Український математичний вісник
Том 13 **(2016)**, № 3, 350 – 360

Removability of isolated singularity for solutions of anisotropic porous medium equation with absorption term

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Abstract. In this article we obtained the removability result for quasilinear equations model of which is

$$u_t - \sum_{i=1}^{n} \left(u^{m_i-1} u_{x_i}\right)_{x_i} + f(u) = 0, \ u \geq 0.$$

2010 MSC. 35B40.

Key words and phrases. Quasilinear parabolic equations, removable isolated singularity.

1. Introduction and main result

In this paper we study solutions to quasilinear parabolic equation in the divergent form

$$u_t - divA(x,t,u,\nabla u) + a_0(u) = 0, \ (x,t) \in \Omega_T, \tag{1.1}$$

satisfying a initial condition

$$u(x,0) = 0, \ \ x \in \Omega \setminus \{(0,0)\} \tag{1.2}$$

in $\Omega_T = \Omega \times (0,T)$, $0 < T < \infty$, where Ω is a bounded domain in $R^n, n > 2$.

The qualitative behaviour of solution to elliptic equations was investigated by many authors starting from the seminal papers of Serrin (see [4–8]). In [1] Brezis and Veron proved that for $q \geq \frac{n}{n-2}$ the isolated singularities of solutions to the elliptic equation

Received 05.09.2016

ISSN 1810 – 3200.

$$-\triangle u + u^q = 0,$$

are removable. The result on the removability of an isolated singularity for the following parabolic equation

$$\frac{\partial u}{\partial t} - \triangle u + |u|^{q-1}u = 0, \quad (x,t) \in \Omega_T \setminus \{(0,0)\}$$

was obtained by Brézis and Friedman [2] in the case $q \geq \frac{n+2}{n}$. The anisotropic elliptic equation with absorption

$$-\sum_{i=1}^{n} \left(|u_{x_i}|^{p_i-2}u_{x_i}\right)_{x_i} + |u|^{q-1}u = 0$$

was studied in [12]. It was proved that the isolated singularity for solution of the this equation is removable if

$$q \geq \frac{n(p-1)}{n-p}, \quad 1 \leq p_1 \leq \ldots \leq p_n \leq \frac{n-1}{n-p}p.$$

For quasilinear elliptic and parabolic equations of special form with absorption similar questions were treated by many authors. A survey of their results and references can be found in Veron's monograph [14]. The removability of isolated singularities for more general elliptic and parabolic equations with absorption were established in [10] and [11].

We suppose that the functions $A = (a_1, ..., a_n)$ and a_0 satisfy the Caratheodory conditions and the following structure conditions hold

$$A(x,t,u,\xi)\xi \geq \nu_1 \sum_{i=1}^{n} |u|^{m_i-1}|\xi_i|^2,$$

$$|a_i(x,t,u,\xi)| \leq \nu_2 u^{\frac{m_i-1}{2}} \left(\sum_{j=1}^{n} |u|^{m_j-1}|\xi_j|^2\right)^{\frac{1}{2}}, \; i = \overline{1,n}, \tag{1.3}$$

$$a_0(u) \geq \nu_1 f(u),$$

with positive constants ν_1, ν_2 and continuous, positive function $f(u)$ and

$$\min_{1\leq i\leq n} m_i > 1, \; \max_{1\leq i\leq n} m_i \leq 1 + \frac{\kappa}{n}, \; p < n, \tag{1.4}$$

where $\kappa = n(m-1)+2$, $d = \frac{1}{n}\sum\limits_{i=1}^{n} \frac{m_i}{2}$, and assume without loss, that $m_n = \max\limits_{1\leq i\leq n} m_i$.

We will write $V_{2,m}(\Omega_T)$ for the class of functions $\varphi \in C(0,T,L^2(\Omega))$ with $\sum\limits_{i=1}^{n} \iint\limits_{\Omega_T} |\varphi|^{m_i-1} |\varphi_{x_i}|^2 \, dxdt < \infty$.

We say that u is a weak solution to the problem (1.1), (1.2) if for an arbitrary $\psi \in C^1(\Omega_T)$, vanishing in a neighborhood of $\{(0,0)\}$, we have an inclusion $u\psi \in V_{2,m}(\Omega_T)$ and for any interval $(t_1,t_2) \subset [0,T)$ the integral identity

$$\int\limits_{\Omega} u\varphi dx \Big|_{t_1}^{t_2} + \int\limits_{t_1}^{t_2} \int\limits_{\Omega} \{-u\varphi_t + A(x,t,u,\nabla u)\nabla\varphi + a_0(u)\varphi\} \, dx \, dt = 0 \quad (1.5)$$

holds for $\varphi = \zeta\psi$ with an arbitrary $\zeta \in \overset{o}{V}_{2,m}(\Omega_T)$.

