*To Uzy Smilansky on his $75\frac{2}{3}$-th birthday*

# Fluctuations of Interlacing Sequences

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In a series of works published in the 1990s, Kerov put forth various applications of the circle of ideas centered at the Markov moment problem to the limiting shape of random continual diagrams arising in representation theory and spectral theory. We demonstrate on several examples that his approach is also adequate to study the fluctuations about the limiting shape.

In the random matrix setting, we compare two continual diagrams: one is constructed from the eigenvalues of the matrix and the critical points of its characteristic polynomial, whereas the second one is constructed from the eigenvalues of the matrix and those of its principal submatrix. The fluctuations of the latter diagram were recently studied by Erdős and Schröder; we discuss the fluctuations of the former, and compare the two limiting processes.

For Plancherel random partitions, the Markov correspondence establishes the equivalence between Kerov's central limit theorem for the Young diagram and the Ivanov–Olshanski central limit theorem for the transition measure. We outline a combinatorial proof of the latter, and compare the limiting process with the ones of random matrices.

*Key words:* interlacing sequences, Markov moment problem, continual diagrams, random matrices, central limit theorem.

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## 1. Overview

**1.1. Markov correspondence.** Two sequences of real numbers $A = (a_1 < \cdots < a_n)$ and $B = (b_1 < \cdots < b_{n-1})$ are called interlacing if

$$a_1 < b_1 < a_2 < \cdots < b_{n-1} < a_n.$$

To a pair of interlacing sequences $(A, B)$ one associates the probability measure

$$\mu = \sum_{j=1}^{n} p_j \delta_{a_j},$$

where $p_j$ are defined by the simple fraction decomposition

$$\frac{\prod_{j=1}^{n-1}(z - b_j)}{\prod_{j=1}^{n}(z - a_j)} = \sum_{j=1}^{n} \frac{p_j}{z - a_j}. \tag{1}$$

Vice versa, a probability measure supported on a finite set of atoms gives rise to a pair of interlacing sequences.

This construction admits numerous generalizations. The relation

$$\exp\left\{ -\int \log(z - x) d(\tau_+(x) - \tau_-(x)) \right\} = \int \frac{d\mu(x)}{z - x}, \tag{2}$$

obtained from (1) by replacing sums with integrals, forms the basis of the solution of the Markov moment problem (see [5, 37, 43]), and is one of the forms of the Markov correspondence (which we further discuss below and in 2.2.2). In the terminology of Kerov [33, 34], $\tau_+$ and $\tau_-$ corresponding to a probability measure $\mu$ form a pair of interlacing measures; such pairs are intrinsically characterized by the inequalities

$$\tau_+[x, \infty) \geq \tau_-[x, \infty) , \quad \tau_+(-\infty, x] \geq \tau_-(-\infty, x] \quad (x \in \mathbb{R}). \tag{3}$$

The equality (2) may be viewed as a connection between an additive and a multiplicative representations of a function from the Nevanlinna class. In this form it admits further generalizations, extensively used starting from the works of Akhiezer and Krein (see, e.g., the appendix to [37]); we also refer to the work of Yuditskii [63] for some recent developments.

In the 1990s, Kerov discovered a number of applications of the Markov correspondence to problems in representation theory and analysis. These applications form a central theme in the monograph [34], see further the survey [33].

Here we follow Kerov and switch to the language of continual diagrams, which are 1-Lipschitz functions coinciding with $|x - a|$ for large values of $|x|$ (such as in Figure 1c and Figure 2; see 2.2.1 for a formal definition). The mapping from pairs of interlacing sequences to continual diagrams is given by $(A, B) \mapsto \omega$,

$$\omega(x) = \sum_{j=1}^{n} |x - a_j| - \sum_{j=1}^{n-1} |x - b_j|; \tag{4}$$

whereas for interlacing measures $(\tau_+, \tau_-)$ as in (2) or (3), one sets

$$\omega(x) = \int |x - a| \, d\tau_+(a) - \int |x - b| \, d\tau_-(b).$$

The Markov correspondence (2) induces a bijection between continual diagrams $\omega$ and probability measures $\mu$, which is called the Markov transform and denoted $\mu = \mathfrak{M}\,\omega$. Some of its properties are listed in 2.2.2.

