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FREE VIBRATION OF THE SYSTEM OF TWO VISCOELASTIC BEAMS COUPLED BY VISCOELASTIC INTERLAYER

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An analytical method of solution of the free vibration problem for composite system of two viscoelastic beams coupled by a viscoelastic interlayer has been proposed. The phenomenon of free vibration has been described using a homogenous system of conjugate partial differential equations. After the separation of variables in the differential equations the boundary problem has been solved and two complex sequences have been obtained: the sequence of frequencies and the sequence of modes of free vibration. The property of orthogonality of complex modes of free vibration has been demonstrated. Polyharmonic free vibration has been expanded into the complex Fourier series with respect to complex eigenfunctions. The coefficients at the eigenfunctions are determined by the initial conditions.

Запропоновано аналiтичний метод розв'язку задачi про вiльнi коливання системи, складеної з двох в'язко-пружних балок, сполучених за допомогою в'язко-пружного промiжного шару. Вiльнi коливання описувались з використанням однорiдної системи спряжених диференцiйних рiвнянь у частинних похiдних. Пiсля роздiлення змiнних у диференцiйних рiвняннях було розв'язано граничну задачу i отримано двi комплекснi послiдовностi: послiдовнiсть частот та послiдовнiсть мод вiльних коливань. Продемонстровано властивiсть ортогональностi комплексних мод вiльних коливань. Полiгармонiчнi вiльнi коливання розкладались у комплекснi ряди Фур'є вiдносно комплексних власних функцiй, коефiцiєнти при яких визначаються початковими умовами.

Предложен аналитический метод решения задачи о свободных колебаниях системы, состоящей из двух вязко-упругих балок, связанных при помощи вязко-упругого промежуточного слоя. Свободные колебания были описаны с использованием однородной системы сопряженных дифференциальных уравнений в частных производных. После разделения переменных в дифференциальных уравнениях была решена граничная задача и получены две комплексные последовательности: последовательность частот и последовательность мод свободных колебаний. Продемонстрировано свойство ортогональности комплексных мод свободных колебаний. Полигармонические свободные колебания раскладывались в комплексные ряды Фурье относительно комплексных собственных функций, коэффициенты при которых определяются начальными условиями.

INTRODUCTION

Strings and systems of beams coupled together by viscoelastic constraints play an important role in various engineering and building structures. They are being used in railway and tram tractions with live load [1, 2]. Such kind of structures can also be found in some ski lifts and cable car systems. Beams can work together with strings, slabs and membranes in various structures. Light roof structure of the sport arena is an example of matching of strings and membranes.

Analysis of vibration of complex structural systems with damping posesses a difficult problem. In the above mentioned cases, especially where the viscosity and discrete elements occur, it is recommended to solve the dynamic problem by representing the amplitudes as the complex functions of real variable [3, 4]. For the first time the property of orthogonality of the complex modes of free vibration has been demonstrated in paper [3] for discrete systems with damping, and in paper [4] for discrete – continuous systems with damping. With the use of complex functions the description of free vibration of the beam supported on viscoelastic continuous Winkler's foundation [5, 6] has been developed in papers [7 – 10]. In the paper [11] the dynamic problem for complex continuous system has been solved by classical method [12] according to complete theory of non-damped vibration. In paper [10] the uniform method of solving the free vibration problems for complex continuous one- and two-dimensional structures with damping for various boundary conditions and different initial conditions has been presented. Verification of this method has been carried out for the system of two strings in the case where no damping occurs [11]. The results for natural frequencies and coefficients of amplitude derived by method presented in paper [10] agree with the results obtained with classical method [11].

The goal of this paper is to carry out the mathematical analysis of solution of the free vibration problem for composite structure consisting of two viscoelastic beams coupled by viscoelastic interlayer. According to proposed method the solution for eigenforms of the system is presented through the set of complex functions. Phase characteristics of the eigenmodes are studies as the fucntions of their orders.

1. FORMULATION OF THE PROBLEM

The physical model of the system under consideration consists of two parallel homogenous beams of equal length coupled together by viscoelastic interlayer (fig. 1). The Bernoulli – Euler's beams are simply supported at the ends. We assume that the beams are made of viscoelastic materials and their physical properties are described in scope of Voigt – Kelvin's model [13 – 15]. We assume that the viscoelastic properties of interlayer also can be described by the Voigt – Kelvin's model [13 – 15]. As to the elastic properties, the interlayer is considered as classical one-directional Winkler's foundation [6].

