On the Conjectures Regarding the 4-Point Atiyah Determinant

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Abstract. For the case of 4 points in Euclidean space, we present a computer aided proof of Conjectures II and III made by Atiyah and Sutcliffe regarding Atiyah's determinant along with an elegant factorization of the square of the imaginary part of Atiyah's determinant.

Key words: Atiyah determinant; Atiyah–Sutcliffe conjectures

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1 Introduction

The Atiyah determinant is a complex-valued determinant function $\mathrm{At}(P_1, \ldots, P_n)$ associated with n distinct points $P_1, \ldots, P_n$ of $\mathbb{R}^3$. It was constructed by M.F. Atiyah in [1] in his attempt at answering a natural geometric question posed in [3] and arising from the study of the spin statistics theorem using classical quantum theory. The original conjecture of Atiyah was that At does not vanish for all configurations of distinct points $P_1, \ldots, P_n \in \mathbb{R}^3$. The conjecture was verified in the linear case (all points lie on a straight line) and in the case $n = 3$ by Atiyah in [1]. However, the case $n \geq 4$ turned out to be notoriously difficult. In a subsequent paper [2], Atiyah and Sutcliffe studied the function At and added two new conjectures (after normalizing At) which imply the original conjecture of Atiyah. They provided compelling numerical evidence of the validity of all three conjectures. The three conjectures can be stated as follows: For all distinct points $P_1, \ldots, P_n$ of $\mathbb{R}^3$ (and all $n \geq 1$) we have:

(I) $\mathrm{At}(P_1, \ldots, P_n) \neq 0$,

(II) $|\mathrm{At}(P_1, \ldots, P_n)| \geq \prod_{i<j}(2r_{ij})$,   where $r_{ij} = ||\overrightarrow{P_i P_j}||$,

(III) $|\mathrm{At}(P_1, \ldots, P_n)|^{n-2} \geq \prod_{k=1}^{n} |\mathrm{At}(P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_n)|$.

From the statement of these conjectures we can see that III $\implies$ II $\implies$ I. The three conjectures have been very resistant since their inauguration time. The first conjecture was proved by Eastwood and Norbury [5] for the case $n = 4$. Other attempts were successful only on special configurations (see [4] and [6]). In this paper, we build on the work of Eastwood and Norbury by presenting a computer aided proof of Conjectures II and III in the case $n = 4$ and we also give an elegant factorization of the square of the imaginary part of the Atiyah determinant.

The construction of the determinant is as follows: One starts with n distinct points $P_1, \ldots, P_n \in \mathbb{R}^3$. By considering P_j as an observer of the other $n-1$ points we obtain $n-1$ vectors $\overrightarrow{P_j P_1}, \ldots, \overrightarrow{P_j P_{j-1}}, \overrightarrow{P_j P_{j+1}}, \ldots, \overrightarrow{P_j P_n}$ in $\mathbb{R}^3$. We lift each of these vectors from $\mathbb{R}^3$ to $\mathbb{C}^2$ using
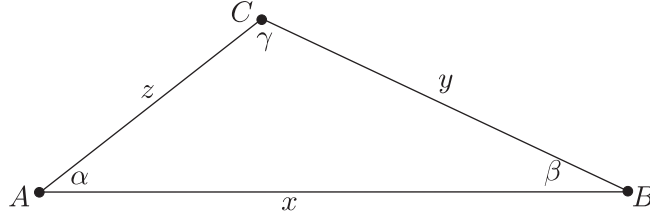
Figure 1. Three points.

the Hopf map $h : \mathbb{C}^2 \to \mathbb{R}^3$ given by $h(z, w) = ((|z|^2 - |w|^2)/2, z\overline{w})$ to obtain $n - 1$ points of $\mathbb{C}^2$. Note that the lifts are not unique and are defined up to phase because $h(\lambda z, \lambda w) = |\lambda|^2 h(z, w)$. Consequently, our lifts can be considered as points of $\mathbb{C}P^1$. Taking the symmetric product of these lifts gives a vector V_j in $\mathbb{C}P^n$ because $\odot_n \mathbb{C}P^1 = \mathbb{C}P^n$. Atiyah's first conjecture was that $\{V_1, \ldots, V_n\}$ is a linearly independent set. In other words, the determinant of the matrix having the vector V_j as its jth column is nonzero. This determinant is well-defined up to a phase factor. To get rid of the phase factor ambiguity, we apply the following normalization imposed by Atiyah: If (z, w) is the chosen lift of $\overrightarrow{P_i P_j}$ and $i < j$, then $(-\overline{w}, \overline{z})$ must be the lift of $\overrightarrow{P_j P_i}$. After this normalization, this determinant is called the Atiyah determinant and is denoted by At.