Continual diagrams appear naturally as scaling limits of Young diagrams. Indeed, a Young diagram rotated by $135°$ (with respect to the English convention) is a continual diagram, see Figure 1 for an illustration and 4.1.1, 4.1.2 for a formal construction.
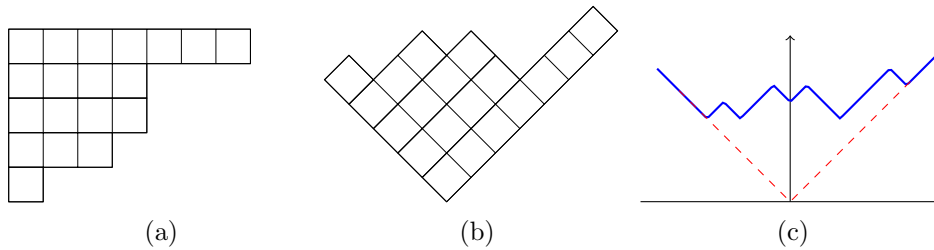


(a)         (b)         (c)

Fig. 1: A Young diagram in English convention (a) and in Russian convention (b), and the corresponding continual diagram (c).

Kerov showed that the Markov transform $\mu_n = \mathfrak{M}\,\omega_n$ of a continual diagram $\omega_n$ obtained in this way encodes the transition probabilities of the Young diagram in a stochastic process called the Plancherel growth (see 4.1.3), and called $\mu_n$ the transition measure of the Young diagram. If $\omega_n$ is a random continual diagram associated with a Young diagram sampled at random from the Plancherel measure, the Logan–Shepp–Vershik–Kerov limit law [40,60,61] asserts that (uniformly almost surely)

$$\frac{1}{\sqrt{n}}\omega_n(\sqrt{n}x) \longrightarrow \Omega_{\mathrm{LSVK}}(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin \frac{x}{2} + \sqrt{4 - x^2}\right), & |x| \le 2, \\ |x|, & |x| > 2. \end{cases} \tag{5}$$

Using the Markov correspondence, Kerov deduced that $\tilde{\mu}_n$ (here and further in the introduction, tildes indicate unspecified scaling) obey the semicircle law [30]

$$d\tilde{\mu}_n(x) \to d\rho_{\mathrm{s.c.}}(x) = \frac{1}{2\pi}\sqrt{(4 - x^2)_+}\,dx; \tag{6}$$

vice versa, (6) implies (5).

Kerov also discovered [31] a random matrix counterpart of these statements; it comes in two flavors. Let $\tilde{P}_n(x)$ be the characteristic polynomial of a Wigner

matrix $\tilde{H}_n$ of dimension $n \times n$ (see 2.3.4); let $\tilde{\lambda}_j$ be the zeros of $\tilde{P}_n$ (which are the eigenvalues of $\tilde{H}_n$). Also let $\tilde{\lambda}_j^*$ be the zeros of $\tilde{P}_n'$, and let $\tilde{\lambda}_j^{**}$ be the eigenvalues of the top-left $(n-1) \times (n-1)$ principal submatrix of $\tilde{H}_n$. Then (see 2.3.4)

$$\sum_{j=1}^{n} |x - \tilde{\lambda}_j| - \sum_{j=1}^{n-1} |x - \tilde{\lambda}_j^*| \to \Omega_{\text{LSVK}}(x) \tag{7}$$

and also (see 2.3.5)

$$\sum_{j=1}^{n} |x - \tilde{\lambda}_j| - \sum_{j=1}^{n-1} |x - \tilde{\lambda}_j^{**}| \to \Omega_{\text{LSVK}}(x). \tag{8}$$

The former was proved by Kerov, the latter was proved by Alexey Bufetov [14] who strengthened the result in [31].

As put forth by Kerov, (7) is equivalent (by the Markov correspondence) to Wigner's semicircle law

$$\widetilde{\rho}_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\tilde{\lambda}_j} \to \rho_{\text{s.c.}}$$

for the normalized eigenvalue counting measure, see further 2.3.5. On the other hand, (8) is equivalent to Wigner's law for the spectral measure associated with a fixed vector in $\mathbb{C}^n$, see 2.3.4.

Another similar looking result was found by Kerov [31] in the setting of Jacobi matrices with regularly varying coefficients. There (see 2.1.1), the counterparts of $\tilde{\lambda}_j$ and $\tilde{\lambda}_j^{**}$ are the (properly rescaled) eigenvalues of an $n \times n$ and an $(n-1) \times (n-1)$ principal submatrix, respectively; then (8) holds, as a consequence of the semicircle limit law for what could be colloquially called the spectral measure at infinity (see 2.1.1 and 2.1.2). On the other hand, the counterpart of (7) is, in general, false (see 2.1.5).

In Section 2, mostly following [33, 34] in substance if not in terminology, we give an overview of these results with some proofs.