Stated assumptions allow to express the forces of interaction between the beams in terms of the deflections of beams and physical characteristics of Winkler's foundation. The small transverse vibrations of considered structure are described with the system of the following conjugate partial differential equations:

$$\begin{aligned} &E_1 I_1 \left(1 + c_1 \frac{\partial}{\partial t}\right) \frac{\partial^4 w_1}{\partial x^4} + \mu_1 \frac{\partial^2 w_1}{\partial t^2} + \\ &\quad + c\frac{\partial}{\partial t}(w_1 - w_2) + k(w_1 - w_2) = 0, \\ &E_2 I_2 \left(1 + c_2 \frac{\partial}{\partial t}\right) \frac{\partial^4 w_2}{\partial x^4} + \mu_2 \frac{\partial^2 w_2}{\partial t^2} + \\ &\quad + c\frac{\partial}{\partial t}(w_1 - w_2) + k(w_1 - w_2) = 0, \end{aligned} \tag{1}$$

where $w_1 = w_1(x,t)$ and $w_2 = w_2(x,t)$ are the deflections of the beams (hereafter the subscripts "1" and "2" mean the beam I and the beam II respectively); E_1 and E_2 are Young's modules of materials of the beams; I_1 and I_2 are the moments of inertia of cross-sections of the beams; μ_1 and μ_2 are the masses of the beams per unit of length; c_1 and c_2 are relative coefficients of viscosity in the beams; c is the coefficient of viscosity of the interlayer; k is the coefficient of elasticity of the interlayer; l is the length of the beams.

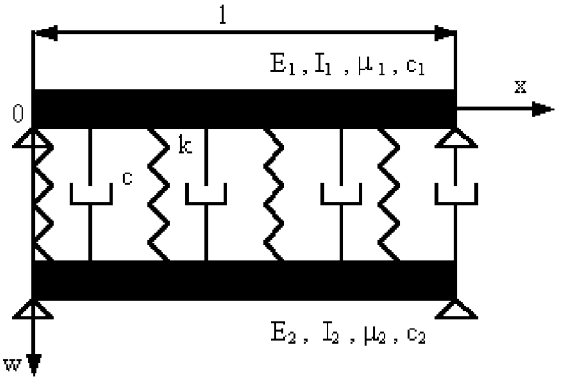


Fig. 1. Dynamic model of the system of two viscoelastic beams coupled by viscoelastic interlayer

2. SEPARATION OF VARIABLES

The goal of this paper is to describe the free vibration of complex visco-elastic system. The process of vibration is not harmonical one because of damping existing in the system. However, in scope of the considered model it is possible to express the time evolution of the system introducing the notion of complex frequency ν. This approach allows to assume the desired deflections to be the harmonical functions of time.

Substituting the expression

$$\begin{aligned} w_1 &= W_1(x)\exp(i\nu t), \\ w_2 &= W_2(x)\exp(i\nu t), \end{aligned} \tag{2}$$

for w_1 and w_2 in the system of differential equations (1) we obtain the homogenous system of conjugate ordinary differential equations describing the complex modes of vibration of the beams:

$$\begin{aligned} &\frac{d^4 W_1}{dx^4} - \left[E_1 I_1 (1 + c_1 \nu)\right]^{-1} \times \\ &\quad \times \left[(\mu_1 \nu^2 - k - ic\nu) W_1 + (k + ic\nu) W_2\right] = 0, \\ &\frac{d^4 W_2}{dx^4} - \left[E_2 I_2 (1 + c_2 \nu)\right]^{-1} \times \\ &\quad \times \left[(\mu_2 \nu^2 - k - ic\nu) W_2 + (k + ic\nu) W_1\right] = 0, \end{aligned} \tag{3}$$

where $W_1(x)$, $W_2(x)$ is the complex mode of vibration of the beams; ν is the complex frequency of vibration of the beams; t is the time.

3. SOLUTION OF THE BOUNDARY VALUE PROBLEM

Searching for particular solution of system of differential equations (3) in the following form [16]:

$$W_1 = Ae^{rx}, \quad W_2 = Be^{rx}, \tag{4}$$

we obtain the homogenous system of linear algebraic equations

$$\begin{aligned} &A\left[R_1(1 + ic_1\nu)r^4 - \mu_1\nu^2 + k + ic\nu\right] - \\ &\quad - B\left[k + ic\nu\right] = 0, \\ &A\left[k + ic\nu\right] - \\ &\quad - B\left[R_2(1 + ic_2\nu)r^4 - \mu_2\nu^2 + k + ic\nu\right] = 0, \end{aligned} \tag{5}$$

where $R_1 = E_1 I_1$, $R_2 = E_2 I_2$. This system has a nontrivial solution only if its determinant is equal to zero. This condition is in essence the dispertion equation linking the wavenumber r and the complex frequency ν. In doing so, we obtain the four-quadratic equation

