$  supersymmetry.

## 5 $\mathcal{N} = 4$ , 3D SQM in a non-Abelian monopole background

### 5.1 Superfield action in HSS

We can choose off-shell  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$  multiplet  $L^{++}(\zeta, u)$  instead of the  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  one  $q^{+a}(\zeta, u)$  as the dynamical (co-ordinate) multiplet and still keep the gauged  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  multiplet  $v^+(\zeta, u)$ ,  $\bar{v}^+(\zeta, u)$  to represent semi-dynamical spin degrees of freedom. This gives rise to  $\mathcal{N} = 4$ , 3D SQM with coupling to non-Abelian 3D gauge background [21].

The corresponding total superfield action is

$$S = \int dt d^4\theta du R_{\text{kin}}(L^{++}, L^{+-}, L^{--}, u) - \frac{ik}{2} \int dt_A dud\theta^+ d\bar{\theta}^+ V^{++} - \frac{1}{2} \int dt_A dud\theta^+ d\bar{\theta}^+ K(L^{++}, u) v^+ \bar{v}^+ \equiv S_1 + S_2 + S_3.$$

The first two pieces produce the kinetic sigma-model type term of the  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$  multiplet and Fayet–Iliopoulos term of the gauge  $\mathcal{N} = 4$  multiplet. The third piece describes the WZ-type superfield coupling of the co-ordinate multiplet to the external gauge background.

### 5.2 Component action

For simplicity, we choose the free action for  $L^{++}$ , with the Lagrangian  $\sim L^{++}(D^{--})^2 L^{++}$ . The component form of the bosonic part of the full action  $S$  is

$$S \rightarrow \int dt \left[ \frac{1}{2} \dot{\ell}_m^2 + \mathcal{A}_m^i T^i \dot{\ell}_m + i\bar{\varphi}^\alpha (\dot{\varphi}_\alpha + iB\varphi_\alpha) + kB + \frac{1}{8} F^2 + \frac{1}{2} F (U^i T^i) \right],$$

where  $i = 1, 2, 3$ ,  $m = 1, 2, 3$  and

$$\begin{aligned} \mathcal{A}_m^i &= -\varepsilon_{mni} \partial_n \ln h, & U^i &= -\partial_i \ln h, & T^i &= \frac{1}{2} \bar{\varphi}^\alpha (\sigma^i)_\alpha^\beta \varphi_\beta, \\ h(\ell) &= \int du K(\ell^{\alpha\beta} u_\alpha^+ u_\beta^+, u_\gamma^\pm), & \Delta h &= 0. \end{aligned}$$

The 3D gauge field  $\mathcal{A}_m^i$  and potential  $U^i$  are particular solutions of the *Bogomolny* equations

$$\mathcal{F}_{mn}^i = \varepsilon_{mns} \nabla_s U^i,$$

with

$$\mathcal{F}_{mn}^i = \partial_m \mathcal{A}_n^i - \partial_n \mathcal{A}_m^i + \varepsilon^{ikl} \mathcal{A}_m^k \mathcal{A}_n^l, \quad \nabla_m U^i = \partial_m U^i + \varepsilon^{ikl} \mathcal{A}_m^k U^l.$$

### 5.3 Quantization and SO(3) example

Quantization follows the same line as in the 4D case:

$$[T^i, T^k] = i\varepsilon^{ikl} T^l, \quad T^i T^i = \frac{k}{2} \left( \frac{k}{2} + 1 \right).$$

The Hamiltonian, in the case with the free kinetic term for  $\ell_m$ , is

$$H = \frac{1}{2} (\hat{p}_m - \mathcal{A}_m)^2 + \frac{1}{2} U^2 + \text{fermionic terms}, \quad U \equiv U^i T^i. \quad (19)$$

A new feature of the 3D case is the appearance of the ‘‘induced’’ potential term  $\sim U^i U^k T^i T^k$  which is generated as a result of elimination of the auxiliary field  $F$ . The system (19) provides a non-Abelian generalization of the  $\mathcal{N} = 4$  SQM model pioneered in [30].

