Український математичний вісник
Том 13 **(2016)**, № 4, 452 – 472

Completion and extension of operators in Kreĭn spaces

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(*Presented by M.M. Malamud*)

Abstract. A generalization of the well-known results of M.G. Kreĭn about the description of selfadjoint contractive extension of a hermitian contraction is obtained. This generalization concerns the situation, where the selfadjoint operator A and extensions $\widetilde{A}$ belong to a Kreĭn space or a Pontryagin space and their defect operators are allowed to have a fixed number of negative eigenvalues. Also a result of Yu. L. Shmul'yan on completions of nonnegative block operators is generalized for block operators with a fixed number of negative eigenvalues in a Kreĭn space.

This paper is a natural continuation of S. Hassi's and author's recent paper [7].

2010 MSC. 46C20, 47A20, 47A63.

Key words and phrases. Completion, extension of operators, Kreĭn and Pontryagin spaces.

1. Introduction

Let A be a densely defined lower semi-bounded operator in a separable Hilbert space $\mathfrak{H}$, $A \geq m_A I$. A problem of existing of selfadjoint extensions preserving the lower bound m_A of A was formulated by J. von Neumann [4]. He solved it for the case of an operator with finite deficiency indices. A solution to this problem for operators with arbitrary deficiency indices was obtained by M. Stone, H. Freudental, and K. Friedrichs [4]. M. G. Kreĭn in his seminal paper [19] (see also [1]) described the set $\operatorname{Ext}_A(0,\infty)$ of all nonnegative selfadjoint extensions $\widetilde{A}$ of $A \geq 0$ as follows

$$(A_F + a)^{-1} \leq (\widetilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > 0, \quad \widetilde{A} \in \operatorname{Ext}_A(0,\infty).$$

Received 27.12.2016

The author thanks his supervisor Seppo Hassi for several detailed discussions on the results of this paper and also Mark Malamud for comments.

ISSN 1810 – 3200.

Here A_F and A_K are the Friedrichs (hard) and Kreĭn (soft) extensions of A, respectively.

To obtain such a description he used a special form of the Cayley transform

$$T_1 = (I - A)(I + A)^{-1}, \quad T = (I - \widetilde{A})(I + \widetilde{A})^{-1},$$

to reduce the study of unbounded operators to the study of contractive selfadjoint extensions T of a Hermitian nondensely defined contraction $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$, where $\mathfrak{H}_1 = \operatorname{ran}(I+A)$. The set of all selfadjoint contractive extensions of T_1 is denoted by $\operatorname{Ext}_{T_1}(-1,1)$. M.G. Kreĭn proved that the set $\operatorname{Ext}_{T_1}(-1,1)$ forms an operator interval with minimal and maximal entries T_m and T_M, respectively,

$$T_m \le T \le T_M, \quad T \in \operatorname{Ext}_{T_1}(-1,1).$$

T. Ando and K. Nishio [2] extended main results of the Kreĭn theory to the case of nondensely defined symmetric operators A. For the case of linear relations (multivalued linear operators) $A \ge 0$ it was done by E.A. Coddington and H.S.V. de Snoo [9].

With respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ a contraction $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ admits a block-matrix representation $T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}$. Block matrix representations of the operators T_m and T_M were obtained in [6, 18], and [16], (see also [4, 12, 13, 27] Namely, it is shown that

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & -I + V(I - T_{11})V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & I - V(I + T_{11})V^* \end{pmatrix}, \end{aligned} \tag{1.1}$$

where $D_{T_{11}} := (I - T_{11}^2)^{1/2}$ and V is given by $V := \operatorname{clos}(T_{21} D_{T_{11}}^{[-1]})$. Based on these formulas a complete parametrization of the set $\operatorname{Ext}_{T_1}(-1,1)$ as well as main results of the Kreĭn theory have also been obtained there. In turn, the proof of formulas for T_m and T_M was based on a result of Yu. L. Shmul'yan [26] (see also [27]) of nonnegative completions of a nonnegative block operator.

Recently, S. Hassi and the author [7] extended the main result of [16] to the case of "quasi-contractive" symmetric operators T_1. Recall, that the "quasi-contractivity" means that $\nu_-(I - T^*T) < \infty$, where

$$\nu_-(K) = \dim(E_K(-\infty, 0)\mathfrak{H}).$$

For this purpose the above mentioned result of Shmul'yan was generalized there. Also an analog of block matrix formulas for the operators T_m and

T_M was established. Formulas T_m and T_M in this case look similar to (1.1) but the entries $V(I \pm T_{11})V^*$ are replaced by $V(I \pm T_{11})JV^*$, where $J = \operatorname{sign}(I - T_{11}^2)$ and $D_{T_{11}} := |I - T_{11}^2|^{1/2}$.

The first result of the present paper is a further generalization of Shmul'yan's result [26] to the case of block operators acting in a Kreĭn space and having a fixed number of negative eigenvalues.

In Section 4 a first Kreĭn space analog of completion problem is formulated and a description of its solutions is found. Namely, we consider classes of "quasi-contractive" symmetric operators T_1 in a Kreĭn space with $\nu_-(I - T_1^* T_1) < \infty$ and describe all possible selfadjoint (in the Kreĭn space sense) extensions T of T_1 which preserve the given negative index $\nu_-(I - T^*T) = \nu_-(I - T_1^* T_1)$. This problem is close to the completion problem studied in [7] and has a similar description for its solutions. For related problems see also [3–5, 10–16, 18, 20, 22–25, 27].

The main result of the present paper is Theorem 5.7. Namely, we consider classes of "quasi-contractive" symmetric operators T_1 in a Pontryagin space $(\mathfrak{H}, J)$ with

$$\nu_-[I - T_1^{[*]} T_1] := \nu_-(J(I - T_1^{[*]} T_1)) < \infty \tag{1.2}$$

and we establish a solvability criterion and a description of all possible selfadjoint extensions T of T_1 (in the Pontryagin space sense) which preserve the given negative index $\nu_-[I - T^{[*]}T] = \nu_-[I - T_1^{[*]} T_1]$. The formulas for T_m and T_M are also extended in an appropriate manner (see (5.16)). It should be emphasized that in this more general setting formulas (5.16) involve so-called link operator L_T which was introduced by Arsene, Constantintscu and Gheondea in [5] (see also [4, 10, 11, 21]).