We say that solution u to the problem (1.1), (1.2) has a removable singularity at $\{(0,0)\}$ if u can be extended to $\{(0,0)\}$ so that the extension $\tilde{u}$ of u satisfies (1.5) with $\psi \equiv 1$ and $\tilde{u} \in V_{2,m}(\Omega_T)$.

Remark 1.1. Condition (1.4) implies the local boundedness of weak solutions to the equation (1.1) ([3]).

The main result of this paper is the following theorem.

Theorem 1.1. *Let the conditions* (1.3), (1.4) *be fulfilled and u be a nonnegative weak solution to the problem* (1.1), (1.2). *Assume also that $f(u) = u^q$ and*

$$q \geq m + \frac{2}{n}, \quad (1.6)$$

then the singularity at the point $\{(0,0)\}$ is removable.

The rest of the paper contains the proof of Theorem 1.1.

2. Integral estimates of solutions

For $0 \leq \lambda < n$ we define the following numbers

$$\kappa(\lambda) = \frac{1}{2 + (n-\lambda)(m-1)}, \; \kappa_i(\lambda) = \frac{2}{2 + (n-\lambda)(m-m_i)}, \; i = \overline{1,n}.$$

Let

$$\rho_\lambda(x,t) = \left(t^{\frac{\kappa(\lambda)}{\kappa_1(\lambda)}} + \sum_{i=1}^{n} |x_i|^{\frac{\kappa_i(\lambda)}{\kappa_1(\lambda)}} \right)^{\kappa_1(\lambda)},$$

assume that $D_\lambda(r) = \{(x,t) : \rho_\lambda(x,t) < r\}$, $D_\lambda(R_0) \subset \Omega_T$ and for $0 < r < R_0$ we set $M(r,\lambda) = \sup\limits_{D_\lambda(R_0)\setminus D_\lambda(r)} u(x,t)$, $E(r,\lambda) = \{(x,t) \in$

$\Omega_T : u(x,t) > M(r,\lambda)\}$, $u_r(r,t,\lambda) = (u(x,t) - M(r,\lambda))_+$ and consider the function $\psi_r(x,t) = \eta_r(\rho_\lambda(x,t))$, where $\eta_r : \mathbb{R}^1 \to \mathbb{R}^1$ is a function taking the following values: $\eta_r(z) = 0$ if $z \le r$, $\eta_r(z) = 1$ if $z \ge R(r)$, $\eta_r(z) = \left[(1-\varepsilon)\ln\ln\frac{1}{r}\right]^{-1}\left(\ln\ln\frac{1}{r} - \ln\ln\frac{1}{z}\right)$, if $r \le z \le R(r)$, here ε is a number from the interval $(0,1)$ specified in what follows and $R(r)$ defined by the equality

$$\ln\frac{1}{R(r)} = \ln^{\varepsilon}\frac{1}{r}. \tag{2.1}$$

Note that by the evident equalities $\frac{1}{q-1} = (n-\lambda)\kappa(\lambda)$, $\frac{2}{q-m_i} = (n-\lambda)\kappa_i(\lambda)$, $i = \overline{1,n}$, with $\lambda \ge 0$ defined by

$$\lambda = n - \frac{2}{q-m}, \tag{2.2}$$

the Keller–Osserman estimate yields

$$M(r,\lambda) \le \gamma r^{\lambda-n}, \ r > 0. \tag{2.3}$$

This estimate is received from Theorems 4.1, 4.2 (Appendix) in the case $p_1 = p_2 = ... = p_n = 2$.

Consider the functions $F_1(r,\lambda), F_2(r,\lambda)$ defined by the following equalities

$$F_1(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{\frac{q-2}{q-1}}\frac{1}{r}, \ \lambda = 0, \ q > 2, \\ \ln\ln\frac{1}{r}, \ \lambda = 0, \ q = 2, \\ \ln^{-\frac{2-q}{q-1}}, \ \lambda = 0, \ q < 2 \end{cases}$$

$$F_2(r,\lambda) = \begin{cases} R^{\lambda}(r), \ \lambda > 0, \\ \ln^{\frac{q-2m_1}{q-m_1}}\frac{1}{r}, \ \lambda = 0, \ q > 2m_1, \\ \ln\ln\frac{1}{r}, \ \lambda = 0, \ q = 2m_1, \\ \ln^{-\frac{2m_1-q}{1-m_1}}, \ \lambda = 0, \ q < 2m_1. \end{cases}$$

To simplify the following calculations we will write $M(r)$, $E(r)$, $u_r(x,t)$ instead of $M(r,\lambda), E(r,\lambda), u_r(x,t,\lambda)$.