$$r^8 - \left[\frac{\mu_1\nu^2 - k - ic\nu}{R_1(1+ic_1\nu)} + \frac{\mu_2\nu^2 - k - ic\nu}{R_2(1+ic_2\nu)}\right] r^4 + \\ +\nu^2 \frac{\mu_1\mu_2\nu^2 - (\mu_1+\mu_2)(k+ic\nu)}{R_1R_2(1+ic_1\nu)(1+ic_2\nu)} = 0, \qquad (6)$$

with the following roots:

$$r_j = \pm i\lambda_v, \quad r_j = \pm\lambda_v, \\ j = 1,2,3,4, \quad v = 1,2, \qquad (7)$$

where

$$\lambda_v = \sqrt[4]{\frac{1}{2}\left[\frac{\mu_1\nu^2 - k - ic\nu}{R_1(1+ic_1\nu)} + \frac{\mu_2\nu^2 - k - ic\nu}{R_2(1+ic_2\nu)}\right] \pm \sqrt{\Delta}}, \qquad (8)$$

and

$$\Delta = \left[\frac{\mu_1\nu^2 - k - ic\nu}{R_1(1+ic_1\nu)} - \frac{\mu_2\nu^2 - k - ic\nu}{R_2(1+ic_2\nu)}\right]^2 + \\ + \frac{4(k+ic\nu)^2}{R_1R_2(1+ic_1\nu)(1+ic_2\nu)} > 0, \qquad (9)$$

is the discriminate of the four-quadratic equation (6).

After applying the Euler's formulas the general solution of the system of differential equations (3) can be represented through the fundamental system of solutions:

$$W_1(x) = \sum_{v=1}^{2} A_v^* \,\mathrm{Sh}\,\lambda_v x + A_v^{**} \mathrm{Ch}\,\lambda_v x + \\ + A_v^{***} \sin\lambda_v x + A_v^{****} \cos\lambda_v x, \\ W_2(x) = \sum_{v=1}^{2} B_v^* \,\mathrm{Sh}\,\lambda_v x + B_v^{**} \mathrm{Ch}\,\lambda_v x + \\ + B_v^{***} \sin\lambda_v x + B_v^{****} \cos\lambda_v x, \qquad (10)$$

where A_v^*, A_v^{**}, A_v^{***}, A_v^{****}, B_v^*, B_v^{**}, B_v^{***}, B_v^{****} are constants.

In agreement with (5) there exist the following relations between the constants from (10):

$$a_v = \frac{B_v^*}{A_v^*} = \frac{B_v^{**}}{A_v^{**}} = \frac{B_v^{***}}{A_v^{***}} = \frac{B_v^{****}}{A_v^{****}}, \qquad (11)$$

where

$$a_v = \frac{R_1(1+ic_1\nu_v)\lambda_v^4 - \mu_1\nu_v^2 + k + ic\nu_v}{k + ic\nu_n} = \\ = \frac{k + ic\nu_n}{R_2(1+ic_2\nu_v)\lambda_v^4 - \mu_2\nu_v^2 + k + ic\nu_v}. \qquad (12)$$

After incorporating the representation (11) in (10) the general solution of the system of differential equations (3) takes the form

$$W_1(x) = \sum_{v=1}^{2} A_v^* \,\mathrm{Sh}\,\lambda_v x + A_v^{**} \mathrm{Ch}\,\lambda_v x + \\ + A_v^{***} \sin\lambda_v x + A_v^{****} \cos\lambda_v x, \\ W_2(x) = \sum_{v=1}^{2} a_v \left(A_v^* \,\mathrm{Sh}\,\lambda_v x + A_v^{**} \mathrm{Ch}\,\lambda_v x + \right. \\ \left. + A_v^{***} \sin\lambda_v x + A_v^{****} \cos\lambda_v x\right). \qquad (13)$$

To determine the amplitudes of the eigenmodes of the system one should specify some kind of boundary conditions. In this scope we assume the ends of the beams to be simply supported:

$$W_1(0) = 0, \qquad W_1(l) = 0, \\ \frac{d^2W_1}{dx^2}(0) = 0, \quad \frac{d^2W_1}{dx^2}(l) = 0 \qquad (14)$$

for beam I and

$$W_2(0) = 0, \qquad W_2(l) = 0, \\ \frac{d^2W_2}{dx^2}(0) = 0, \quad \frac{d^2W_2}{dx^2}(l) = 0 \qquad (15)$$

for beam II. Substituting the general solution (13) into boundary conditions (14), (15) we obtain the homogenous system of linear algebraic equations, which in the matrix notation has the following form:

$$\mathbf{YX} = 0, \qquad (16)$$

where

$$\mathbf{X} = [A_1^*, A_1^{**}, A_1^{***}, A_1^{****}, A_2^*, A_2^{**}, A_2^{***}, A_2^{****}]^T$$

is the vector of the amplitude coefficients and

$$\mathbf{Y} = [Y_{ij}]_{8\times 8} \qquad (17)$$

is the characteristic matrix of the system of equations (16). The elements of this matrix are presented in table 1, where

$$SS_1 = \mathrm{Sh}\,\lambda_1 l, \quad SS_2 = \mathrm{Sh}\,\lambda_2 l, \\ CC_1 = \mathrm{Ch}\,\lambda_1 l, \quad CC_2 = \mathrm{Ch}\,\lambda_2 l, \\ ss_1 = \sin\lambda_1 l, \quad ss_2 = \sin\lambda_2 l, \\ cc_1 = \cos\lambda_1 l, \quad cc_2 = \cos\lambda_2 l, \\ ll_1 = \lambda_1^2, \quad ll_2 = \lambda_2^2.$$

Table 1. Matrix of coefficients Y_{ij}

i / j	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0	0	1	1	0	0	1	1
2	0	0	ll_1	ll_2	0	0	$-ll_1$	$-ll_2$
3	0	0	a_1	a_2	0	0	a_1	a_2
4	0	0	$a_1 ll_1$	$a_2 ll_2$	0	0	$-a_1 ll_1$	$-a_2 ll_2$
5	SS_1	SS_2	CC_1	CC_2	ss_1	ss_2	cc_1	cc_2
6	$ll_1 SS_1$	$ll_2 SS_2$	$ll_1 CC_1$	$ll_2 CC_2$	$-ll_1 ss_1$	$-ll_2 ss_2$	$-ll_1 cc_1$	$-ll_2 cc_2$
7	$a_1 SS_1$	$a_2 SS_2$	$a_1 CC_1$	$a_2 CC_2$	$a_1 ss_1$	$a_2 ss_2$	$a_1 cc_1$	$a_2 cc_2$
8	$a_1 ll_1 SS_1$	$a_2 ll_2 SS_2$	$a_1 ll_1 CC_1$	$a_2 ll_2 CC_2$	$-a_1 ll_1 ss_1$	$-a_2 ll_2 ss_2$	$-a_1 ll_1 cc_1$	$-a_2 ll_2 cc_2$

The condition of solvability for the system of equations (16) is the vanishing of the characteristic determinant, i. e.

$$\det \mathbf{Y} = 0. \qquad (18)$$

The identity $A_1^* = A_1^* = A_1^{**} = A_1^{**} = A_1^{****} = A_1^{****} = 0$, obtained from the system (16) leads to the reduction of the characteristic equations (18) to the following form:

$$\begin{vmatrix} \sin\lambda_1 l & \sin\lambda_2 l \\ a_1 \sin\lambda_1 l & a_2 \sin\lambda_2 l \end{vmatrix} = 0, \qquad (19)$$

where

$$\begin{aligned} \lambda_1 &= \alpha_1 + i\beta_1, \\ \lambda_2 &= \alpha_2 + i\beta_2 \end{aligned} \qquad (20)$$

are in the general case the complex numbers.

Vanishing of the determinant in (19) is equivalent to the following transcential equation:

$$\sin(\alpha_1 + i\beta_1)l \, \sin(\alpha_2 + i\beta_2)l = 0, \qquad (21)$$

having the roots

$$\begin{aligned} \alpha_{1n} &= \alpha_{2n} = \alpha_n = \frac{s\pi}{l}, \\ \beta_{1n} &= \beta_{2n} = \beta_n = 0, \\ s &= 1, 2, 3, \ldots, \end{aligned} \qquad (22)$$

where

$$n = 2s - \delta_{n,(2s-1)}, \qquad (23)$$

and $\delta_{n,(2s-1)}$ is the Kronecker's number.