2. A completion problem for block operators in Kreĭn spaces

By definition the modulus $|C|$ of a closed operator C is the nonnegative selfadjoint operator $|C| = (C^*C)^{1/2}$. Every closed operator admits a polar decomposition $C = U|C|$, where U is a (unique) partial isometry with the initial space $\overline{\operatorname{ran}}\,|C|$ and the final space $\overline{\operatorname{ran}}\,C$, cf. [17]. For a selfadjoint operator $H = \int_{\mathbb{R}} t\, dE_t$ in a Hilbert space $\mathfrak{H}$ the partial isometry U can be identified with the signature operator, which can be taken to be unitary: $J = \operatorname{sign}(H) = \int_{\mathbb{R}} \operatorname{sign}(t)\, dE_t$, in which case one should define $\operatorname{sign}(t) = 1$ if $t \geq 0$ and otherwise $\operatorname{sign}(t) = -1$.

Let $\mathcal{H}$ be a Hilbert space, and let $J_{\mathcal{H}}$ be a signature operator in it, i.e., $J_{\mathcal{H}} = J_{\mathcal{H}}^* = J_{\mathcal{H}}^{-1}$. We interpret the space $\mathcal{H}$ as a Kreĭn space $(\mathcal{H}, J_{\mathcal{H}})$

(see [6,8]) in which the indefinite scalar product is defined by the equality

$$[\varphi, \psi]_{\mathcal{H}} = (J_{\mathcal{H}}\varphi, \psi)_{\mathcal{H}}.$$

Let us introduce a partial ordering for selfadjoint Kreĭn space operators. For selfadjoint operators A and B with the same domains $A \geq_J B$ if and only if $[(A-B)f, f] \geq 0$ for all $f \in \operatorname{dom} A$. If not otherwise indicated the word "smallest" means the smallest operator in the sense of this partial ordering.

Consider a bounded incomplete block operator

$$A^0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & * \end{pmatrix} \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix} \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix} \tag{2.1}$$

in the Kreĭn space $\mathfrak{H} = (\mathfrak{H}_1 \oplus \mathfrak{H}_2, J)$, where $(\mathfrak{H}_1, J_1)$ and $(\mathfrak{H}_2, J_2)$ are Kreĭn spaces with fundamental symmetries J_1 and J_2, and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$.

Theorem 2.1. *Let $\mathfrak{H} = (\mathfrak{H}_1 \oplus \mathfrak{H}_2, J)$ be an orthogonal decomposition of the Kreĭn space $\mathfrak{H}$ and let A^0 be an incomplete block operator of the form (2.1). Assume that $A_{11} = A_{11}^{[*]}$ and $A_{21} = A_{12}^{[*]}$ are bounded, the numbers of negative squares of the quadratic form $[A_{11}f, f]$ ($f \in \operatorname{dom} A_{11}$) $\nu_-[A_{11}] := \nu_-(J_1 A_{11}) = \kappa < \infty$, where $\kappa \in \mathbb{Z}_+$, and let us introduce $J_{11} := \operatorname{sign}(J_1 A_{11})$ the (unitary) signature operator of $J_1 A_{11}$. Then:*

(i) *There exists a completion $A \in [(\mathfrak{H}, J)]$ of A^0 with some operator $A_{22} = A_{22}^{[*]} \in [(\mathfrak{H}_2, J_2)]$ such that $\nu_-[A] = \nu_-[A_{11}] = \kappa$ if and only if*

$$\operatorname{ran} J_1 A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}.$$

(ii) *In this case the operator $S = |A_{11}|^{[-1/2]} J_1 A_{12}$, where $|A_{11}|^{[-1/2]}$ denotes the (generalized) Moore–Penrose inverse of $|A_{11}|^{1/2}$, is well defined and $S \in [(\mathfrak{H}_2, J_2), (\mathfrak{H}_1, J_1)]$. Moreover, $S^{[*]} J_1 J_{11} S$ is the "smallest" operator in the solution set*

$$\mathcal{A} := \left\{ A_{22} = A_{22}^{[*]} \in [(\mathfrak{H}_2, J_2)] : \, A = (A_{ij})_{i,j=1}^2 : \nu_-[A] = \kappa \right\}$$

and this solution set admits a description

$$\mathcal{A} = \Big\{ A_{22} \in [(\mathfrak{H}_2, J_2)] : \, A_{22} = J_2(S^* J_{11} S + Y) = S^{[*]} J_1 J_{11} S + J_2 Y, \\ \text{where } Y = Y^* \geq 0 \Big\}.$$

Proof. Let us introduce a block operator

$$\widetilde{A}^0 = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & * \end{pmatrix} = \begin{pmatrix} J_1 A_{11} & J_1 A_{12} \\ J_2 A_{21} & * \end{pmatrix}.$$

The blocks of this operator satisfy the identities $\widetilde{A}_{11} = \widetilde{A}_{11}^*$, $\widetilde{A}_{21}^* = \widetilde{A}_{12}$ and

$$\begin{aligned} \operatorname{ran} J_1 A_{11} &= \operatorname{ran} \widetilde{A}_{11} \subset \operatorname{ran} |\widetilde{A}_{11}|^{1/2} = \operatorname{ran} (\widetilde{A}_{11}^* \widetilde{A}_{11})^{1/4} \\ &= \operatorname{ran} (A_{11}^* A_{11})^{1/4} = \operatorname{ran} |A_{11}|^{1/2}. \end{aligned}$$

Then due to [7, Theorem 1] a description of all selfadjoint operator completions of $\widetilde{A}^0$ admits representation $\widetilde{A} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}$ with $\widetilde{A}_{22} = \widetilde{S}^* J_{11} \widetilde{S} + Y$, where $\widetilde{S} = |\widetilde{A}_{11}|^{[-1/2]} \widetilde{A}_{12}$ and $Y = Y^* \geq 0$.

This yields description for the solutions of the completion problem. The set of completions has the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$$\begin{aligned} A_{22} &= J_2 \widetilde{A}_{22} = J_2 A_{21} J_1 |A_{11}|^{[-1/2]} J_{11} |A_{11}|^{[-1/2]} J_1 A_{12} + J_2 Y \\ &= J_2 S^* J_{11} S + J_2 Y = S^{[*]} J_1 J_{11} S + J_2 Y. \end{aligned}$$

