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Quantum Gravity as a Broken Symmetry Phase of a BF Theory*

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Abstract. We explain how General Relativity with a cosmological constant arises as a broken symmetry phase of a BF theory. In particular we show how to treat de Sitter and anti-de Sitter cases simultaneously. This is then used to formulate a quantisation of General Relativity through a spin foam perturbation theory. We then briefly discuss how to calculate the effective action in this quantization procedure.

Key words: de Sitter; anti-de Sitter; spin foams

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1 Introduction

Quantization of General Relativity (GR) is still one of the outstanding problems in theoretical physics. Although the string theory has made a significant progress on this problem [1], the string theory approach is essentially a perturbation theory around the flat spacetime, which then makes it difficult to study the quantum cosmology problems. On the other hand, the Loop Quantum Gravity (LQG) approach is nonperturbative and background metric independent [2]; however, obtaining the perturbative and semiclassical results is difficult. This is related to the fact that LQG is not manifestly covariant under four-dimensional diffeomorphisms. This problem has been overcome by introducing the spin foam formalism [3], where the basic object of LQG, the spin network, a colored graph which lives in the space, is generalised to a spin foam (SF), a colored two-complex which lives in the spacetime. The SF models are typically obtained from the path-integral quantization of a BF theory [3], and the reason for this is that the Einstein–Hilbert action in the Palatini formalism

$$S_{EH} = \int_M \epsilon^{abcd} e_a \wedge e_b \wedge R_{cd},$$

where the e_a denote the tetrads and $R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c_a$ is the curvature two-form for the spin connection ω_{ab} on the four-manifold M, can be represented as a constrained $SO(3,1)$ BF theory

$$S_{EH} = \int_M B^{ab} \wedge F_{ab},$$

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$$B^{ab} = \epsilon^{abcd} e_c \wedge e_d . \tag{1}$$

The BF theories are topological, and their path-integral quantization is well understood [3]. Let the BF theory group be a Lie group G and let Λ label the finite-dimensional irreducible representations (irreps) of G, then the partition function can be written as a sum over the irreps of the spin foam amplitudes associated to the dual 2-complex Γ of a triangulation of M

$$Z_{BF} = \sum_{\Lambda_1,\dots,\Lambda_F} \sum_{\iota_1,\dots,\iota_L} \prod_{f=1}^{F} \dim \Lambda_f \prod_{l=1}^{L} A_\tau(\Lambda_{f(l)}) \prod_{v=1}^{V} A_\sigma(\Lambda_{f(v)}, \iota_{l(v)}), \tag{2}$$

where the Λ's label the faces f of Γ, the ι's are the corresponding intertwiners which label the edges l of Γ, A_τ are the tetrahedron amplitudes and A_σ are the four-simplex amplitudes. The A_l is a function of the dimensions of the four irreps that meet at an edge l, while A_v is given by an evaluation of the four-symplex spin network where the ten edges carry the irreps Λ_f and the five vertices carry the intertwiners ι_l [3]. The infinite sums in (2) are divergent and the typical regularization is to replace the Lie group G by the quantum group G_q where q is a root of unity [4]. In this case the set of Λ_f becomes finite so that (2) becomes finite.

However, the quantization of constrained BF theories is not that well understood, and in the GR case there is a proposal by Barrett and Crane [5, 6] to implement the constraint (1) in the SF formalism by restricting the irreducible representations of $SO(3,1)$ that color the faces of the spin-foam complex as

$$\epsilon^{abcd} J_{ab}^{(\Lambda)} J_{cd}^{(\Lambda)} = 0, \tag{3}$$

where $J^{(\Lambda)}$ are the Lorentz group generators in the representation Λ. Furthermore, one can argue that the admissible irreps have to be unitary, so that (3) selects the so called simple unitary irreps of the Lorentz group $\Lambda = (j, 0)$ or $\Lambda = (0, \rho)$ where $2j \in \mathbb{Z}_+$ and $\rho \in (0, \infty)$ [6]. Given that a triangulation of a four-dimensional pseudo-Riemannian manifold can be always chosen such that all the triangles are spacelike, one can work with only the $(0, \rho)$ irreps. In this way one obtains a spin foam model where the GR path integral is defined as a multiple integral over the face irreps of the corresponding spin foam amplitudes and (2) becomes

$$Z_{GR} = \int_0^\infty d\rho_1 \cdots \int_0^\infty d\rho_F \prod_{f=1}^{F} \rho_f \prod_{l=1}^{L} \theta^{-1}(\rho_{f(l)}) \prod_{v=1}^{V} A_\sigma^{\mathrm{rel}}(\rho_{f(v)}),$$

where θ is a function of the four edge irreps and A_σ^{rel} is the relativistic evaluation of a four-simplex spin network.

One can show that for each non-degenerate triangulation of M the corresponding BC spin foam state sum is finite, provided that the θ amplitudes are appropriately chosen [7]. In this way one obtains a finite theory of quantum gravity; however, there are problems with this theory. First, the choice of the θ amplitudes is not unique, since there are other choices which also lead to a finite state sum [8]. Second, coupling fermionic matter is difficult because fermion fields couple to individual tetrads, while in the BC model one can couple the fermions only to a specific quadratic combination of the tetrads (the B field). And the last, and the most difficult problem is to find the semiclassical limit of this quantum gravity theory.