Substitution of (22) in (20) leads to the equality

$$\lambda_{1n} = \lambda_{2n} = \lambda_n = \alpha_n = \frac{s\pi}{l}. \qquad (24)$$

Substituting $r^4 = \lambda^4$ in the equation (6) and carrying out all the transformations we readily obtain the following equation with respect to frequency:

$$\begin{aligned} &\nu^4 - (\mu_1\mu_2)^{-1} \Big\{ ic(\mu_1+\mu_2)\nu^3 + \\ &+ \Big[\big(R_1(1+ic_1\nu)\lambda_n^4 + k\big)\mu_2 + \\ &+ \big(R_2(1+ic_2\nu)\lambda_n^4 + k\big)\mu_1 \Big] \nu^2 - \\ &- ic\big(R_1(1+ic_1\nu) + R_2(1+ic_2\nu)\big)\lambda_n^4 \nu + \\ &- \Big[\big(R_1 R_2 (1+ic_1\nu)(1+ic_2\nu)\big)\lambda_n^4 + \\ &+ k\big(R_1(1+ic_1\nu) + R_2(1+ic_2\nu)\big) \Big] \lambda_n^4 \Big\} = 0, \end{aligned} \qquad (25)$$

from which the sequence of complex eigenfrequencies can be determined:

$$\nu_n = i\eta_n \pm \omega_n. \qquad (26)$$

In fact the equation (25) has four roots. In (26) we left only two of them that correspond to physically consistent vibration of the system decaying with time.

Using the representation (26) in the expression for a_v (12) we obtain the final formulas for coefficients of amplitudes that are in fact the relative ratios of amplitudes of the two beams vibrating on the certain mode:

$$\begin{aligned} a_n &= \frac{R_1(1+ic_1\nu_n)\lambda_n^4 - \mu_1\nu_n^2 + k + ic\nu_n}{k + ic\nu_n} = \\ &= \frac{k + ic\nu_n}{R_2(1+ic_2\nu_n)\lambda_n^4 - \mu_2\nu_n^2 + k + ic\nu_n}. \end{aligned} \qquad (27)$$

Incorporation of the sequences of λ_n and a_n into (13) gives two sequences of modes of free vibration for beams I and II:

$$\begin{aligned} W_{1n}(x) &= \sin\lambda_n x, \\ W_{2n}(x) &= a_n \sin\lambda_n x. \end{aligned} \qquad (28)$$

4. SOLUTION OF THE INITIAL VALUE PROBLEM

The complex equation of motion

$$T = \Phi \exp(i\nu t), \qquad (29)$$

in the case of $\nu = \nu_n$ can be written as

$$T_n = \Phi_n \exp(i\nu_n t), \qquad (30)$$

where Φ_n is the Fourier coefficient.

It is natural to present the free vibration of beams in form of the Fourier series based on complex eigenfunctions, i. e.

$$\begin{aligned} w_{1n}(x,t) &= \sum_{n=1}^{\infty} W_{1n}(x)\Phi_n \exp(i\nu_n t), \\ w_{2n}(x,t) &= \sum_{n=1}^{\infty} W_{2n}(x)\Phi_n \exp(i\nu_n t). \end{aligned} \qquad (31)$$

When omitting the damping in the beams, from the system (3) after performing the algebraic transformations of its equations, adding them together and integration of the final expression in limits from 0 to l we obtain the property of orthogonality of eigenfunctions [8 – 10].

$$\begin{aligned} &\int_0^l \Big\{ \xi_1 (W_{1m} V_{1n} + W_{1n} V_{1m}) + \\ &\xi_2 (W_{2m} V_{2n} + W_{2n} V_{2m}) + \\ &+2\eta \big[(W_{1n} - W_{2n})(W_{1n} - W_{2m}) \big] \Big\} dx = N_n \delta_{mn}, \end{aligned} \qquad (32)$$

where δ_{nm} is Kronecker's delta,

$$\begin{aligned} N_n = 2 \int_0^l \big[&\xi_1 W_{1n} V_{1n} + \xi_2 W_{2n} V_{2n} + \\ &+\eta (W_{1n} - W_{2n})^2 \big] dx, \end{aligned} \qquad (33)$$

$$\begin{aligned} &V_{1n}(x) = i\nu_n W_{1n}(x), \quad V_{2n}(x) = i\nu_n W_{2n}(x), \\ &V_{1m}(x) = i\nu_m W_{1m}(x), \; V_{2m}(x) = i\nu_m W_{2m}(x), \\ &\xi_1 = \mu_1/\mu, \quad \xi_2 = \mu_2/\mu, \quad \eta = \mu_1/(2\mu). \end{aligned} \qquad (34)$$

The problem on free vibration of beams is solved by applying the following conditions:

$$\begin{aligned} w_1(x,0) &= w_{01}, \quad \dot{w}_1(x,0) = \dot{w}_{01}, \\ w_2(x,0) &= w_{02}, \quad \dot{w}_2(x,0) = \dot{w}_{02}. \end{aligned} \qquad (35)$$