Lemma 2.1. *Let the assumptions of Theorem 1.1 be fulfilled, then for every $l \ge \frac{2q}{q-m_n}$ and for every $2r < \rho \le \frac{R_0}{2}$ the following estimate holds*

$$\sup_{0<t<T} \int\limits_{E\left(\frac{\rho}{2}\right)\times\{t\}} \int\limits_{M\left(\frac{\rho}{2}\right)}^{u} \ln_+ \frac{s}{M\left(\frac{\rho}{2}\right)} ds\, \psi_r^l\, dx + \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i-2}|u_{x_i}|^2 \psi_r^l dxdt$$
$$+ \iint\limits_{E\left(\frac{\rho}{2}\right)} u^q \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \psi_r^l dxdt \le \gamma \left(F_1(r,\lambda) + F_2(r,\lambda)\right). \quad (2.4)$$

Proof. Testing (1.5) by $\varphi = \ln_+ \frac{u}{M\left(\frac{\rho}{2}\right)} \psi_r^l$, using (1.3) and the Young inequality we get

$$\sup_{0<t<T} \int\limits_{E\left(\frac{\rho}{2}\right)\times\{t\}} \int\limits_{M\left(\frac{\rho}{2}\right)}^{u} \ln_+ \frac{s}{M\left(\frac{\rho}{2}\right)} ds\, \psi_r^l\, dx + \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i-2}|u_{x_i}|^2 \psi_r^l dxdt$$
$$+ \iint\limits_{E\left(\frac{\rho}{2}\right)} u^q \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \psi_r^l dxdt \le \gamma \iint\limits_{E\left(\frac{\rho}{2}\right)} u \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial t}\right| \psi_r^{l-1} dxdt$$
$$+ \gamma \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i} \ln^2 \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial x_i}\right|^2 \psi_r^{l-2} dxdt.$$

From this, by the Young inequality we obtain

$$\sup_{0<t<T} \int\limits_{E\left(\frac{\rho}{2}\right)\times\{t\}} \int\limits_{M\left(\frac{\rho}{2}\right)}^{u} \ln_+ \frac{s}{M\left(\frac{\rho}{2}\right)} ds\, \psi_r^l\, dx + \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i-2}|u_{x_i}|^2 \psi_r^l dxdt$$
$$+ \iint\limits_{E\left(\frac{\rho}{2}\right)} u^q \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \psi_r^l dxdt \le \gamma \iint\limits_{E\left(\frac{\rho}{2}\right)} \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial t}\right|^{\frac{q}{q-1}} dxdt$$
$$+ \gamma \iint\limits_{E\left(\frac{\rho}{2}\right)} \ln^{\frac{2q-m_i}{q-m_i}} \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial x_i}\right|^{\frac{2q}{q-m_i}} dxdt = \gamma \left(J_1 + J_2\right). \quad (2.5)$$

By (2.3) we have

$$J_1 + J_2 \le \gamma \iint\limits_{D_\lambda(R(r))\setminus D_\lambda(r)} \ln^{-\frac{1}{q-1}} \frac{1}{\rho_\lambda}\, \rho_\lambda^{-\frac{1}{\kappa(\lambda)}\frac{q}{q-1}} dxdt$$
$$+ \gamma \sum_{i=1}^{n} \iint\limits_{D_\lambda(R(r))\setminus D_\lambda(r)} \ln^{-\frac{m_i}{q-m_i}} \frac{1}{\rho_\lambda}\, \rho_\lambda^{-\frac{2q}{\kappa_i(\lambda)(q-m_i)}} dxdt$$

$$\leq \gamma \int\limits_r^{R(r)} \ln^{-\frac{1}{q-1}} \frac{1}{z}\, z^{\lambda-1} dz + \gamma \int\limits_r^{R(r)} \ln^{-\frac{m_1}{q-m_1}} \frac{1}{z}\, z^{\lambda-1} dz \leq \gamma \left(F_1(r,\lambda) + F_2(r,\lambda)\right). \tag{2.6}$$

Combining (2.5), (2.6) we obtain (2.4), which completes the proof of the lemma. □