2 GR as a symmetry breaking of a BF theory

The first two problems of the BC model suggest that one should look for a spin foam model of GR which arises from the quantization of a BF theory which is not constrained and includes the tetrads in the BF theory connection one-form. This also means that GR will appear through a symmetry breaking mechanism, since the larger symmetry of a bigger connection has to be broken to a Lorentz group connection plus the tetrads. Interestingly, such a mechanism was found by MacDowell and Mansouri [9] in the context of $OSp(n|4)$ supergravity theories. The bosonic spatial symmetry group was $Sp(4)$, which is the covering group of the anti-de Sitter group $SO(3,2)$. Their action was quadratic in the field strengths and gave EH action with a negative cosmological constant. A BF theory formulation for a positive cosmological constant EH action was found by Smolin and Starodubtsev [10], and corresponds to the de Sitter group case, i.e. $SO(4,1)$.

Extending the Smolin–Starodubtsev result to the anti-de Sitter case is easy, since one can treat the both cases simultaneously in the following way. The Lie algebras of both groups can be represented as

$$[J^{ab},J^{cd}] = \eta^{[a[c}J^{d]b]}, \qquad [J^{ab},P^c] = -\eta^{c[a}P^{b]}, \qquad [P^a,P^b] = \pm J^{ab},$$

where $\pm$ corresponds to anti-de Sitter and de Sitter cases respectively. The corresponding connection can be written as

$$\mathcal{A} = \omega_{ab}J^{ab} + \lambda e_a P^a,$$

where ω is the spin-connection, e_a are the tetrads and λ is a dimensionful parameter. The curvature 2-form $\mathcal{F} = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ is then given by

$$\mathcal{F} = T_a P^a + (R_{ab} \pm \lambda^2 e_a\wedge e_b)J^{ab},$$

where $T_a = de_a + \omega_a^{\ b}\wedge e_b$ is the torsion. Let us introduce the Lie algebra valued 2-form

$$\mathcal{B} = b_a P^a + B_{ab}J^{ab},$$

then consider the following action

$$S = \int_M \left(Tr\,(\mathcal{B}\wedge\mathcal{F}) - \frac{\alpha}{2}\epsilon^{abcd}B_{ab}\wedge B_{cd}\right). \qquad (4)$$

This is the BF theory action which is perturbed by a symmetry breaking term, since the quadratic in B term is invariant only under the Lorentz subgroup. This action can be written as

$$S = \int_M \left(b^a\wedge T_a + B^{ab}\wedge(R_{ab}\pm\lambda^2 e_a\wedge e_b) - \frac{\alpha}{2}\epsilon^{abcd}B_{ab}\wedge B_{cd}\right),$$

so that the b and B equations of motion imply vanishing of the torsion and

$$B^{ab} = \frac{1}{\alpha}\epsilon^{abcd}(R_{cd}\pm\lambda^2 e_c\wedge e_d).$$

The remaining equations correspond to the action

$$S^* = \frac{1}{2\alpha}\int_M \epsilon^{abcd}(R_{ab}\pm\lambda^2 e_a\wedge e_b)\wedge(R_{cd}\pm\lambda^2 e_c\wedge e_d).$$

The plus sign gives the MacDowell–Mansouri action, while the minus sign corresponds to the theory studied in [10]. Either way, it is easy to see that S^* can be written as

$$S^* = \pm\frac{1}{G_N}\int_M \epsilon^{abcd}\, e_a\wedge e_b\wedge(R_{cd}\pm\Lambda e_c\wedge\epsilon_d) + S_{\text{top}},$$

where $G_N = \alpha\lambda^{-2}$ is the Newton constant and $\Lambda = \frac{\lambda^2}{2}$ is the cosmological constant and S_{top} is proportional to the Euler class of M, which does not affect the equations of motion. Hence one obtains the Einstein equations for positive/negative cosmological constant. Since $\alpha = 2G_N\Lambda$, this gives that α is an extremally small number, which then justifies the view that GR with a small cosmological constant is a small deformation of a BF theory.

3 Spin foam perturbation theory

The action (4) is well-suited for a perturbative quantization. The standard approach is to calculate perturbatively the generating functional

$$Z[j,J]=\int \mathcal{D}\mathcal{A}\,\mathcal{D}\mathcal{B}\exp\left(i\int_M \mathrm{Tr}\,(\mathcal{B}\wedge\mathcal{F}+j\wedge\mathcal{A}+J\wedge\mathcal{B})+U(\mathcal{A},\mathcal{B})\right),$$

where J and j are the sources for the $\mathcal{A}$ and the $\mathcal{B}$ fields and $U(\mathcal{A},\mathcal{B})$ is the perturbative interaction. This can be done via the formula

$$Z[j,J]=\exp\left(i\int_M U\left(\frac{1}{i}\frac{\delta}{\delta j},\frac{1}{i}\frac{\delta}{\delta J}\right)\right)Z_0[j,J],$$

where

$$Z_0[j,J]=\int \mathcal{D}\mathcal{A}\mathcal{D}\mathcal{B}\exp\left(i\int_M \mathrm{Tr}\,(\mathcal{B}\wedge\mathcal{F}+j\wedge\mathcal{A}+J\wedge\mathcal{B})\right).$$