Application of conditions (35) in the series (31) and taking into consideration the property of orthogonality (32) leads to the formula for the Fourier coefficient [8 – 10]:

$$\begin{aligned} \Phi_n = \frac{1}{N_n} \int_0^l \Big\{ &\xi_1 (V_{1n} w_{01} + W_{1n} \dot{w}_{01}) + \\ &+\xi_2 (V_{2n} w_{02} + W_{2n} \dot{w}_{02}) + \\ &+2\eta \big[(W_{1n} - W_{2n})(w_{01} - w_{02}) \big] \Big\} dx. \end{aligned} \qquad (36)$$

After the substitution of (28), (30) and (36) into (31) and obvious transformations finally one can obtain the expressions for free vibration of beams:

$$\begin{aligned} w_{1n} = \sum_{n=1}^{\infty} & e^{-\eta t} |\Phi_n| \, W_{1n}(x) \big[\cos(\omega_n t + \varphi_n) + \\ &+ i \sin(\omega_n t + \varphi_n) \big], \\ w_{2n} = \sum_{n=1}^{\infty} & e^{-\eta t} |\Phi_n| \, W_{2n}(x) \big[\cos(\omega_n t + \varphi_n) + \\ &+ i \sin(\omega_n t + \varphi_n) \big], \end{aligned} \qquad (37)$$

where $\varphi_n = \arg \Phi_n$.

5. NUMERICAL RESULTS AND DISCUSSION

Two beams coupled by viscoelastic interlayer have been considered. Calculations have been carried out for the following data: $E_1 = 1010$ N m^{-2}, $E_2 = 1010$ N m^{-2}, $I_1 = 4.5 \cdot 10^{-4}$ m^4, $I_2 = 8.9 \cdot 10^{-4}$ m^4, $\mu_1 = 1.2 \cdot 10^2$ kg m^{-1}, $\mu_2 = 1.75 \cdot 10^2$ kg m^{-1}, $k = 2 \cdot 10^5$ N m^{-2}, $l = 10$ m, $c = 0.75$, $c_1 = 0$, $c_2 = 0$. From teh above values it is easy to see that in calculations both the beams were regarded as elastic with no internal damping.

The Fourier coefficients Φ_n (36) were derived for the following initial conditions:

$$w_{01} = A_s \sin(\pi x/l), \quad \dot{w}_{01} = A_s \omega_1 \sin(2\pi x/l),$$

$$w_{02} = 0, \quad \dot{w}_{02} = 0, \quad A_s = 0.01 l.$$

For the system of two beams coupled by viscoelastic interlayer the calculations were performed in the program “MATHEMATICA” [17, 18]. The viscoelastic Bernoulli – Euler's beams coupled together by viscoelastic material can be assumed to be the thin beams, so that there don't occur the angles of rotation of cross-sections of the beams. As to the viscoelastic one-directional Winkler's interlayer, it was

Table 2. Natural frequencies and ratios of amplitudes
for lower modes of vibrations of the system

Number of mode		ν_n	a_n
$s=1$	$n=1$	$21.0245+0.000002.84721i$	$0.953953+0.00000355862i$
	$n=2$	$56.8328+0.00526501i$	$-1.391128-0.0000140292i$
$s=2$	$n=3$	$81.93308+0.000635179i$	$0.479134+0.0000975824i$
	$n=4$	$99.4811+0.00463268i$	$-0.698737-0.000172791i$
$s=3$	$n=5$	$176.268+0.00242943i$	$0.110562+0.0000740459i$
	$n=6$	$203.96+0.00283843i$	$-0.161222-0.000123323i$