Define a function $u^{(\rho)}(x,t)$ and a set $E\left(\frac{\rho}{2}, 2\rho\right)$ as follows

$$u^{(\rho)}(x,t) = \min\left(M\left(\frac{\rho}{2}\right) - M(2\rho),\, u_{2\rho}(x,t)\right),$$

$$E\left(\frac{\rho}{2}, 2\rho\right) = \{x \in E(2\rho) : u < M\left(\frac{\rho}{2}\right)\}.$$

Lemma 2.2. *Under the assumptions of Lemma 2.1 next inequality holds*

$$\iint\limits_{E(2\rho)} u^{(\rho)} u^q \psi_r^l dxdt \leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right)$$

$$\times \left\{F_3(r,\lambda) + \left(F_1(r,\lambda) + F_2(r,\lambda)\right)^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda)\right\}, \tag{2.7}$$

where

$$F_3(r,\lambda) = \begin{cases} R^\lambda(r), \ \lambda > 0, \\ \ln^{-\frac{1}{q-1}} \frac{1}{r}, \ \lambda = 0, \end{cases} \quad F_4(r,\lambda) = \begin{cases} R^\lambda(r), \ \lambda > 0, \\ \ln^{-1} \frac{1}{r}, \ \lambda = 0. \end{cases}$$

Proof. Testing (1.5) by $\varphi = u^{(\rho)} \psi_r^l$, using (1.3) and the Young inequality we get

$$\iint\limits_{E(2\rho)} u^{(\rho)} u^q \psi_r^l dxdt \leq \gamma \iint\limits_{E(2\rho)} u^{(\rho)} \left|\frac{\partial \psi_r}{\partial t}\right|^{\frac{q}{q-1}} dxdt$$

$$+\gamma \sum_{i=1}^n \iint\limits_{E(2\rho)} \left(\sum_{j=1}^n u^{m_j-1} |u_{x_j}|^2\right)^{\frac{1}{2}} u^{\frac{m_i-1}{2}} u^{(\rho)} \left|\frac{\partial \psi_r}{\partial x_i}\right| \psi_r^{l-1} dxdt$$

$$= \gamma \left(J_3 + J_4\right). \tag{2.8}$$

By the Hölder inequality, (2.3) and Lemma 2.1 the integrals in the right-hand side of (2.8) are estimated as follows

$$J_3 \leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) \iint\limits_{E(2\rho)} \left|\frac{\partial \psi_r}{\partial t}\right|^{\frac{q}{q-1}} dxdt$$

$$\leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho) \right) \int\limits_{D_\lambda(R(\lambda))\setminus D_\lambda(r)} \ln^{-\frac{q}{q-1}} \frac{1}{\rho_\lambda} \; \rho_\lambda^{-\frac{q}{(q-1)\kappa(\lambda)}} dxdt$$

$$\leq \gamma\left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right)\int\limits_{r}^{R(\lambda)} \ln^{-\frac{q}{q-1}} \frac{1}{z} \, z^{\lambda-1} dz \leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) F_3(r,\lambda). \quad (2.9)$$

Similarly

$$J_4 \leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \iint\limits_{E(2\rho)} u^{m_j-2} |u_{x_j}|^2 \psi_r^l dxdt \right)^{\frac{1}{2}}$$

$$\times \left(\iint\limits_{E(2\rho)} u^{m_i} \left| \frac{\partial \psi_r}{\partial x_i} \right|^2 \psi_r^l dxdt \right)^{\frac{1}{2}} \leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) \times$$

$$\times \left(F_1(r,\lambda) + F_2(r,\lambda)\right)^{\frac{1}{2}} \sum_{i=1}^{n} \left(\iint\limits_{D_\lambda(R(\lambda))\setminus D_\lambda(r)} \ln^{-2} \frac{1}{\rho_\lambda} \rho_\lambda^{-m_i(n-\lambda)-\frac{2}{\kappa_i(\lambda)}} dxdt \right)^{\frac{1}{2}}$$

$$\leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) \left(F_1(r,\lambda) + F_2(r,\lambda)\right)^{\frac{1}{2}} \left(\int\limits_{r}^{R(r)} \ln^{-2} \frac{1}{z} \; z^{\lambda-1} dz \right)^{\frac{1}{2}}$$

$$\leq \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) \left(F_1(r,\lambda) + F_2(r,\lambda)\right)^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda). \qquad (2.10)$$