It is immediate from the above construction that At is coordinate free and is independent of solid motion. In other words, the determinant function At is invariant under translations and rotations in $\mathbb{R}^3$. Furthermore, the Atiyah determinant is built so that it is independent of the order of the points. In other words, if $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$ then $\mathrm{At}(P_{j_1}, \ldots, P_{j_n}) = \mathrm{At}(P_1, \ldots, P_n)$. Another property is that At gets conjugated under a plane reflection of the points (see [1]). As a consequence, At must be real-valued if the set of points $\{P_1, \ldots, P_n\}$ is symmetric relative to a plane (e.g. if the points are co-planar) since a reflection in the plane leaves the set of points unchanged.

Let us start computing At in the cases $n = 2$ and $n = 3$. For the case $n = 2$, we have two distinct points A and B. We can identify $\mathbb{R}^3$ with $\mathbb{R} \times \mathbb{C}$ and assume (possibly after a solid motion) that A and B have coordinates $(0, 0)$ and $(0, x)$ respectively, where $x > 0$ is the distance from A to B. By choosing $(\sqrt{x}, \sqrt{x})$ as a lift of $\overrightarrow{AB}$, we are forced to take $(-\sqrt{x}, \sqrt{x})$ as a lift of $\overrightarrow{BA}$. Consequently, Atiyah's determinant is:

$$\mathrm{At}(A, B) = \begin{vmatrix} \sqrt{x} & -\sqrt{x} \\ \sqrt{x} & \sqrt{x} \end{vmatrix} = 2x, \qquad \text{where} \quad x = ||\overrightarrow{AB}||.$$

Let us now consider the case $n = 3$. Assume (possibly after a solid motion) that $A = (0, 0)$, $B = (0, x)$, and $C = (0, ze^{I\alpha})$ where I denotes $\sqrt{-1}$, $y = ||\overrightarrow{BC}||$, $z = ||\overrightarrow{AC}||$, $x = ||\overrightarrow{AB}||$ and α, β, γ are the angles indicated in Fig. 1. When the first point is considered as a vision point we obtain $\overrightarrow{AB} = (0, x)$, $\overrightarrow{AC} = (0, ze^{I\alpha})$ whose lifts under the Hopf map h are $(\sqrt{x}, \sqrt{x})$ and $(\sqrt{z}, \sqrt{z}e^{-I\alpha})$. And when $B = (0, x)$ is the vision point, we obtain the vectors $\overrightarrow{BA} = (0, -x)$ and $\overrightarrow{BC} = (0, -ye^{-I\beta})$ whose lifts are $(-\sqrt{x}, \sqrt{x})$ and $(-\sqrt{y}, \sqrt{y}e^{I\beta})$. Similarly, the lifts corresponding to the vision point C are $(-\sqrt{z}e^{I\alpha}, \sqrt{z})$ and $(-\sqrt{y}e^{-I\beta}, -\sqrt{y})$. The symmetric tensor product of the vectors are then $\sqrt{xz}(1, 1 + e^{-I\alpha}, e^{-I\alpha})$, $\sqrt{xy}(1, -1 - e^{I\beta}, e^{I\beta})$ and $\sqrt{yz}(e^{I(\alpha-\beta)}, e^{I\alpha} - e^{-I\beta}, -1)$, respectively. Consequently, we obtain the Atiyah determinant for three points as

$$\mathrm{At}(A, B, C) = xyz \begin{vmatrix} 1 & 1 & e^{I(\alpha-\beta)} \\ 1 + e^{-I\alpha} & -1 - e^{I\beta} & e^{I\alpha} - e^{-I\beta} \\ e^{-I\alpha} & e^{I\beta} & -1 \end{vmatrix}$$
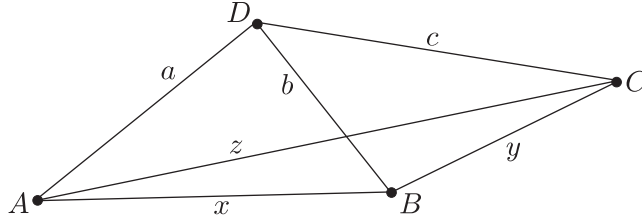
Figure 2. Four points.