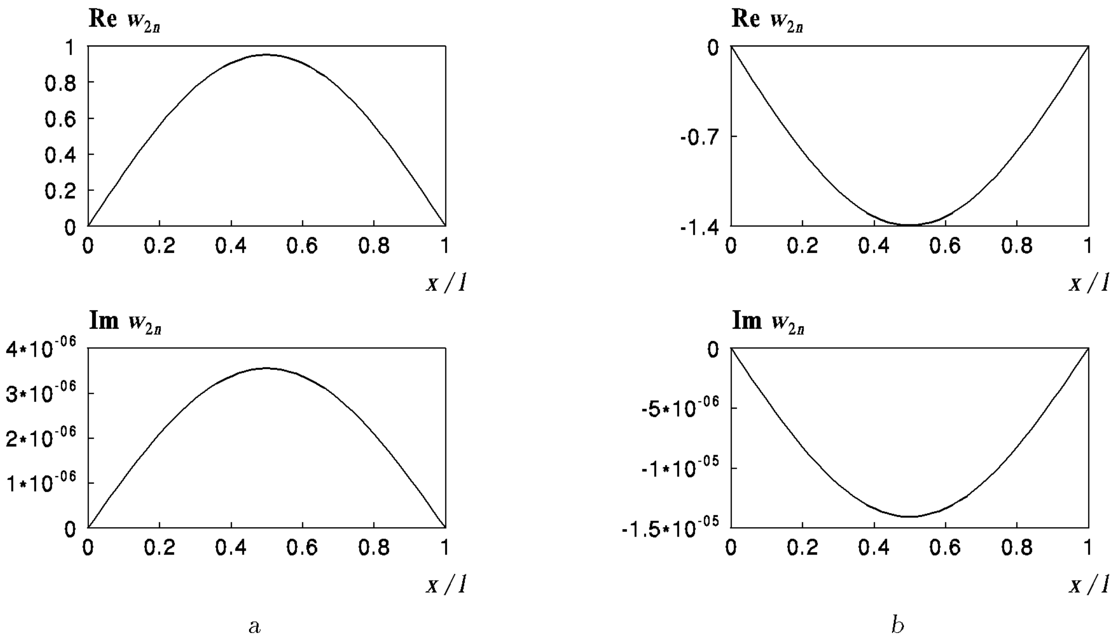


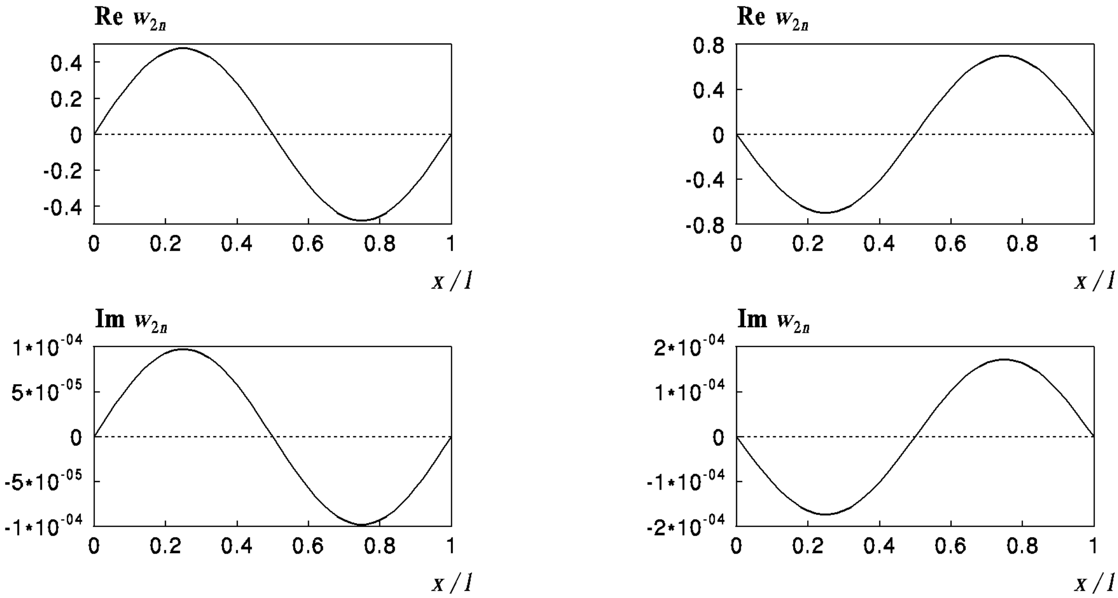
Fig. 2. Modes of vibration of beam II for $s=1$, $(n=1,2)$:
a – $n=1$; b – $n=2$

considered as one of small thickness, in which the longitudinal interaction between the interlayer and the Bernoulli – Euler's beam don't occur. Table 2 contains the complex natural frequencies ν_n and complex coefficients of modes of free vibration a_n for the system with damping. The results are given for $s=1$, $(n=1,2)$, $s=2$, $(n=3,4)$, $s=3$, $(n=5,6)$.

Investigation of complex modes for beams I and II has shown that relative amplitude of vibration of beam II (normalized to that of beam I) decreases with the increase of s and number of mode n, respecively. So, if at $s=1$ $(n=1,2)$ the amplitudes of vibrations of the both beams are of the same order, with the increase s to 3 $(n=5,6)$ their ratio decreases to 10 – 15 %.