As an example of the gauge-field background, let us quote the  $SO(3)$  invariant one:

$$h_{\text{so}(3)}(\ell) = c_0 + c_1 \frac{1}{\sqrt{\ell^2}} \quad \Rightarrow \quad \mathcal{A}_m^i = \varepsilon_{mni} \frac{\ell_n}{\ell^2} \frac{c_1}{c_1 + c_0 \sqrt{\ell^2}}, \quad U^i = \frac{\ell_i}{\ell^2} \frac{c_1}{c_1 + c_0 \sqrt{\ell^2}}.$$

In the limit  $c_0 = 0$  the background gauge field becomes *Wu–Yang* monopole [31]; the  $\mathcal{N} = 4$  SQM for this case was earlier constructed in [22] in a different approach<sup>7</sup>.

## 6 Summary and outlook

Let us summarize the basic contents of this contribution.

One of its incentives was to provide more evidence that the  $\mathcal{N} = 4$ ,  $d = 1$  harmonic superspace [8] is a useful tool of constructing and analyzing SQM models with  $\mathcal{N} = 4$  supersymmetry. It allows one to construct off-shell invariant actions, to establish interrelations between different multiplets, to reveal the relevant target geometries, and so on.

As one of the recent uses of the  $d = 1$  HSS, off-shell  $\mathcal{N} = 4$  supersymmetric couplings of the multiplets  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  and  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$  to the external non-Abelian gauge backgrounds were presented. They essentially exploit the auxiliary (iso)spin  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  multiplet. The background should be self-dual and be described by the 't Hooft ansatz or its static  $3D$  reduction. HSS is indispensable for setting up the relevant off-shell actions. It should be noticed that, for the time being, our *off-shell* superfield construction is limited to the 't Hooft ansatz and the gauge group  $SU(2)$ . It is still an open question how to extend it to the  $SU(N)$  gauge group and to the general self-dual backgrounds, e.g., to the renowned ADHM one.

Surprisingly, the *on-shell* actions, with all auxiliary fields eliminated, admit a direct extension to the gauge group  $SU(N)$  and general self-dual backgrounds [19, 20, 21]. This is attainable at cost of *on-shell* realization of  $\mathcal{N} = 4$  supersymmetry. It is interesting to inquire if it is possible to derive these models from some *off-shell* superfield approach.

We finish by indicating possible applications and further directions of study.

It would be interesting to extend our construction of couplings to the external non-Abelian gauge fields to the case of higher  $\mathcal{N}$ ,  $d = 1$  supersymmetries, e.g. to  $\mathcal{N} = 8$ . Also, an obvious task is to exploit some other  $\mathcal{N} = 4$  multiplets to represent the coordinate and/or spin variable sectors, e.g. nonlinear versions of the multiplets  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  and  $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ , the  $(\mathbf{2}, \mathbf{4}, \mathbf{2})$  multiplets, etc.

As for applications, it would be tempting to make use of the techniques based on the semi-dynamical spin supermultiplets for the explicit calculations of the world-line superextensions of non-Abelian Wilson loops, with the evolution parameter along the loop as a “time”. One more possible area of using the models constructed and their generalizations includes superextensions of Landau problem and higher-dimensional quantum Hall effect, as well as supersymmetric black-hole stuff.

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<sup>7</sup>In [22] and in some other works of these authors, it was suggested to describe the spin variables, originally introduced in [4] as a bosonic sector of the multiplet  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ , by the fermionic multiplet  $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ . On the road from the superfield action to the component one, these authors make non-canonical replacements of the time derivatives of the fermionic fields by new auxiliary fermionic fields. This procedure basically amounts to the construction of “composite”  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$  multiplet from the  $(\mathbf{0}, \mathbf{4}, \mathbf{4})$  one, as explained in the end of Section 2.3. In view of existence of the direct  $\mathcal{N} = 4$  off-shell superfield formulation of the multiplet  $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ , the use of the auxiliary fermionic multiplet as the starting point looks artificial and superfluous.