This determinant expands to $xyz[6+2(\cos\alpha+\cos\beta+\cos\gamma)]$, which can be written as $xyz[8+8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}]$. Using the identity $\sin\frac{\alpha}{2}=\frac{1}{2}\sqrt{\frac{(a+b-c)(a+c-b)}{bc}}$ and similar identities for $\sin\frac{\beta}{2}$ and $\sin\frac{\gamma}{2}$, we can rewrite the Atiyah determinant for three points as

$$\mathrm{At}(A,B,C) = 8xyz + d_3(x,y,z), \tag{1.1}$$

where d_3 is the polynomial defined by $d_3(x,y,z) = (-x+y+z)(x+y-z)(x+y-z)$. From the triangle inequality it follows that $d_3(x,y,z)$ is nonnegative, and so Conjecture III is verified for three points. (Note that there is no need to use $|\,\mathrm{At}\,|$ because in this case At is real.)

2 The case of four points

Given four points A, B, C, D in $\mathbb{R}^3$, the vector $u = U(A,B,C,D)$ in $\mathbb{R}^6$, called the *vector of pair-wise distances*, is defined by $u = (a,b,c,x,y,z)$ where $a = ||\overrightarrow{AD}||$, $b = ||\overrightarrow{BD}||$, $c = ||\overrightarrow{CD}||$, $x = ||\overrightarrow{AB}||$, $y = ||\overrightarrow{BC}||$, $z = ||\overrightarrow{AC}||$ (see Fig. 2). The function U, as defined above, maps $\mathbb{R}^{3\times4}$ into $\mathbb{R}^6$, and it is clear that U is neither injective nor surjective. A vector $u \in \mathbb{R}^6$ is said to be *geometric* if it belongs to the range of U. For convenience, we adopt the convention that $\mathrm{At}(A,B,C,D)$ equals 0 when the points A, B, C, D are not distinct, and Conjecture II (for four points) becomes

(II) $|\,\mathrm{At}(A,B,C,D)\,| \geq 64abcxyz$ for all points $A,B,C,D \in \mathbb{R}^3$.

Atiyah's determinant is designed to be invariant under permutations of the points. Each of the 24 possible permutations of the four points A, B, C, D results in a permutation of the pair-wise distances a, b, c, x, y, z. Specifically, if $u = (a,b,c,x,y,z) \in \mathbb{R}^6$, then the 24 resultant permutations are

$$
\begin{aligned}
&u_0 = (a,b,c,x,y,z), &&u_1 = (a,x,z,b,y,c), &&u_2 = (b,c,a,y,z,x), &&u_3 = (x,b,y,a,c,z),\\
&u_4 = (c,a,b,z,x,y), &&u_5 = (z,y,c,x,b,a), &&u_6 = (y,z,c,x,a,b), &&u_7 = (c,b,a,y,x,z),\\
&u_8 = (x,y,b,z,c,a), &&u_9 = (a,c,b,z,y,x), &&u_{10} = (z,x,a,y,b,c), &&u_{11} = (b,a,c,x,z,y),\\
&u_{12} = (z,c,y,a,b,x), &&u_{13} = (x,z,a,y,c,b), &&u_{14} = (x,a,z,b,c,y), &&u_{15} = (y,x,b,z,a,c),\\
&u_{16} = (y,b,x,c,a,z), &&u_{17} = (y,c,z,b,a,x), &&u_{18} = (c,y,z,b,x,a), &&u_{19} = (z,a,x,c,b,y),\\
&u_{20} = (b,x,y,a,z,c), &&u_{21} = (c,z,y,a,x,b), &&u_{22} = (a,z,x,c,y,b), &&u_{23} = (b,y,x,c,z,a).
\end{aligned}
$$

A function $f : \mathbb{R}^6 \to \mathbb{R}$ is said to be *symmetric* if $f(u) = f(u_i)$ for $i = 0,1,\ldots,23$ and is *skew-symmetric* if $f(u) = (-1)^i f(u_i)$ for $i = 0,1,\ldots,23$. The *symmetric average* of f is the symmetric function $\mathrm{av}[f]$ defined by

$$\mathrm{av}[f](u) = \frac{1}{24}\sum_{i=0}^{23} f(u_i).$$

Using Maple, Eastwood and Norbury have found that the real part of $\mathrm{At}(A, B, C, D)$ can be expressed as $\Re \mathrm{At}(A, B, C, D) = d_4(u)$, where d_4 is the homogeneous polynomial of degree 6 given by