□

3. Some inertia formulas

Some simple inertia formulas are now recalled. The factorization $H = B^{[*]} E B$ clearly implies that $\nu_\pm[H] \leq \nu_\pm[E]$, cf. (1.2). If H_1 and H_2 are selfadjoint operators in a Kreĭn space, then

$$H_1 + H_2 = \begin{pmatrix} I \\ I \end{pmatrix}^{[*]} \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix}$$

shows that $\nu_\pm[H_1 + H_2] \leq \nu_\pm[H_1] + \nu_\pm[H_2]$. Consider the selfadjoint block operator $H \in [(\mathfrak{H}_1, J_1) \oplus (\mathfrak{H}_2, J_2)]$, where $J_i = J_i^* = J_i^{-1}$, $(i = 1, 2)$ of the form

$$H = H^{[*]} = \begin{pmatrix} A & B^{[*]} \\ B & I \end{pmatrix},$$

By applying the above mentioned inequalities shows that

$$\nu_\pm[A] \leq \nu_\pm[A - B^{[*]} B] + \nu_\pm(J_2). \tag{3.1}$$

Assuming that $\nu_-[A - B^* J_2 B]$ and $\nu_-(J_2)$ are finite, the question when $\nu_-[A]$ attains its maximum in (3.1), or equivalently, $\nu_-[A - B^* J_2 B] \geq$

$\nu_-[A] - \nu_-(J_2)$ attains its minimum, turns out to be of particular interest. The next result characterizes this situation as an application of Theorem 2.1. Recall that if $J_1 A = J_A|A|$ is the polar decomposition of $J_1 A$, then one can interpret $\mathfrak{H}_A = (\overline{\operatorname{ran}}\, J_1 A, J_A)$ as a Kreĭn space generated on $\overline{\operatorname{ran}}\, J_1 A$ by the fundamental symmetry $J_A = \operatorname{sign}(J_1 A)$.

Theorem 3.1. *Let* $A \in [(\mathfrak{H}_1, J_1)]$ *be selfadjoint,* $B \in [(\mathfrak{H}_1, J_1), (\mathfrak{H}_2, J_2)]$, $J_i = J_i^* = J_i^{-1} \in [\mathfrak{H}_i]$, $(i = 1, 2)$, *and assume that* $\nu_-[A], \nu_-(J_2) < \infty$. *If the equality*

$$\nu_-[A] = \nu_-[A - B^{[*]}B] + \nu_-(J_2)$$

holds, then $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$ *and* $J_1 B^{[*]} = |A|^{1/2} K$ *for a unique operator* $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$ *which is* J*-contractive:* $J_2 - K^* J_A K \geq 0$.

Conversely, if $B^{[*]} = |A|^{1/2} K$ *for some* J*-contractive operator* $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$, *then the equality* (3.1) *is satisfied.*

Proof. Assume that (3.1) is satisfied. The factorization

$$H = \begin{pmatrix} A & B^{[*]} \\ B & I \end{pmatrix} = \begin{pmatrix} I & B^{[*]} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - B^{[*]}B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$$

shows that $\nu_-[H] = \nu_-[A - B^{[*]}B] + \nu_-(J_2)$, which combined with the equality (3.1) gives $\nu_-[H] = \nu_-[A]$. Therefore, by Theorem 2.1 one has $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$ and this is equivalent to the existence of a unique operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$ such that $J_1 B^{[*]} = |A|^{1/2} K$; i.e. $K = |A|^{[-1/2]} J_1 B^{[*]}$. Furthermore, $K^{[*]} J_1 J_A K \leq_{J_2} I$ by the minimality property of $K^{[*]} J_1 J_A K$ in Theorem 2.1, in other words K is a J-contraction.

Converse, if $J_1 B^{[*]} = |A|^{1/2} K$ for some J-contractive operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$, then clearly $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$. By Theorem 2.1 the completion problem for H^0 has solutions with the minimal solution $S^{[*]} J_1 J_A S$, where

$$S = |A|^{[-1/2]} J_1 B^{[*]} = |A|^{[-1/2]} |A|^{1/2} K = K.$$

Furthermore, by J-contractivity of K one has $K^{[*]} J_1 J_A K \leq_{J_2} I$, i.e. I is also a solution and thus $\nu_-[H] = \nu_-[A]$ or, equivalently, the equality (3.1) is satisfied. □

4. A pair of completion problems in a Kreĭn space

In this section we introduce and describe the solutions of a Kreĭn space version of a completion problem that was treated in [7].

Let $(\mathfrak{H}_i, (J_i\cdot,\cdot))$ and $(\mathfrak{H}, (J\cdot,\cdot))$ be Kreĭn spaces, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$, and J_i are fundamental symmetries ($i = 1, 2$), let $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ be an operator such that $\nu_-(I - T_{11}^* T_{11}) = \kappa < \infty$. Denote $\widetilde{T}_{11} = J_1 T_{11}$, then $\widetilde{T}_{11} = \widetilde{T}_{11}^*$ in the Hilbert space $\mathfrak{H}_1$. Rewrite $\nu_-(I - T_{11}^* T_{11}) = \nu_-(I - \widetilde{T}_{11}^2)$. Denote

$$J_+ = \operatorname{sign}(I - \widetilde{T}_{11}),\ J_- = \operatorname{sign}(I + \widetilde{T}_{11}),\ \text{and } J_{11} = \operatorname{sign}(I - \widetilde{T}_{11}^2),$$

and let $\kappa_+ = \nu_-(J_+)$ and $\kappa_- = \nu_-(J_-)$. It is easy to get that $J_{11} = J_- J_+ = J_+ J_-$. Moreover, there is an equality $\kappa = \kappa_- + \kappa_+$ (see [7, Lemma 5.1]). We recall the results for the operator $\widetilde{T}_{11}$ from the paper [7] and after that reformulate them for the operator T_{11}. We recall completion problem and its solutions that was investigated in a Hilbert space setting in [7]. The problem concerns the existence and a description of selfadjoint operators $\widetilde{T}$ such that $\widetilde{A}_+ = I + \widetilde{T}$ and $\widetilde{A}_- = I - \widetilde{T}$ solve the corresponding completion problems

$$\widetilde{A}_\pm^0 = \begin{pmatrix} I \pm \widetilde{T}_{11} & \pm \widetilde{T}_{21}^* \\ \pm \widetilde{T}_{21} & * \end{pmatrix}, \tag{4.1}$$

under *minimal index conditions* $\nu_-(I + \widetilde{T}) = \nu_-(I + \widetilde{T}_{11})$, $\nu_-(I - \widetilde{T}) = \nu_-(I - \widetilde{T}_{11})$, respectively. The solution set is denoted by $\operatorname{Ext}_{\widetilde{T}_1,\kappa}(-1, 1)$.

The next theorem gives a general solvability criterion for the completion problem (4.1) and describes all solutions to this problem.