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## References

- [1] Witten E., Dynamical breaking of supersymmetry, *Nuclear Phys. B* **188** (1981), 513–554.
- [2] Smilga A.V., Low-dimensional sisters of Seiberg–Witten effective theory, in From Fields to Strings: Circumnavigating Theoretical Physics, Vol. 1, Editors M. Shifman et al., World Sci. Publ., Singapore, 2005, 523–528, [hep-th/0403294](#).
- [3] Wyllard N., (Super)conformal many-body quantum mechanics with extended supersymmetry, *J. Math. Phys.* **41** (2000), 2826–2838, [hep-th/9910160](#).  
Galajinsky A., Polovnikov K., Lechtenfeld O.,  $\mathcal{N} = 4$  superconformal Calogero models, *J. High Energy Phys.* **2007** (2007), no. 11, 008, 23 pages, [arXiv:0708.1075](#).  
Krivonos S., Lechtenfeld O., Polovnikov K.,  $\mathcal{N} = 4$  superconformal  $n$ -particle mechanics via superspace, *Nuclear Phys. B* **817** (2009), 265–283, [arXiv:0812.5062](#).
- [4] Fedoruk S., Ivanov E., Lechtenfeld O., Supersymmetric Calogero models by gauging, *Phys. Rev. D* **79** (2009), 105015, 6 pages, [arXiv:0812.4276](#).
- [5] Ivanov E., Mezincescu L., Townsend P.K., Planar super-Landau models, *J. High Energy Phys.* **2006** (2006), no. 1, 143, 23 pages, [hep-th/0510019](#).  
Curtright T., Mezincescu L., Ivanov E., Townsend P.K., Planar super-Landau models revisited, *J. High Energy Phys.* **2007** (2007), no. 4, 020, 25 pages, [hep-th/0612300](#).
- [6] Gates S.J., Jr., Rana L., Ultra-multiplets: a new representation of rigid  $2D$ ,  $\mathcal{N} = 8$  supersymmetry, *Phys. Lett. B* **342** (1995), 132–137, [hep-th/9410150](#).  
Pashnev A., Toppan F., On the classification of  $N$ -extended supersymmetric quantum mechanical systems, *J. Math. Phys.* **42** (2001), 5257–5271, [hep-th/0010135](#).
- [7] Ivanov E., Krivonos S., Lechtenfeld O.,  $\mathcal{N} = 4$ ,  $d = 1$  supermultiplets from nonlinear realizations of  $D(2, 1; \alpha)$ , *Classical Quantum Gravity* **21** (2004), 1031–1050, [hep-th/0310299](#).
- [8] Ivanov E., Lechtenfeld O.,  $\mathcal{N} = 4$  supersymmetric mechanics in harmonic superspace, *J. High Energy Phys.* **2003** (2003), no. 9, 073, 33 pages, [hep-th/0307111](#).
- [9] Galperin A., Ivanov E., Ogievetsky V., Sokatchev E., Harmonic superspace: key to  $\mathcal{N} = 2$  supersymmetric theories, *JETP Lett.* **40** (1984), 912–916.  
Galperin A., Ivanov E., Kalitzin S., Ogievetsky V., Sokatchev E., Unconstrained  $\mathcal{N} = 2$  matter, Yang–Mills and supergravity theories in harmonic superspace, *Classical Quantum Gravity* **1** (1984), 469–498.
- [10] Galperin A.S., Ivanov E.A., Ogievetsky V.I., Sokatchev E.S., Harmonic superspace, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, 2001.
- [11] Ivanov E.A., Supersymmetry in superspace: 35 years of the research activity in LTP, *Phys. Part. Nuclei* **40** (2009), 291–306, [hep-th/0609176](#).
- [12] Delduc F., Ivanov E., Gauging  $\mathcal{N} = 4$  supersymmetric mechanics, *Nuclear Phys. B* **753** (2006), 211–241, [hep-th/0605211](#).
- [13] Delduc F., Ivanov E., Gauging  $\mathcal{N} = 4$  supersymmetric mechanics. II.  $(1, 4, 3)$  models from the  $(4, 4, 0)$  ones, *Nuclear Phys. B* **770** (2007), 179–205, [hep-th/0611247](#).
- [14] Delduc F., Ivanov E., The common origin of linear and nonlinear chiral multiplets in  $\mathcal{N} = 4$  mechanics, *Nuclear Phys. B* **787** (2007), 176–197, [arXiv:0706.0706](#).
- [15] Sonner J., Tong D., Berry phase and supersymmetry, *J. High Energy Phys.* **2009** (2009), no. 1, 063, 9 pages, [arXiv:0810.1280](#).
- [16] Gonzales M., Kuznetsova Z., Nersessian A., Toppan F., Yeghikyan V., Second Hopf map and supersymmetric mechanics with a Yang monopole, *Phys. Rev. D* **80** (2009), 025022, 13 pages, [arXiv:0902.2682](#).
- [17] Dunne G.V., Jackiw R., Trugenberger C.A., “Topological” (Chern–Simons) quantum mechanics, *Phys. Rev. D* **14** (1990), 661–666.  
Howe P.S., Townsend P.K., Chern–Simons quantum mechanics, *Classical Quantum Gravity* **7** (1990), 1655–1668.
- [18] Feher L., Horváthy P.A., O’Raifeartaigh L., Applications of chiral supersymmetry to spin fields in selfdual backgrounds, *Internat. J. Modern Phys. A* **4** (1989), 5277–5285, [arXiv:0903.2920](#).