$$d_4(u) = 60p_4(u) + 4n_4(u) + 2z_4(u) + 12 \operatorname{av}\left[a\left((b+c)^2 - y^2\right)d_3(x, y, z)\right], \tag{2.1}$$

where $p_4(u) = abcxyz$, d_3 is defined in (1.1), $n_4(u) = p_4(u) - d_3(xc, ay, bz)$ and

$$
\begin{aligned}
z_4(u) &= a^2y^2\left(b^2 + c^2 + x^2 + z^2\right) + b^2z^2\left(a^2 + c^2 + x^2 + y^2\right) + c^2x^2\left(a^2 + b^2 + y^2 + z^2\right) \\
&\quad - \left(a^4y^2 + a^2y^4 + b^4z^2 + b^2z^4 + c^4x^2 + c^2x^4\right) \\
&\quad - \left(a^2b^2x^2 + a^2c^2z^2 + b^2c^2y^2 + x^2y^2z^2\right).
\end{aligned}
$$

Eastwood and Norbury use the notation $144V^2$ in place of $z_4(u)$. If $u = U(A, B, C, D)$, the value $z_4(u)$ equals $144V^2$, where V denotes the volume of the tetrahedron formed by the points A, B, C, D, and it therefore follows that $z_4(u) \geq 0$. It would be erroneous to infer from this that the polynomial z_4 is nonnegative on all of $\mathbb{R}^6$; the above statement implies only that z_4 is nonnegative on geometric vectors.

Having expressed $\Re \mathrm{At}(A, B, C, D) = d_4(u)$ as in (2.1), Eastwood and Norbury then invoke the inequalities $z_4(u) \geq 0$, $(b+c)^2 \geq y^2$, $d_3(x, y, z) \geq 0$ and $abcxyz \geq d_3(xc, ay, bz)$ (i.e. $n_4(u) \geq 0$) to conclude that

$$|\mathrm{At}(A, B, C, D)| \geq \Re \mathrm{At}(A, B, C, D) = d_4(u) \geq 60p_4(u),$$

which proves Conjecture I and comes close to proving Conjecture II.

Regarding the imaginary part of $\mathrm{At}(A, B, C, D)$, Eastwood and Norbury have shown that its square can be written as $(\Im \mathrm{At}(A, B, C, D))^2 = F_4(u)$, where F_4 is a symmetric homogeneous polynomial of degree 12. Whereas d_4 seems unwilling to be expressed in a simple manner, we have found that F_4 factors elegantly as

$$F_4 = w_4^2 z_4,$$

where w_4 is the skew-symmetric homogeneous polynomial of degree 3 given by

$$
\begin{aligned}
w_4(u) &= \left(a^2 + y^2\right)(b - c - x + z) + \left(b^2 + z^2\right)(-a + c + x - y) + \left(c^2 + x^2\right)(a - b + y - z) \\
&\quad + 2(cx + yz)(-a + b) + 2(ay + xz)(-b + c) + 2(bz + xy)(a - c).
\end{aligned}
$$

Note that since w_4 is skew-symmetric it follows that w_4^2 is symmetric. As mentioned in the introduction, the imaginary part of $\mathrm{At}(A, B, C, D)$ vanishes whenever the set of four points $\{A, B, C, D\}$ is symmetric about a plane. Interestingly, this property can be derived from the above factorization: Assuming $\{A, B, C, D\}$ is symmetric about a plane, it then follows that $u = U(A, B, C, D)$ is invariant under some odd permutation of the four points A, B, C, D. Since w_4 is skew-symmetric, we must have $w_4(u) = 0$ and hence $F_4(u) = 0$.

3   A linear program related to Conjecture II

Since $|\mathrm{At}(A, B, C, D)| \geq \Re \mathrm{At}(A, B, C, D) = d_4(u)$, in order to prove Conjecture II, it suffices to show that the polynomial d_4 satisfies

$$d_4(u) \geq 64p_4(u) \quad \text{for all geometric vectors} \quad u. \tag{3.1}$$

If one has in hand a collection $f_1, f_2, \ldots, f_k$ of symmetric homogeneous polynomials of degree 6 which are known to be nonnegative on geometric vectors, then one can 'have a go' at (3.1) by solving the linear program

$$\text{maximize} \quad \alpha,$$
$$\text{subject to} \quad d_4 = \alpha p_4 + \sum_{j=1}^{k} \lambda_j f_j, \quad \text{with} \quad \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0. \tag{3.2}$$

If (3.2) is feasible and if the optimal objective value is $\alpha = 64$ (we will see later that $\alpha > 64$ is impossible), then we immediately obtain (3.1). The remaining difficulty is that of finding suitable polynomials $\{f_j\}$. One means of generating a large collection of such polynomials, which we now describe, stems from the triangle inequality.