Theorem 4.1. *([7, Theorem 5]) Let* $\widetilde{T}_1 = \begin{pmatrix} \widetilde{T}_{11} \\ \widetilde{T}_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}$ *be a symmetric operator with* $\widetilde{T}_{11} = \widetilde{T}_{11}^* \in [\mathfrak{H}_1]$ *and* $\nu_-(I - \widetilde{T}_{11}^2) = \kappa < \infty$, *and let* $J_{11} = \operatorname{sign}(I - \widetilde{T}_{11}^2)$. *Then the completion problem for* $\widetilde{A}_\pm^0$ *in* (4.1) *has a solution* $I \pm \widetilde{T}$ *for some* $\widetilde{T} = \widetilde{T}^*$ *with* $\nu_-(I - \widetilde{T}^2) = \kappa$ *if and only if the following condition is satisfied:*

$$\nu_-(I - \widetilde{T}_{11}^2) = \nu_-(I - \widetilde{T}_1^* \widetilde{T}_1). \tag{4.2}$$

If this condition is satisfied then the following facts hold:

(i) *The completion problems for* $\widetilde{A}_\pm^0$ *in* (4.1) *have minimal solutions* $\widetilde{A}_\pm$.

(ii) *The operators* $\widetilde{T}_m := \widetilde{A}_+ - I$ *and* $\widetilde{T}_M := I - \widetilde{A}_- \in \operatorname{Ext}_{\widetilde{T}_1,\kappa}(-1, 1)$.

(iii) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *have the block form*

$$\begin{aligned}\widetilde{T}_m &= \begin{pmatrix} \widetilde{T}_{11} & D_{\widetilde{T}_{11}}V^* \\ VD_{\widetilde{T}_{11}} & -I+V(I-\widetilde{T}_{11})J_{11}V^* \end{pmatrix},\\ \widetilde{T}_M &= \begin{pmatrix} \widetilde{T}_{11} & D_{\widetilde{T}_{11}}V^* \\ VD_{\widetilde{T}_{11}} & I-V(I+\widetilde{T}_{11})J_{11}V^* \end{pmatrix},\end{aligned} \tag{4.3}$$

where $D_{\widetilde{T}_{11}} := |I-\widetilde{T}_{11}^2|^{1/2}$ *and* V *is given by* $V := \operatorname{clos}(\widetilde{T}_{21}D_{\widetilde{T}_{11}}^{[-1]})$.

(iv) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *are extremal extensions of* $\widetilde{T}_1$*:*

$$\widetilde{T} \in \operatorname{Ext}_{\widetilde{T}_1,\kappa}(-1,1) \quad \textit{iff} \quad \widetilde{T} = \widetilde{T}^* \in [\mathfrak{H}], \quad \widetilde{T}_m \le \widetilde{T} \le \widetilde{T}_M.$$

(v) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *are connected via*

$$(-\widetilde{T})_m = -\widetilde{T}_M, \quad (-\widetilde{T})_M = -\widetilde{T}_m.$$

For what follows it is convenient to reformulate the above theorem in a Kreĭn space setting. Consider the Kreĭn space $(\mathfrak{H}, J)$ and a selfadjoint operator T in this space. Now the problem concerns selfadjoint operators $A_+ = I+T$ and $A_- = I-T$ in the Kreĭn space $(\mathfrak{H}, J)$ that solve the completion problems

$$A_\pm^0 = \begin{pmatrix} I\pm T_{11} & \pm T_{21}^{[*]} \\ \pm T_{21} & * \end{pmatrix}, \tag{4.4}$$

under *minimal index conditions* $\nu_-(I+JT) = \nu_-(I+J_1T_{11})$ and $\nu_-(I-JT) = \nu_-(I-J_1T_{11})$, respectively. The set of solutions T to the problem (4.4) will be denoted by $\operatorname{Ext}_{J_2T_1,\kappa}(-1,1)$.

Denote

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}, \tag{4.5}$$

so that T_1 is symmetric (nondensely defined) operator in the Kreĭn space $[(\mathfrak{H}_1, J_1)]$, i.e. $T_{11} = T_{11}^{[*]}$.

Theorem 4.2. *Let* T_1 *be a symmetric operator in a Kreĭn space sense as in* (4.5) *with* $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ *and* $\nu_-(I-T_{11}^*T_{11}) = \kappa < \infty$, *and let* $J = \operatorname{sign}(I-T_{11}^*T_{11})$. *Then the completion problems for* $A_\pm^0$ *in* (4.4) *have a solution* $I\pm T$ *for some* $T = T^{[*]}$ *with* $\nu_-(I-T^*T) = \kappa$ *if and only if the following condition is satisfied:*

$$\nu_-(I-T_{11}^*T_{11}) = \nu_-(I-T_1^*T_1). \tag{4.6}$$

If this condition is satisfied then the following facts hold:

(i) *The completion problems for* $A^0_\pm$ *in* (4.4) *have "minimal" (*J_2*-minimal) solutions* $A_\pm$*.*

(ii) *The operators* $T_m := A_+ - J$ *and* $T_M := J - A_- \in \operatorname{Ext}_{J_2 T_1,\kappa}(-1,1)$*.*

(iii) *The operators* T_m *and* T_M *have the block form*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & -J_2 + J_2 V(I - J_1 T_{11}) J_{11} V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & J_2 - J_2 V(I + J_1 T_{11}) J_{11} V^* \end{pmatrix}, \end{aligned} \tag{4.7}$$

where $D_{T_{11}} := |I - T_{11}^* T_{11}|^{1/2}$ *and* V *is given by* $V := \operatorname{clos}(J_2 T_{21} D_{T_{11}}^{[-1]})$*.*

(iv) *The operators* T_m *and* T_M *are* J_2*-extremal extensions of* T_1*:*

$$T \in \operatorname{Ext}_{J_2 T_1,\kappa}(-1,1) \quad \textit{iff} \quad T = T^{[*]} \in [(\mathfrak{H}, J)], \quad T_m \le_{J_2} T \le_{J_2} T_M.$$

(v) *The operators* T_m *and* T_M *are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m.$$

Proof. The proof is obtained by systematic use of the equivalence that T is a selfadjoint operator in a Kreĭn space if and only if $\widetilde{T}$ is a selfadjoint in a Hilbert space. In particular, T gives solutions to the completion problems (4.4) if and only if $\widetilde{T}$ solves the completion problems (4.4). In view of

$$I - T_{11}^* T_{11} = I - T_{11}^* J J T_{11} = I - \widetilde{T}_{11}^2,$$

we are getting formula (4.6) from (4.2). Then formula (4.7) follows by multiplying the operators in (4.3) by the fundamental symmetry. □