This generating functional can be calculated by using the spin foam technology [11], and the result can be written as a state sum

$$Z_0[j,J]=\sum_{\Lambda_f,\lambda_l,\iota_l}\prod_f \dim\Lambda_f\left\langle\prod_l \mu(\lambda_l,j_l)\prod_v A_\sigma(\Lambda_{f(v)},\lambda_{l(v)},\iota_{l(v)},J_{f(v)})\right\rangle, \tag{5}$$

where

$$\mu(\lambda_l,j_l)=\int_G dg_l\left(D^{(\lambda_l)}(g_l)\right)^* e^{i\,\mathrm{Tr}\,(A_l j_l)},$$

is the insertion at an edge l of Γ while $A_\sigma(\lambda,J)$ is the modified 4-symplex amplitude, with the λ-edges attachments at its vertices and $D^{(\Lambda_f)}(e^{J_f})$ insertions at its edges. When the sources vanish, the state sum (5) reduces to the state sum (2). Notice that the state sum (5) is not of the same type as (2), because one has to label both the edges and the faces of Γ with the irreps of G. The expression (5) is an infinite sum, and has to be regularized. When $G=SO(5)$, i.e. the Euclidian gravity case, the regularization consists of replacing the category of irreps of G by the category of irreps of the quantum group G_q where q is a root of unity. When $G=SO(3,2)$ or $G=SO(4,1)$, one has to use the category of unitary irreps, which are infinite-dimensional and typically have discrete and continuos series. This means that one will obtain both the state sums and the state integrals (like in the BC model case). Convergence properties of these models have not been yet investigated, and the hope is that the corresponding categories of quantum group irreps will yield convergent sums. Alternatively, one could try to generalize the discrete gauge fixing procedure introduced by Freidel and Louapre for the three-dimensional spin foam models [12, 13].

Given a regularized $Z_0[j,J]$ one can calculate the generating functional perturbatively as

$$Z[j,J]=Z_0[j,J]+\alpha Z_1[j,J]+\alpha^2 Z_2[j,J]+\cdots.$$

The semiclassical properties of the theory where Z_0 is given by (5) can be explored by analyzing the effective action. The effective action can be calculated via the Legendre transform of $Z[J,j]$

$$\Gamma(\bar{\mathcal{A}}_l,\bar{\mathcal{B}}_f)=W(j_l,J_f)-\sum_l \mathrm{Tr}\,(j_l\bar{\mathcal{A}}_l)-\sum_f \mathrm{Tr}\,(J_f\bar{\mathcal{B}}_f),$$

where

$$\bar{\mathcal{A}}_l=\frac{\partial W}{\partial j_l},\qquad \bar{\mathcal{B}}_f=\frac{\partial W}{\partial J_f},\qquad iW(j_l,J_f)=\log Z(j_l,J_f).$$

This can be done perturbatively in α, and if we denote $(\bar{\mathcal{A}}_l, \bar{\mathcal{B}}_f)$ as X_I, then

$$\Gamma(X) = \sum_{m\geq 0} \alpha^m \sum_{n\geq 0} \frac{1}{n!} \sum_{I_1,\dots,I_n} C_{mn}(I_1 \cdots I_n) X_{I_1} \cdots X_{I_n} = \sum_{m\geq 0} \alpha^m \Gamma_m(X).$$

In this way one can explore the semiclassical limit of the theory, and the crucial test for the physical relevance of the theory is whether or not $\Gamma_0 + \alpha\Gamma_1$ gives the discretized classical action (4)

$$S = \sum_f \mathrm{Tr}\,(\mathcal{B}_f \mathcal{F}_f) - \frac{\alpha}{2} \sum_{f,f'} C(f,f')\epsilon^{abcd} B_f^{ab} B_{f'}^{cd}.$$

4 Conclusions

The spin foam perturbation theory is a promising approach for defining a viable quantisation of GR. However, one must resolve first the technical questions related with the regularization of the SF generating functional. This is something which may be complicated, but it can be done. Then one can calculate the effective action by the method outlined here and verify the classical limit. If this limit is GR, one can then proceed to calculate the higher order in α corrections. The matter can be coupled by using again the same method of the generating functional [11].

On the mathematical side, one can explore the state sum (5) for the case of compact groups and study what happens with the topological invariance. In the non-compact group case, one will need first to study the unitary irreps of quantum de Sitter and anti-de Sitter groups. One can also study $Z_0(j,J)$ for three-dimensional spacetimes, in which case the relevant groups are $SO(4)$, $SO(3,1)$, $SO(2,2)$ and their covering groups. In the case of two-dimensional spacetimes the relevant groups are $SO(3)$, $SO(2,1)$ and their covering groups.

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