Combining (2.8)–(2.10) we arrive at the required (2.7), this proves the lemma. □

2.1. Pointwise estimates of solutions

Similarly to [13], using the De Giorgi type iteration, we prove the following estimate

$$(M(\rho) - M(2\rho))^{1+m+m\frac{n+2}{2}}$$

$$\leq \gamma \left(M\left(\frac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i} \left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right)^{\frac{n+2}{2}} \iint\limits_{D_\lambda(R_0)\setminus D_\lambda\left(\frac{\rho}{2}\right)} u_{2\rho}^{1+m} dxdt.$$

Since $u_{2\rho} \le M\left(\frac{\rho}{2}\right) - M(2\rho)$ for $(x,t) \in D_\lambda(R_0) \setminus D_\lambda\left(\frac{\rho}{2}\right)$ by the Hölder inequality and Lemma 2.2 we get

$$\begin{aligned}
&(M(\rho) - M(2\rho)^{1+m+m\frac{n+2}{2}} \\
&\le \gamma M^{\frac{m+1}{q+1}}\left(\frac{\rho}{2}\right)\left(M\left(\frac{\rho}{2}\right)\rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i}\left(\frac{\rho}{2}\right)\rho^{-\frac{2}{\kappa_i(\lambda)}}\right)^{\frac{n+2}{2}} \\
&\times\left\{F_3(r,\lambda) + (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda)\right\} |D_\lambda(R_0)|^{\frac{q-m}{q+1}}. \qquad (2.11)
\end{aligned}$$

In the inequality (2.11) we will pass to the limit as $r \to 0$. By (2.1) the following relations are valid for $\lambda = 0$

$$F_1(r,0)F_4(r,0) = \ln^{\frac{q-2}{q-1}}\frac{1}{r}\ \ln^{-1}\frac{1}{R(r)} = \ln^{\frac{q-2}{q-1}-\varepsilon}\frac{1}{r},\ \text{if } q > 2,$$
$$F_2(r,0)F_4(r,0) = \ln^{\frac{q-2m_1}{q-m_1}}\frac{1}{r}\ \ln^{-1}\frac{1}{R(r)} = \ln^{\frac{q-2m_1}{q-m_1}-\varepsilon}\frac{1}{r},\ \text{if } q > 2m_1,$$

choose ε from the condition $\max\left(\frac{1}{2}, \frac{q-2}{q-1}, \frac{q-2m_1}{q-m_1}\right) < \varepsilon < 1$, now passing to the limit as $r \to 0$ in (2.11) we obtain for any $\rho \le \frac{R_0}{2}$

$$M(\rho) - M(2\rho) \le 0,$$

iterating last inequality we get for any $\rho \le \frac{R_0}{2}$

$$M(\rho) \le M(R_0),$$

this proves the boundedness of solutions.

3. End of the proof of Theorem 1.1

Let K be a compact subset in Ω, and $\xi = 0$ in $\partial\Omega \times (0,T)$, such that $\xi = 1$ for $(x,t) \in K \times (0,T)$. Testing (1.5) by $\varphi = u\xi^2\psi_r$, $\psi = \psi_r$, using conditions (1.3), the Young inequality, the boundedness of u and passing to the limit $r \to 0$ we get

$$\sup_{0<t<T}\int_K u^2 dx + \sum_{i=1}^{n}\int_0^T\int_K u^{m_i-1}|u_{x_i}|^2 dxdt + \int_0^T\int_K u^{q+1} dxdt \le \gamma. \quad (3.1)$$

Testing (1.5) by $\varphi\psi_r$, where φ is an arbitrary function which belongs to $\overset{o}{V}_{2,m}(\Omega_T)$, using (3.1), the boundedness of solution, and passing to the limit $r \to 0$, we obtain the integral identity (1.5) with an arbitrary $\varphi \in \overset{o}{V}_{2,m}(\Omega_t)$ and $\psi \equiv 1$. Thus Theorem 1.1 is proved.

4. Appendix

Let $(x^{(0)}, t^{(0)}) \in \Omega_T$, for any $\tau, \theta_1, \theta_2, \ldots, \theta_n > 0$, $\theta = (\theta_1, \ldots, \theta_n)$ we define $Q_{\theta,\tau}(x^{(0)}, t^{(0)}) := \{(x,t) : |t - t^{(0)}| < \tau,\ |x_i - x_i^{(0)}| < \theta_i,\ i = \overline{1,n}\}$ and set

$$M(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} u, \delta(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} \delta(u),$$

$$\Phi(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} \Phi(u), \Phi(u) = \int_0^u \varphi(s)ds, \varphi(s) = s^{m_n-1} f(s).$$

We say that nondecreasing continuous function ψ satisfies the condition (A) if for any $\varepsilon \in (0,1)$ there exists $u_0(\varepsilon) \geq 1$ such that

$$\psi(\varepsilon u) \leq \varepsilon^{\mu} \psi(u), \tag{A}$$

with some $\mu > 0$ and for all $u \geq u_0(\varepsilon)$.