5. Completion problem in a Pontryagin space

5.1. Defect operators and link operators

Let $(\mathfrak{H}, (\cdot,\cdot))$ be a Hilbert space and let J be a symmetry in $\mathfrak{H}$, i.e. $J = J^* = J^{-1}$, so that $(\mathfrak{H}, (J\cdot,\cdot))$, becomes a Pontryagin space. Then associate with $T \in [\mathfrak{H}]$ the corresponding defect and signature operators

$$D_T = |J - T^* J T|^{1/2}, \quad J_T = \operatorname{sign}(J - T^* J T), \quad \mathfrak{D}_T = \overline{\operatorname{ran}}\, D_T,$$

where the so-called defect subspace $\mathfrak{D}_T$ can be considered as a Pontryagin space with the fundamental symmetry J_T. Similar notations are used with T^*:

$$D_{T^*} = |J - T J T^*|^{1/2}, \quad J_{T^*} = \operatorname{sign}(J - T J T^*), \quad \mathfrak{D}_{T^*} = \overline{\operatorname{ran}}\, D_{T^*}.$$

By definition $J_T D_T^2 = J - T^* JT$ and $J_T D_T = D_T J_T$ with analogous identities for D_{T^*} and J_{T^*}. In addition,

$$(J - T^* JT)JT^* = T^* J(J - TJT^*),\ (J - TJT^*)JT = TJ(J - T^* JT).$$

Recall that $T \in [\mathfrak{H}]$ is said to be a J-contraction if $J - T^* JT \geq 0$, i.e. $\nu_-(J - T^* JT) = 0$. If, in addition, T^* is a J-contraction, T is termed as a J-bicontraction.

For the following consideration an indefinite version of the commutation relation of the form $TD_T = D_{T^*} T$ is needed; these involve so-called link operators introduced in [5, Section 4] (see also [7]).

Definition 5.1. *There exist unique operators $L_T \in [\mathfrak{D}_T, \mathfrak{D}_{T^*}]$ and $L_{T^*} \in [\mathfrak{D}_{T^*}, \mathfrak{D}_T]$ such that*

$$D_{T^*} L_T = TJD_T \restriction \mathfrak{D}_T, \quad D_T L_{T^*} = T^* JD_{T^*} \restriction \mathfrak{D}_{T^*}; \tag{5.1}$$

in fact, $L_T = D_{T^*}^{[-1]} TJD_T \restriction \mathfrak{D}_T$ and $L_{T^*} = D_T^{[-1]} T^* JD_{T^*} \restriction \mathfrak{D}_{T^*}$.

The following identities can be obtained with direct calculations; see [5, Section 4]:

$$\begin{gathered} L_T^* J_{T^*} \restriction \mathfrak{D}_{T^*} = J_T L_{T^*}; \\ (J_T - D_T J D_T) \restriction \mathfrak{D}_T = L_T^* J_{T^*} L_T; \\ (J_{T^*} - D_{T^*} J D_{T^*}) \restriction \mathfrak{D}_{T^*} = L_{T^*}^* J_T L_{T^*}. \end{gathered} \tag{5.2}$$

Now let T be selfadjoint in Pontryagin space $(\mathfrak{H}, J)$, i.e. $T^* = JTJ$. Then connections between D_{T^*} and D_T, J_{T^*} and J_T, L_{T^*} and L_T can be established.

Lemma 5.1. *Assume that $T^* = JTJ$. Then $D_T = |I - T^2|^{1/2}$ and the following equalities hold:*

$$D_{T^*} = JD_T J, \tag{5.3}$$

in particular,

$$\mathfrak{D}_{T^*} = J\mathfrak{D}_T \text{ and } \mathfrak{D}_T = J\mathfrak{D}_{T^*};$$

$$J_{T^*} = JJ_T J; \tag{5.4}$$

$$L_{T^*} = JL_T J. \tag{5.5}$$

Proof. The defect operator of T can be calculated by the formula

$$D_T = \left(\left(I - (T^*)^2 \right) JJ(I - T^2) \right)^{1/4} = \left(\left(I - (T^*)^2 \right) (I - T^2) \right)^{1/4}.$$

Then

$$\begin{aligned} D_{T^*} &= \left(J\left(I-(T^*)^2\right)(I-T^2)J\right)^{1/4} = J\left(\left(I-(T^*)^2\right)(I-T^2)\right)^{1/4} J \\ &= JD_TJ \end{aligned}$$

i.e. (5.3) holds. This implies

$$J\mathfrak{D}_{T^*} \subset \mathfrak{D}_T \text{ and } J\mathfrak{D}_T \subset \mathfrak{D}_{T^*}.$$

Hence from the last two formulas we get

$$\mathfrak{D}_{T^*} = J(J\mathfrak{D}_{T^*}) \subset J\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$$

and similarly

$$\mathfrak{D}_T = J(J\mathfrak{D}_T) \subset J\mathfrak{D}_{T^*} \subset \mathfrak{D}_T.$$

The formula

$$\begin{aligned} J_TD_T^2 &= J - T^*JT = J(J-TJT^*)J = JJ_{T^*}D_{T^*}^2J = JJ_{T^*}JD_T^2JJ \\ &= JJ_{T^*}JD_T^2 \end{aligned}$$

yields the equation (5.4).

The relation (5.5) follows from

$$D_TL_{T^*} = T^*JD_{T^*}\restriction\mathfrak{D}_{T^*} = JTJD_TJ\restriction\mathfrak{D}_{T^*} = JD_{T^*}L_TJ = D_TJL_TJ.$$

□

5.2. Lemmas on negative indices of certain block operators

The first two lemmas are of preparatory nature for the last two lemmas, which are used for the proof of the main theorem.

Lemma 5.2. *Let* $\begin{pmatrix} J & T \\ T & J \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix}$ *be a selfadjoint operator in the Hilbert space* $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$. *Then*

$$\left|\begin{pmatrix} J & T \\ T & J \end{pmatrix}\right|^{1/2} = U \begin{pmatrix} |J+T|^{1/2} & 0 \\ 0 & |J-T|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ *is a unitary operator.*

Proof. It is easy to check that

$$\begin{pmatrix} J & T \\ T & J \end{pmatrix} = U \begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix} U^*. \tag{5.6}$$

Then by taking the modulus one gets

$$\left| \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right|^2 = \left(\begin{pmatrix} J & T \\ T & J \end{pmatrix}^* \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right) = U \begin{pmatrix} |J+T|^2 & 0 \\ 0 & |J-T|^2 \end{pmatrix} U^*.$$

The last step is to extract the square roots (twice) from the both sides of the equation:

$$\left| \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right|^{1/2} = U \begin{pmatrix} |J+T|^{1/2} & 0 \\ 0 & |J-T|^{1/2} \end{pmatrix} U^*.$$

The right hand side can be written in this form because U is unitary. □

Lemma 5.3. *Let $T = T^* \in \mathfrak{H}$ be a selfadjoint operator in a Hilbert space $\mathfrak{H}$ and let $J = J^* = J^{-1}$ be a fundamental symmetry in $\mathfrak{H}$ with $\nu_-(J) < \infty$. Then*

$$\nu_-(J - TJT) + \nu_-(J) = \nu_-(J - T) + \nu_-(J + T). \tag{5.7}$$

In particular, $\nu_-(J - TJT) < \infty$ if and only if $\nu_-(J \pm T) < \infty$.