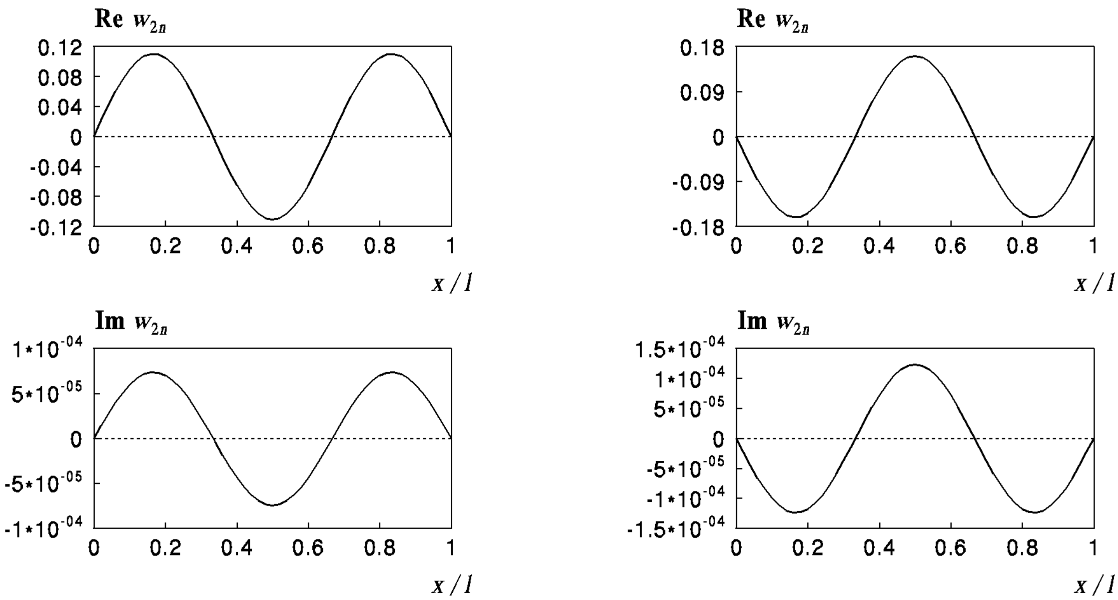
The another peculiarity is some phase shift between the modes with corresponding numbers for beams I and II. Obviously, this phenomenon is conditioned by presence of viscousity in the elements of considered structure. This relative phase shift can be expressed by the existence of nonzero imaginary parts of spatial modes of beam II while for all modes of beam I the imaginary part can be regarded as identical to zero. Eigenmodes for the both beams with $s=1,2,3$ are represented on fig. 2 – fig. 4.

Presence of the above-mentioned phase shift between the deflections of the beams is natural for visco-elastic systems, and its the value stronly depends on the material parameters of components of system. The analysis shows that for lower modes of vibration this shift of phases is extremely low, but demonstrating the trend to growth: for $n=1$ $\max|\mathbf{Im}\, w_{2n}/\mathbf{Re}\, w_{2n}|$ is less then $5\cdot10^{-4}$ % and for $n=6$ it grows to 0.077 %.



a b

Fig. 3. Modes of vibration of beam *II* for $s=2$, $(n=3,4)$:
a – $n=3$; b – $n=4$



a b

Fig. 4. Modes of vibration of beam *II* for $s=3$, $(n=5,6)$:
a – $n=5$; b – $n=6$

Also, it should be noted that amplitude ratios of modes for both beams significantly decrease with the increase of s. It is interesting that modes for beams *I* and *II* with odd numbers are excited almost in phase, but modes having even numbers are counterphase ones.

To substantiate the reliability and marketability of the offered method of the solution the verification of numerical results for the system of two Bernoulli – Euler's beams without damping has been carried out. The results for natural frequencies and ratios of amplitudes derived by means of the presented method, have been compared with the results obtained using the classical method [11]. The reasonable agreement betwen both mentioned results has been noted.

For the solution of the problem with damping the new analytical uniform method presented in this paper may be used. The basis of this method is the property of orthogonality of the complex modes of free vibration of two Bernoulli – Euler's beams coupled together by viscoelastic interlayer (32). This property is identical to that of system of two strings coupled by viscoelastic interlayer [10].

CONCLUSIONS

1. The analytical method of the solution of problems on free vibration of continuous system of two viscoelastic beams coupled by viscoelastic interlayer is introduced in this paper. According to this method the set of complex modes of free vibration forms has been found. Mentioned set can be regarded as functional basis for representation of arbitrary vibration of the system.

2. The calculations for presented system have been carried out for two complex sequences: the sequence of frequency, and the sequence of modes of free vibration.

3. Spatial modes of vibration of beams I and II are slightly shifted in phase. With the increase of number of mode n it is oserved the decrease of relative amplitude of vibration of beam II with respect of that of beam I. The absolute values of amplitude ratios for both beams are also decreasing with the increase of n.

4. The method presented in this paper can be applied to solutions of free vibration of different engineering structures consisting of two viscoelastic beams coupled by viscoelastic interlayer (girder, road or railway bridges).

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