The four points A, B, C, D contain four (possibly degenerate) triangles and each triangle, by means of the triangle inequality, gives rise to three linear polynomials which are nonnegative when $u = (a, b, c, x, y, z)$ is geometric. For example, the triangle A, B, C yields $-x + y + z$, $x - y + z$ and $x + y - z$. In all, there are twelve such linear polynomials which we refer to as *triangular variables* and use the notation $t = (t_1, t_2, \ldots, t_{12})$, where

$$
\begin{array}{llll}
t_1 = -a + b + x, & t_4 = -b + c + y, & t_7 = -a + c + z, & t_{10} = -x + y + z, \\
t_2 = a - b + x, & t_5 = b - c + y, & t_8 = a - c + z, & t_{11} = x - y + z, \\
t_3 = a + b - x, & t_6 = b + c - y, & t_9 = a + c - z, & t_{12} = x + y - z.
\end{array}
$$

A vector $\alpha \in \mathbb{Z}_+^{12}$ is called a *multi-index* with *order* $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{12}$. Employing the standard notation $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_{12}^{\alpha_{12}}$, we see that t^α represents a homogeneous polynomial of degree $|\alpha|$ in the variables (a, b, c, x, y, z). Applying the symmetric average, we conclude that $\text{av}[t^\alpha]$ represents a symmetric homogeneous polynomial of degree $|\alpha|$ which is nonnegative on geometric vectors. For integers $\ell \geq 0$, we define $\mathbb{T}_\ell$ to be the set of all polynomials $\text{av}[t^\alpha]$ with $|\alpha| = \ell$:

$$\mathbb{T}_\ell = \{\text{av}[t^\alpha] : |\alpha| = \ell\}.$$

Numerically, we have found that if one chooses $\{f_j\}$ equal to $\mathbb{T}_6$, then the linear program (3.2) is feasible and has optimal objective value $\alpha = 32$. The formulation (2.1) of Eastwood and Norbury can be understood in the context of (3.2) as the result of including, in addition to $\mathbb{T}_6$, the two symmetric polynomials z_4 and n_4 which are nonnegative on geometric vectors. Numerically solving (3.2) with $\{f_j\}$ equal to $\{z_4, n_4\} \cup \mathbb{T}_6$, we have found that the optimal objective value is $\alpha = 60$, and (2.1) is indeed an optimal solution of (3.2) as the term $\text{av}[a((b+c)^2 - y^2)d_3(x, y, z)]$ can be written as a nonnegative linear combination of polynomials in $\mathbb{T}_6$.

In order to further increase the optimal objective value α in (3.2), we need other symmetric polynomials which are nonnegative on geometric vectors. In pursuit of this, we have identified the following twenty-one geometric vectors u where $d_4(u) = 64p_4(u)$ (all are obtained as $u = U(A, B, C, D)$ with A, B, C, D collinear or non-distinct):

$$
\begin{array}{lll}
(0, 1, 4, 1, 4, 4), & (0, 4, 8, 4, 7, 8), & (0, 6, 0, 6, 6, 0), \\
(0, 1, 1, 1, 2, 1), & (0, 5, 5, 5, 5, 5), & (0, 8, 8, 8, 1, 8), \\
(0, 1, 3, 1, 4, 3), & (0, 6, 3, 6, 8, 3), & (0, 6, 7, 6, 3, 7), \\
(0, 6, 6, 6, 9, 6), & (0, 1, 1, 1, 0, 1), & (0, 5, 3, 5, 3, 3), \\
(3, 3, 1, 0, 2, 2), & (9, 9, 7, 0, 2, 2), & (13, 13, 7, 0, 6, 6), \\
(19, 11, 7, 8, 4, 12), & (17, 13, 4, 4, 9, 13), & (15, 8, 7, 7, 1, 8), \\
(9, 8, 1, 1, 7, 8), & (11, 9, 8, 2, 1, 3), & (17, 9, 2, 8, 7, 15).
\end{array} \tag{3.3}
$$