Proof. Consider block operators $\begin{pmatrix} J & T \\ T & J \end{pmatrix}$ and $\begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$. Equality (5.6) yields $\nu_- \begin{pmatrix} J & T \\ T & J \end{pmatrix} = \nu_- \begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$. The negative index of $\begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$ equals $\nu_-(J-T)+\nu_-(J+T)$ and the negative index of $\begin{pmatrix} J & T \\ T & J \end{pmatrix}$ is easy to find by using the equality

$$\begin{pmatrix} J & T \\ T & J \end{pmatrix} = \begin{pmatrix} I & 0 \\ TJ & I \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J - TJT \end{pmatrix} \begin{pmatrix} I & JT \\ 0 & I \end{pmatrix}. \tag{5.8}$$

Then one gets (5.7). □

Let $(\mathfrak{H}_i, (J_i \cdot, \cdot))$ $(i = 1, 2)$ and $(\mathfrak{H}, (J \cdot, \cdot))$ be Pontryagin spaces, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$. Consider an operator $T_{11} = T_{11}^{[*]} \in$

$[(\mathfrak{H}_1, J_1)]$ such that $\nu_-[I - T_{11}^2] = \kappa < \infty$; see (1.2). Denote $\widetilde{T}_{11} = J_1 T_{11}$, then $\widetilde{T}_{11} = \widetilde{T}_{11}^*$ in the Hilbert space $\mathfrak{H}_1$. Rewrite

$$\begin{aligned}\nu_-[I - T_{11}^2] &= \nu_-(J_1(I - T_{11}^2)) = \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) \\ &= \nu_-((J_1 - \widetilde{T}_{11}) J_1 (J_1 + \widetilde{T}_{11})).\end{aligned}$$

Furthermore, denote

$$\begin{aligned} J_+ &= \operatorname{sign}(J_1(I - T_{11})) = \operatorname{sign}(J_1 - \widetilde{T}_{11}), \\ J_- &= \operatorname{sign}(J_1(I + T_{11})) = \operatorname{sign}(J_1 + \widetilde{T}_{11}), \\ J_{11} &= \operatorname{sign}(J_1(I - T_{11}^2)) \end{aligned} \tag{5.9}$$

and let $\kappa_+ = \nu_-[I - T_{11}]$ and $\kappa_- = \nu_-[I + T_{11}]$. Notice that $|I \mp T_{11}| = |J_1 \mp \widetilde{T}_{11}|$ and one has polar decompositions

$$I \mp T_{11} = J_1 J_\pm |I \mp T_{11}|. \tag{5.10}$$

Lemma 5.4. *Let* $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ *and* $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in [(\mathfrak{H}, J)]$ *be a selfadjoint extension of the operator* T_{11} *with* $\nu_-[I \pm T_{11}] < \infty$ *and* $\nu_-(J) < \infty$. *Then the following statements*

(i) $\nu_-[I \pm T_{11}] = \nu_-[I \pm T]$;

(ii) $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$;

(iii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$

are connected by the implications $(i) \Leftrightarrow (ii) \Rightarrow (iii)$.

Proof. The Lemma can be formulated in an equivalent way for the Hilbert space operators: the block operator $\widetilde{T} = JT = \begin{pmatrix} \widetilde{T}_{11} & \widetilde{T}_{12} \\ \widetilde{T}_{21} & \widetilde{T}_{22} \end{pmatrix}$ is a selfadjoint extension of $\widetilde{T}_{11} = \widetilde{T}_{11}^* \in [\mathfrak{H}_1]$. Then the following statements

(i') $\nu_-(J_1 \pm \widetilde{T}_{11}) = \nu_-(J \pm \widetilde{T})$

(ii') $\nu_-(J - \widetilde{T} J \widetilde{T}) = \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) - \nu_-(J_2)$;

(iii') $\operatorname{ran} \widetilde{T}_{12} \subset \operatorname{ran} |J_1 \pm \widetilde{T}_{11}|^{1/2}$

are connected by the implications $(i') \Leftrightarrow (ii') \Rightarrow (iii')$.

Hence it's sufficient to prove this form of the Lemma.

Let us prove the equivalence $(i') \Leftrightarrow (ii')$. Condition (ii') is equivalent to

$$\nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix}. \tag{5.11}$$

Indeed, in view of (5.8)

$$\nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_-(J_1) + \nu_-(J_1 - \widetilde{T}_{11}J_1\widetilde{T}_{11})$$

and

$$\begin{aligned}\nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix} &= \nu_-(J) + \nu_-(J - \widetilde{T}J\widetilde{T}) \\ &= \nu_-(J_1) + \nu_-(J_2) + \nu_-(J - \widetilde{T}J\widetilde{T}).\end{aligned}$$

By using Lemma 5.3, equality (5.11) is equivalent to

$$\nu_-(J_1 - \widetilde{T}_{11}) + \nu_-(J_1 + \widetilde{T}_{11}) = \nu_-(J - \widetilde{T}) + \nu_-(J + \widetilde{T}). \tag{5.12}$$

Hence, $(i') \Rightarrow (ii')$.

Because $\nu_-(J_1 \pm \widetilde{T}_{11}) \leq \nu_-(J \pm \widetilde{T})$, then (5.12) shows that $(ii') \Rightarrow (i')$.

Now we prove implication $(ii') \Rightarrow (iii')$;the arguments here will be useful also for the proof of Lemma 5.5 below. Use a permutation to transform the matrix in the right hand side of (5.11):

$$\nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & 0 & \widetilde{T}_{11} & \widetilde{T}_{12} \\ 0 & J_2 & \widetilde{T}_{21} & \widetilde{T}_{22} \\ \widetilde{T}_{11} & \widetilde{T}_{12} & J_1 & 0 \\ \widetilde{T}_{21} & \widetilde{T}_{22} & 0 & J_2 \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} & 0 & \widetilde{T}_{12} \\ \widetilde{T}_{11} & J_1 & \widetilde{T}_{12} & 0 \\ 0 & \widetilde{T}_{21} & J_2 & \widetilde{T}_{22} \\ \widetilde{T}_{21} & 0 & \widetilde{T}_{22} & J_2 \end{pmatrix}.$$

Then condition (5.11) implies to the condition

$$\operatorname{ran}\begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} \subset \operatorname{ran}\left|\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix}\right|^{1/2};$$

(see Theorem 2.1). By Lemma 5.2 the last inclusion can be rewritten as

$$\operatorname{ran}\begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} \subset \operatorname{ran} U \begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is a unitary operator. This inclusion is equivalent to

$$\begin{aligned} \operatorname{ran} U^* \begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} U &= \operatorname{ran} \begin{pmatrix} \widetilde{T}_{12} & 0 \\ 0 & -\widetilde{T}_{12} \end{pmatrix} \\ &\subset \operatorname{ran} \begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} \end{aligned}$$

and clearly this is equivalent to condition (iii').