Theorem 4.1 ([9]). *Let the conditions* (1.3), (1.4) *be fulfilled and u be a nonnegative weak solution to equation* (1.1), *assume also that $f \in C^1(R^1_+)$ and $f'(u) \geq 0$. Let $(x^{(0)}, t^{(0)}) \in \Omega_T$, fix $\sigma \in (0,1)$, $\tau \in (0, \min(\theta_n^{p_n}, t^{(0)}, T - t^{(0)}))$, $\theta_i \in (0, \theta_n)$ for $i \in I' = \{i = \overline{1,n} : m_i(p_i - 1) < m_n(p_n - 1)\}$ and $\theta_i = \theta_n$ for $i \in I'' = \{i = \overline{1,n} : m_i(p_i - 1) = m_n(p_n - 1)\}$, then there exist positive numbers c_8, c_9 depending only on $n, \nu_1, \nu_2, m_1, \ldots, m_n, p_1, \ldots, p_n$ such that either*

$$u(x^{(0)}, t^{(0)}) \leq (\tau^{-1}\rho^{p_n})^{\frac{1}{m_n(p_n-1)-1}} + \sum_{i \in I'} (\theta_i^{-1}\theta_n^{\frac{p_n}{p_i}})^{\frac{p_i}{m_n(p_n-1)-m_i(p_i-1)}}, \tag{4.1}$$

or

$$\Phi(\sigma\theta, \sigma\tau) \leq c_8(1-\sigma)^{-c_9}\theta_n^{-p_n}\delta(\theta,\tau)M^{m_n p_n - 1}(\theta,\tau). \tag{4.2}$$

On the other hand, if I' is empty, i.e. $m_1(p_1 - 1) = m_2(p_2 - 1) = \cdots = m_n(p_n - 1)$, then either

$$u(x^{(0)}, t^{(0)}) \leq (\tau^{-1}\theta_n^{p_n})^{\frac{1}{m_n(p_n-1)-1}}, \tag{4.3}$$

or (4.2) *holds true.*

Theorem 4.2 ([9]). *Let the conditions* (1.3), (1.4) *be fulfilled, u be a nonnegative weak solution to* (1.1), *$f \in C^1(R^1_+)$ and $f'(u) \geq 0$. Let $\partial\Omega_T$ be the parabolic boundary of Ω_T, assume also that $\lim\limits_{(x,t)\to\partial\Omega_T} u(x,t) = +\infty$ and with some $0 \leq a \leq 1$ and $c > 0$ there holds*

$$\delta(u) \leq c u^a.$$

Let $\psi(u) = u^{-1}\Phi^{\frac{1}{m_n p_n + a - 1}}(u)$ *satisfies condition* (A)*. Let* $(x^{(0)}, t^{(0)}) \in \Omega_T$ *and* $8\rho = dist(x^{(0)}, \partial\Omega)$*. Fix* $\tau \in (0, \min(\rho^{p_n}, t^{(0)}, T - t^{(0)}))$ *and* $\theta_i \in (0, \rho)$ *for* $i \in I'$*, then there exists a positive number* c_{10} *depending only on* $n, \nu_1, \nu_2, m_1, ..., m_n, p_1, ..., p_n$ *and c, such that either (4.1) holds, or*

$$\Phi(u(x^{(0)}, t^{(0)})) \leq c_{10}\theta_n^{-p_n} u^{m_n p_n + a - 1}(x^{(0)}, t^{(0)}). \tag{4.4}$$

On the other hand if I' *is empty, i.e.* $m_1(p_1 - 1) = m_2(p_2 - 1) = ... = m_n(p_n - 1)$ *and* $\psi(u) = u^{-1}\Phi^{\frac{1}{m_n p_n + a - 1}}(u)$ *satisfies condition* (A)*, then either* (4.3) *holds, or* (4.4) *holds true.*

Acknowledgements

This work is supported by grant of Ministry of Education and Science of Ukraine (project number is 0115 U 000 136) and it is based on the research provided by the grant support of the State Fund For Fundamental Research (project number is 0116U007160).

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