Both d_4 and p_4 vanish on the first fifteen of these vectors (counting horizontally), but are nonzero on the remaining six. In particular, since $d_4(9,8,1,1,7,8) = 64p_4(9,8,1,1,7,8) = 258048 > 0$, it follows that there are no feasible solutions of (3.2) with $\alpha > 64$. On the other hand, if a feasible solution of (3.2) has been obtained with $\alpha = 64$, then it follows that f_j vanishes on all of the vectors in (3.3), whenever $\lambda_j > 0$. It has been verified that z_4 vanishes on all of these vectors, but n_4 does not. Therefore, the coefficient of n_4 will be 0 if (3.2) has been solved with $\alpha = 64$. We have considered numerous symmetric homogeneous polynomials of degree 6 which vanish on the vectors in (3.3), but only one of these has resulted in an improvement. Let v_4 denote the skew-symmetric homogeneous polynomial of degree 3 defined by

$$v_4(u) = (b + z - c - x)(c + x - a - y)(a + y - b - z).$$

Then v_4 vanishes on the vectors in (3.3), and numerically solving (3.2) with $\{f_j\}$ equal to $\{z_4, n_4, v_4^2\} \cup \mathbb{T}_6$, we have found that the optimal objective value is $\alpha = 188/3$. Our obtained identity, which has been verified in Maple[1], is the following:

$$d_4(u) = \frac{188}{3}p_4(u) + \frac{10}{3}z_4(u) + \frac{4}{3}n_4(u) + \frac{2}{3}v_4^2(u) + \frac{1}{3}\sum_{|\alpha|=6}\lambda_\alpha\,\mathrm{av}[t^\alpha],$$

where the six nonzero coefficients λ_α and corresponding multi-indices α are given by

α	λ_α	α	λ_α	α	λ_α
$000, 001, 010, 112$	6	$000, 001, 011, 111$	18	$000, 001, 110, 102$	6
$001, 001, 001, 111$	14	$001, 001, 010, 111$	24	$001, 011, 100, 110$	24

4   Proof of Conjecture II for four points

Let m_4 be the symmetric homogeneous polynomial of degree 6 defined by $m_4 = d_4 - (64p_4 + 4z_4 + v_4^2)$, so that

$$d_4 = 64p_4 + 4z_4 + v_4^2 + m_4. \tag{4.1}$$

We will show that m_4 is nonnegative on geometric vectors, but unfortunately, we have been unable to formulate a proof using only polynomials of degree 6. Rather, we have had to multiply m_4 by p_4 and then work with polynomials of degree 12.

Theorem 1. *The product p_4m_4 can be written as a nonnegative linear combination of polynomials in $\mathbb{T}^{12}$.*

Proof. Using Maple, we have verified that $64p_4m_4$ can be written as

$$64p_4(u)m_4(u) = \sum_{|\alpha|=12}\lambda_\alpha\,\mathrm{av}[t^\alpha], \tag{4.2}$$

where the sixty-four nonzero coefficients $\{\lambda_\alpha\}$ are all positive integers as given in the following table:

α	λ_α				
		$011, 021, 201, 112$	6	$011, 111, 220, 102$	6
$001, 012, 211, 112$	12	$011, 021, 201, 121$	6	$011, 112, 102, 210$	12
$001, 012, 211, 121$	12	$011, 021, 201, 211$	18	$011, 112, 120, 210$	39

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[1]The sources of our codes are available at http://www.emis.de/journals/SIGMA/2014/070/codes.zip.

$001, 112, 112, 012$	12	$011, 021, 221, 110$	54	$011, 112, 210, 210$	42
$001, 121, 121, 012$	21	$011, 022, 011, 121$	6	$011, 120, 011, 212$	27
$002, 110, 111, 212$	9	$011, 022, 111, 102$	84	$011, 120, 011, 221$	24
$002, 110, 111, 221$	9	$011, 022, 111, 210$	18	$011, 120, 012, 112$	3
$011, 011, 021, 221$	30	$011, 022, 211, 110$	6	$011, 121, 021, 201$	24
$011, 011, 101, 222$	56	$011, 101, 021, 212$	54	$011, 121, 102, 210$	6
$011, 011, 110, 222$	48	$011, 102, 012, 211$	6	$011, 121, 120, 210$	24
$011, 011, 112, 220$	6	$011, 102, 022, 111$	36	$011, 201, 012, 211$	3
$011, 011, 120, 212$	18	$011, 102, 112, 102$	18	$011, 201, 021, 211$	45
$011, 011, 122, 102$	57	$011, 102, 112, 201$	24	$012, 012, 012, 111$	24
$011, 011, 201, 221$	36	$011, 102, 201, 121$	33	$012, 012, 102, 111$	8
$011, 011, 222, 011$	6	$011, 102, 210, 112$	75	$012, 111, 012, 021$	36
$011, 012, 102, 112$	120	$011, 102, 210, 121$	12	$011, 021, 220, 210$	6
$011, 012, 112, 210$	18	$011, 110, 021, 221$	48	$011, 120, 012, 202$	21
$011, 012, 120, 121$	12	$011, 111, 012, 202$	21	$011, 120, 021, 202$	24
$011, 012, 211, 102$	84	$011, 111, 021, 202$	18	$011, 201, 012, 202$	18
$011, 012, 211, 201$	72	$011, 111, 021, 220$	54	$011, 210, 021, 202$	3
$011, 021, 011, 122$	3	$011, 111, 022, 102$	75	$012, 012, 120, 102$	3
$011, 021, 112, 012$	69	$011, 111, 202, 210$	12		∎