Note that if $\widetilde{T}_{11}$ has a selfadjoint extension $\widetilde{T}$ satisfying (i'). Then by applying Theorem 2.1 (or [7, Theorem 1]) it yields (iii'). □

Lemma 5.5. *Let $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ be an operator and let*

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}$$

be an extension of T_{11} with $\nu_-[I - T_{11}^2] < \infty$, $\nu_-(J_1) < \infty$, and $\nu_-(J_2) < \infty$. Then for the conditions

(i) $\nu_-[I_1 - T_{11}^2] = \nu_-[I_1 - T_1^{[*]}T_1] + \nu_-(J_2)$*;*

(ii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}^2|^{1/2}$*;*

(iii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$

the implications (i) ⇒ (ii) and (i) ⇒ (iii) hold.

Proof. First we prove that (i)⇒(ii). In fact, this follows from Theorem 3.1 by taking $A = I - T_{11}^2$ and $B = T_{21}$.

A proof of (i)⇒(iii) is quite similar to the proof used in Lemma 5.4. Statement (i) is equivalent the following equation:

$$\nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix}.$$

Indeed,

$$\begin{aligned} \nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} &= \nu_- \begin{pmatrix} J_1 & 0 \\ 0 & J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11} \end{pmatrix} \\ &= \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) + \nu_-(J_1) < \infty \end{aligned}$$

and

$$\nu_-\begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix} = \nu_-\begin{pmatrix} J & 0 \\ 0 & J_1 - \widetilde{T}_1^* J \widetilde{T}_1 \end{pmatrix}$$
$$= \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11} - \widetilde{T}_{21}^* J_2 \widetilde{T}_{21}) + \nu_-(J_1) + \nu_-(J_2).$$

Due to (i) the right hand sides coincide and then the left hand sides coincide as well.

Now let us permutate the matrix in the latter equation.

$$\nu_-\begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & 0 & \widetilde{T}_{11} \\ 0 & J_2 & \widetilde{T}_{21} \\ \widetilde{T}_{11} & \widetilde{T}_{21}^* & J_1 \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} & 0 \\ \widetilde{T}_{11} & J_1 & \widetilde{T}_{21}^* \\ 0 & \widetilde{T}_{21} & J_2 \end{pmatrix}.$$

It follows from [7, Theorem 1] that the condition (i) implies the condition

$$\operatorname{ran}\begin{pmatrix} 0 \\ \widetilde{T}_{21}^* \end{pmatrix} \subset \operatorname{ran}\left|\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix}\right|^{1/2}$$
$$= \operatorname{ran} U \begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is a unitary operator (see Lemma 5.2). Then, equivalently,

$$\operatorname{ran}\widetilde{T}_{21}^* \subset \operatorname{ran}|J_1 \pm \widetilde{T}_{11}|^{1/2}.$$

□

5.3. Contractive extensions of contractions with minimal negative indices

Following to [7, 16, 18] we consider the problem of existence and a description of selfadjoint operators T in the Pontryagin space $\begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}$ such that $A_+ = I + T$ and $A_- = I - T$ solve the corresponding completion problems

$$A_\pm^0 = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^{[*]} \\ \pm T_{21} & * \end{pmatrix}, \tag{5.13}$$

under *minimal index conditions* $\nu_-[I + T] = \nu_-[I + T_{11}]$, $\nu_-[I - T] = \nu_-[I - T_{11}]$, respectively. Observe, that by Lemma 5.4 the two minimal index conditions above are equivalent to single condition $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$.

It is clear from Theorem 2.1 that the conditions $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}|^{1/2}$ and $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I + T_{11}|^{1/2}$ are necessary for the existence of solutions; however as noted already in [7] they are not sufficient even in the Hilbert space setting.

The next theorem gives a general solvability criterion for the completion problem (5.13) and describes all solutions to this problem. As in the definite case, there are minimal solutions A_+ and A_- which are connected to two extreme selfadjoint extensions T of

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}, \tag{5.14}$$

now with finite negative index $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2) > 0$. The set of solutions T to the problem (5.13) will be denoted by $\operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$.

Theorem 5.1. *Let T_1 be a symmetric operator as in (5.14) with $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ and $\nu_-[I - T_{11}^2] = \kappa < \infty$, and let $J_{T_{11}} = \operatorname{sign}(J_1(I - T_{11}^2))$. Then the completion problem for $A_\pm^0$ in (5.13) has a solution $I \pm T$ for some $T = T^{[*]}$ with $\nu_-[I - T^2] = \kappa - \nu_-(J_2)$ if and only if the following condition is satisfied:*

$$\nu_-[I - T_{11}^2] = \nu_-[I - T_1^{[*]} T_1] + \nu_-(J_2). \tag{5.15}$$

If this condition is satisfied then the following facts hold:

(i) *The completion problems for $A_\pm^0$ in (5.13) have "minimal" solutions $A_\pm$ (for the partial ordering introduced in the first section).*

(ii) *The operators $T_m := A_+ - I$ and $T_M := I - A_- \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$.*

(iii) *The operators T_m and T_M have the block form*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & -I + J_2 V (I - L_T^* J_1) J_{11} V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & I - J_2 V (I + L_T^* J_1) J_{11} V^* \end{pmatrix}, \end{aligned} \tag{5.16}$$

where $D_{T_{11}} := |I - T_{11}^2|^{1/2}$ and V is given by $V := \operatorname{clos}(J_2 T_{21} D_{T_{11}}^{[-1]})$.

(iv) *The operators T_m and T_M are "extremal" extensions of T_1:*

$$T \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2} \quad \textit{iff} \quad T = T^{[*]} \in [(\mathfrak{H}, J)], \quad T_m \le_{J_2} T \le_{J_2} T_M. \tag{5.17}$$

(v) *The operators T_m and T_M are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m. \tag{5.18}$$

Proof. It is easy to see by (3.1) that $\kappa = \nu_-[I - T_{11}^2] \leq \nu_-[I - T_1^{[*]}T_1] + \nu_-(J_2) \leq \nu_-[I-T^2]+\nu_-(J_2)$. Hence the condition $\nu_-[I-T^2] = \kappa-\nu_-(J_2)$ implies (5.15). The sufficiency of this condition is obtained when proving the assertions (i)–(iii) below.