- 
- [19] Konyushikhin M., Smilga A.V., Self-duality and supersymmetry, *Phys. Lett. B* **689** (2010), 95–100, [arXiv:0910.5162](#).
  - [20] Ivanov E.A., Konyushikhin M.A., Smilga A.V., SQM with non-Abelian self-dual fields: harmonic superspace description, *J. High Energy Phys.* **2010** (2010), no. 5, 033, 14 pages, [arXiv:0912.3289](#).
  - [21] Ivanov E., Konyushikhin M.,  $\mathcal{N} = 4$ , 3D supersymmetric quantum mechanics in non-Abelian monopole background, *Phys. Rev. D* **82** (2010), 085014, 8 pages, [arXiv:1004.4597](#).
  - [22] Bellucci S., Krivonos S., Sutulin A., Three dimensional  $\mathcal{N} = 4$  supersymmetric mechanics with Wu–Yang monopole, *Phys. Rev. D* **81** (2010), 105026, 9 pages, [arXiv:0911.3257](#).
  - [23] Krivonos S., Lechtenfeld O., Sutulin A.,  $\mathcal{N} = 4$  supersymmetry and the Belavin–Polyakov–Shvarts–Tyupkin instanton, *Phys. Rev. D* **81** (2010), 085021, 7 pages, [arXiv:1001.2659](#).
  - [24] Kirchberg A., Länge J.D., Wipf A., Extended supersymmetries and the Dirac operator, *Ann. Physics* **315** (2005), 467–487, [hep-th/0401134](#).
  - [25] Ivanov E., Niederle J., Bi-harmonic superspace for  $\mathcal{N} = 4$  mechanics, *Phys. Rev. D* **80** (2009), 065027, 23 pages, [arXiv:0905.3770](#).
  - [26] Rocek M., Verlinde E.P., Duality, quotients, and currents, *Nuclear Phys. B* **373** (1992), 630–646, [hep-th/9110053](#).
  - [27] Galperin A., Ivanov E., Ogievetsky V., Sokatchev E., Gauge field geometry from complex and harmonic analyticities. I. Kähler and self-dual Yang–Mills cases, *Ann. Physics* **185** (1988), 1–21.
  - [28] Yang C.N., Generalization of Dirac’s monopole to  $SU_2$  gauge fields, *J. Math. Phys.* **19** (1978), 320–328.
  - [29] Polychronakos A.P., Integrable systems from gauged matrix models, *Phys. Lett. B* **266** (1991), 29–34.
  - [30] de Crombrugghe M., Rittenberg V., Supersymmetric quantum mechanics, *Ann. Physics* **151** (1983), 99–126.
  - [31] Wu T.T., Yang C.N., Some solutions of the classical isotopic gauge field equations, in *Properties of Matter Under Unusual Conditions*, Editors H. Mark and S. Fernbach, Interscience, New York, 1969, 345–349.