Corollary 1. *The polynomial m_4 is nonnegative on geometric vectors and consequently* (3.1) *holds, which proves Conjecture* II *for four points.*

Proof. Let $u = U(A, B, C, D)$ be a geometric vector. It follows from (4.2) that $p_4(u)m_4(u) \geq 0$. If the points A, B, C, D are distinct, then $p_4(u) > 0$ and hence $m_4(u) \geq 0$. On the other hand, if A, B, C, D are not distinct, then they can be approximated by distinct points A', B', C', D' and it will then follow from the continuity of m_4 that $m_4(u) \geq 0$. ∎

5   Proof of Conjecture III for four points

Let P_4 denote the symmetric homogeneous polynomial of degree 12 given by

$$P_4(u) := (8xyz + d_3(x, y, z))(8abx + d_3(a, b, x))(8acz + d_3(a, c, z))(8bcy + d_3(b, c, y)),$$

whereby $\operatorname{At}(A, B, C)\operatorname{At}(A, B, D)\operatorname{At}(A, C, D)\operatorname{At}(B, C, D) = P_4(u)$ when $u = U(A, B, C, D)$. Since $|\operatorname{At}(A, B, C, D)|^2 \geq (\Re \operatorname{At}(A, B, C, D))^2 = d_4^2(u)$, in order to prove Conjecture III, it suffices to show that

$$d_4^2(u) \geq P_4(u) \quad \text{for all geometric vectors} \quad u. \tag{5.1}$$

Recall from (4.1) that d_4 has been written as $d_4 = 64p_4 + m_4 + (4z_4 + v_4^2)$, so it follows that

$$
\begin{aligned}
d_4^2 &= \left(4z_4 + v_4^2\right)d_4 + (64p_4 + m_4)d_4 \\
&= \left(4z_4 + v_4^2\right)d_4 + (64p_4 + m_4)^2 + (64p_4 + m_4)\left(4z_4 + v_4^2\right) \\
&= \left(4z_4 + v_4^2\right)(d_4 + 32p_4 + m_4) + (64p_4 + m_4)^2 + 32p_4\left(4z_4 + v_4^2\right).
\end{aligned}
$$

With M_4 denoting the symmetric homogeneous polynomial of degree 12 defined by $M_4 = (64p_4 + m_4)^2 + 32p_4(4z_4 + v_4^2) - P_4$, we then have

$$d_4^2 = P_4 + \left(4z_4 + v_4^2\right)(d_4 + 32p_4 + m_4) + M_4. \tag{5.2}$$

Theorem 2. *The polynomial M_4 is nonnegative on geometric vectors, and consequently* (5.1) *holds, which proves Conjecture* III *for four points.*

Proof. Using Maple, we have verified that $128M_4$ can be written as

$$128M_4(u) = \left(4z_4(u) + v_4^2(u)\right) \sum_{|\alpha|=6} \mu_\alpha \, \mathrm{av}[t^\alpha] + \sum_{|\alpha|=12} \nu_\alpha \, \mathrm{av}[t^\alpha], \tag{5.3}$$

where the coefficients $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ are nonnegative integers: The 6 nonzero coefficients μ_α and corresponding monomials α are given in the following table:

$000, 001, 111, 110$	1236	$000, 101, 101, 101$	3594	$001, 010, 011, 011$	300
$000, 100, 101, 111$	60	$000, 101, 110, 101$	114	$001, 011, 101, 001$	1014

The 114 nonzero coefficients ν_α and corresponding monomials α are given in the following table:

$000, 112, 121, 112$	2019	$011, 120, 021, 112$	2184	$001, 120, 202, 121$	76
$001, 012, 121, 112$	369	$011, 121, 021, 201$	228	$001, 121, 201, 220$	6
$001, 012, 211, 121$	138	$011, 121, 210, 210$	72	$001, 121, 220, 021$	3174
$001, 012, 211, 211$	666	$011, 201, 021, 211$	936	$001, 121, 220, 120$	1266
$001, 021, 211, 121$	3087	$012, 012, 012, 111$	3072	$001, 122, 210, 120$	1428
$001, 022, 111, 112$	3009	$012, 111, 021, 201$	1308	$001, 201, 220, 112$	822
$001, 022, 111, 121$	1074	$012, 111, 210, 012$	5374	$001, 210, 221, 021$	612
$001, 022, 112, 111$	42	$000, 012, 111, 222$	60	$001, 211, 220, 012$	300
$001, 022, 211, 111$	12114	$000, 012, 112, 212$	1776	$001, 211, 220, 120$	3072
$001, 101, 211, 122$	240	$001, 001, 112, 222$	138	$002, 002, 112, 112$	1536
$001, 111, 122, 102$	4056	$001, 001, 122, 212$	1398	$002, 011, 221, 021$	1176
$001, 111, 210, 221$	444	$001, 002, 122, 112$	3072	$002, 012, 110, 212$	1662
$001, 120, 121, 112$	144	$001, 002, 211, 122$	1236	$002, 012, 210, 121$	5136
$001, 121, 210, 112$	714	$001, 010, 112, 222$	768	$002, 012, 210, 211$	762
$001, 121, 210, 211$	1146	$001, 010, 122, 212$	384	$002, 012, 211, 021$	2178
$001, 211, 220, 111$	1866	$001, 010, 212, 122$	384	$002, 022, 110, 112$	1188
$002, 011, 111, 122$	2808	$001, 011, 102, 222$	2568	$002, 022, 110, 121$	150
$002, 011, 211, 121$	3207	$001, 011, 222, 012$	1224	$002, 022, 211, 011$	2178
$002, 012, 111, 112$	3654	$001, 012, 110, 222$	2634	$002, 101, 101, 222$	1245
$002, 012, 111, 121$	3252	$001, 012, 120, 212$	66	$002, 101, 122, 102$	246
$002, 021, 111, 112$	276	$001, 012, 122, 102$	822	$002, 101, 221, 012$	528
$002, 022, 111, 111$	720	$001, 021, 112, 220$	840	$002, 102, 121, 102$	222
$002, 111, 212, 011$	516	$001, 022, 120, 112$	4542	$002, 110, 122, 102$	7548
$002, 112, 210, 111$	1206	$001, 022, 211, 120$	2928	$002, 121, 210, 012$	1074
$002, 112, 211, 011$	1662	$001, 022, 220, 111$	6546	$002, 122, 210, 011$	60
$011, 011, 012, 212$	10110	$001, 101, 102, 222$	2898	$011, 012, 120, 220$	3072
$011, 011, 021, 212$	1164	$001, 101, 201, 222$	804	$011, 021, 210, 220$	342
$011, 011, 201, 212$	5472	$001, 101, 220, 122$	1398	$011, 021, 220, 120$	4668
$011, 012, 012, 211$	342	$001, 102, 110, 222$	690	$011, 021, 220, 210$	4608

$011, 012, 112, 201$	2178	$001, 102, 122, 102$	1770	$011, 022, 120, 102$	768
$011, 012, 122, 101$	192	$001, 102, 202, 112$	5532	$011, 022, 120, 210$	1536
$011, 021, 211, 120$	5472	$001, 102, 202, 121$	612	$011, 022, 201, 120$	1890
$011, 022, 111, 120$	2376	$001, 102, 210, 212$	390	$011, 022, 210, 102$	1152
$011, 022, 121, 110$	696	$001, 102, 221, 102$	2634	$011, 022, 210, 210$	6648
$011, 101, 022, 112$	2628	$001, 110, 201, 222$	192	$011, 120, 022, 102$	10752
$011, 102, 201, 112$	372	$001, 112, 220, 120$	600	$011, 210, 012, 202$	168
$011, 110, 012, 122$	3558	$001, 112, 221, 002$	60	$011, 210, 120, 202$	522
$011, 112, 021, 201$	948	$001, 120, 201, 221$	774	$012, 120, 012, 201$	4440

It now follows from (5.3) that M_4 is nonnegative on geometric vectors and we obtain (5.1) as a consequence of (5.2). ∎

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