(i) If the condition (5.15) is satisfied then by using Lemma 5.5 one gets the inclusions $\operatorname{ran} J_1T_{21}^{[*]} \subset \operatorname{ran}|I \pm T_{11}|^{1/2}$, which by Theorem 2.1 means that each of the completion problems, $A_\pm^0$ in (5.13), is solvable. It follows that the operators

$$S_- = |I + T_{11}|^{[-1/2]} J_1 T_{21}^{[*]}, \quad S_+ = |I - T_{11}|^{[-1/2]} J_1 T_{21}^{[*]} \tag{5.19}$$

are well defined and they provide the minimal solutions $A_\pm$ to the completion problems for $A_\pm^0$ in (5.13).

(ii) & (iii) By Lemma 5.5 the inclusion $\operatorname{ran} J_1T_{21}^{[*]} \subset \operatorname{ran}|I - T_{11}^2|^{1/2}$ holds. This inclusion is equivalent to the existence of a (unique) bounded operator $V^* = D_{T_{11}}^{[-1]} J_1 T_{21}^{[*]}$ with $\ker V \supset \ker D_{T_{11}}$, such that $J_1T_{21}^{[*]} = D_{T_{11}}V^*$. The operators $T_m := A_+ - I$ and $T_M := I - A_-$ (see proof of (i)) by using (5.1), (5.2), and 5.1 can be now rewritten as in (5.16). Indeed, observe that (see Theorem 2.1, (5.9), and (5.10))

$$\begin{aligned}
J_2S_-^*J_-S_- &= J_2VD_{T_{11}}|I + T_{11}|^{[-1/2]}J_-|I + T_{11}|^{[-1/2]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}(J_1(I + T_{11}))^{[-1]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}D_{T_{11}}^{[-1]}(I + L_{T_{11}}^*J_1)^{[-1]}D_{T_{11}}J_1D_{T_{11}}V^* \\
&= J_2V(I + L_{T_{11}}^*J_1)^{[-1]}(J_{11} - L_{T_{11}}^*J_{T_{11}^*}L_{T_{11}})V^* \\
&= J_2V(I + L_{T_{11}}^*J_1)^{[-1]}(J_{11} - (L_{T_{11}}^*J_1)^2J_{11})V^* \\
&= J_2V(I + L_{T_{11}}^*J_1)^{[-1]}(I + L_{T_{11}}^*J_1)(I - L_{T_{11}}^*J_1)J_{11}V^* \\
&= J_2V(I - L_{T_{11}}^*J_1)J_{11}V^*,
\end{aligned}$$

where the third equality follows from (5.1) and the fourth from (5.2).

And similarly for

$$\begin{aligned}
J_2S_+^*J_+S_+ &= J_2VD_{T_{11}}|I-T_{11}|^{[-1/2]}J_+|I-T_{11}|^{[-1/2]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}(J_1(I-T_{11}))^{[-1]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}D_{T_{11}}^{[-1]}(I-L_{T_{11}}^*J_1)^{[-1]}D_{T_{11}}J_1D_{T_{11}}V^* \\
&= J_2V(I-L_{T_{11}}^*J_1)^{[-1]}(J_{11}-L_{T_{11}}^*J_{T_{11}^*}L_{T_{11}})V^* \\
&= J_2V(I-L_{T_{11}}^*J_1)^{[-1]}(J_{11}-(L_{T_{11}}^*J_1)^2J_{11})V^* \\
&= J_2V(I-L_{T_{11}}^*J_1)^{[-1]}(I-L_{T_{11}}^*J_1)(I+L_{T_{11}}^*J_1)J_{11}V^* \\
&= J_2V(I+L_{T_{11}}^*J_1)J_{11}V^*,
\end{aligned}$$

which implies the representations for T_m and T_M in (5.16). Clearly, T_m and T_M are selfadjoint extensions of T_1, which satisfy the equalities

$$\nu_-[I+T_m]=\kappa_-, \quad \nu_-[I-T_M]=\kappa_+.$$

Moreover, it follows from (5.16) that

$$T_M-T_m=\begin{pmatrix}0 & 0\\ 0 & 2(I-J_2VJ_{11}V^*)\end{pmatrix}. \tag{5.20}$$

Now the assumption (5.15) will be used again. Since $\nu_-[I-T_1^{[*]}T_1]=\nu_-[I-T_{11}^2]-\nu_-(J_2)$ and $T_{21}=J_2VD_{T_{11}}$ it follows from Theorem 3.1 that $V^*\in[\mathfrak{H}_2,\mathfrak{D}_{T_{11}}]$ is J-contractive: $J_2-VJ_{11}V^*\geq 0$. Therefore, (5.20) shows that $T_M\geq_{J_2}T_m$ and $I+T_M\geq_{J_2}I+T_m$ and hence, in addition to $I+T_m$, also $I+T_M$ is a solution to the problem A_+^0 and, in particular, $\nu_-[I+T_M]=\kappa_-=\nu_-[I+T_m]$. Similarly, $I-T_M\leq_{J_2}I-T_m$ which implies that $I-T_m$ is also a solution to the problem A_-^0, in particular, $\nu_-[I-T_m]=\kappa_+=\nu_-[I-T_M]$. Now by applying Lemma 5.4 we get

$$\nu_-[I-T_m^2]=\kappa-\nu_-(J_2),$$

$$\nu_-[I-T_M^2]=\kappa-\nu_-(J_2).$$

Therefore, $T_m,T_M\in\operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$ which in particular proves that the condition (5.15) is sufficient for solvability of the completion problem (5.13).

(iv) Observe, that $T\in\operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$ if and only if $T=T^{[*]}\supset T_1$ and $\nu_-[I\pm T]=\kappa_\mp$. By Theorem 2.1 this is equivalent to

$$J_2S_-^*J_-S_--I\leq_{J_2}T_{22}\leq_{J_2}I-J_2S_+^*J_+S_+. \tag{5.21}$$

The inequalities (5.21) are equivalent to (5.17).

(v) The relations (5.18) follow from (5.19) and (5.16). □

Remark 5.1. In case of a contraction operator T_1 this result coincides with the main result of [16] and in case of a "quasi-contraction" operator T_1 with finite negative index it coincides with the result of [7, Theorem 5].

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