**1.2. Fluctuations about the limiting shape.** In the main part of the paper (Sections 3 and 4), we use the Markov correspondence to study the deviations of diagrams and of interlacing sequences from the limiting shape. We observe that, although (2) is highly nonlinear, it can be linearized about the limiting shape. Therefore one can study the fluctuations of the left-hand side via the right-hand side, and vice versa. Several forms of this assertion are proved in Section 3.1, for example, in the case of the limiting shape $\Omega_{\text{LSVK}}$ we have:

**Proposition 3.1.4.** *Let $\mu_n$ be probability measures, and let $\omega_n = \mathfrak{M} \mu_n$ be the corresponding continual diagrams. Let $(\epsilon_n)_n$ and $(\alpha_k)_{k \geq 2}$ be two sequences*

*such that $\epsilon_n \to 0$ and $\frac{1}{k} \log |\alpha_k| \to 0$. Then*

$$\lim_{n\to\infty} \int_{-2}^{2} \phi(x) \frac{d\mu_n(x) - d\rho_{\text{s.c.}}(x)}{\epsilon_n} = \sum_{k\geq 2} \alpha_k \int_{-2}^{2} \phi(x) \, 2T_k(x/2) \, d\rho_{\text{arcsin}}(x) \quad (9)$$

*for all test functions $\phi$ which are analytic in $[-2,2]$ if and only if*

$$\lim_{n\to\infty} \int_{-2}^{2} \phi(x) \frac{\omega_n(x) - \Omega_{\text{LSVK}}(x)}{\epsilon_n} dx = \sum_{k\geq 1} \frac{4\alpha_{k+2}}{k+1} \int_{-2}^{2} \phi(x) U_k(x/2) d\rho_{\text{s.c.}}(x) \quad (10)$$

*for all such $\phi$.*

Here

$$d\rho_{\text{arcsin}}(x) = \frac{\mathbb{1}_{[-2,2]}(x) dx}{\pi\sqrt{4 - x^2}}$$

is the arcsine distribution, $T_k$ and $U_k$ are the Chebyshev polynomials (43), and the Markov transform $\mathfrak{M}$ is formally defined in 2.2.2. A marginally more general formulation is given in 3.1.4, and the stochastic setting (as opposed to deterministic deviations from the limit shape) is commented upon in 3.2.1. Note that the integrals are exactly the coefficients of $\phi$ in the Chebyshev expansions:

$$\phi(x) = \sum_{k\geq 0} T_k(x/2) \int_{-2}^{2} \phi(y) \, (2 - \delta_{k0}) T_k(y/2) \, d\rho_{\text{arcsin}}(y)$$

$$= \sum_{k\geq 0} U_k(x/2) \int_{-2}^{2} \phi(y) U_k(y/2) d\rho_{\text{s.c.}}(y).$$

In particular, the assumption (9) implies that (and is almost equivalent to)

$$\int T_k(x/2)(d\mu_n(x) - d\rho_{\text{s.c.}}(x)) = \epsilon_n \alpha_k + o(\epsilon_n), \quad n \to \infty. \quad (11)$$

We use Proposition 3.1.4 to study the fluctuations of random diagrams appearing in the theory of random matrices and in the representation theory of the symmetric group.

The study of fluctuations of diagrams was initiated by a theorem, proved by Kerov [32] in the 1990s, which describes the fluctuations of a (Plancherel) random Young diagram about the limiting shape. Informally,

$$\frac{1}{\sqrt{n}} \omega_n(\sqrt{n}x) \approx \Omega_{\text{LSVK}}(x) + \frac{1}{\sqrt{n}} \sum_{k\geq 1} \frac{2g_{k+2}}{\sqrt{k+1}} \frac{U_k(x/2)\sqrt{4 - x^2}}{2\pi}, \quad (12)$$

where $g_k$ are independent, identically distributed standard Gaussian variables (see 4.1.5). Another proof, based on Kerov's unpublished notes, was given by Ivanov

and Olshanski in [25]. We refer to the works [15,44,45] for various generalizations, not discussed here.

Ivanov and Olshanski also described the fluctuations of the transition measure $\tilde{\mu}_n$ associated with $\omega_n$ (see 4.1.3):

$$d\tilde{\mu}_n(x) \approx d\rho_{\text{s.c.}}(x) + \frac{1}{\sqrt{n}} \sum_{k \geq 3} \frac{\sqrt{k-1}}{2} g_k \frac{2T_k(x/2)dx}{\pi\sqrt{4-x^2}}. \tag{13}$$

As observed in [25], (13) bears a similarity to Johansson's central limit theorem [27] for the Gaussian Unitary Ensemble:

$$d\tilde{\rho}_n(x) \approx d\rho_{\text{s.c.}}(x) + \frac{1}{n} \sum_{k \geq 1} \frac{\sqrt{k}}{2} g_k \frac{2T_k(x/2)dx}{\pi\sqrt{4-x^2}} \tag{14}$$

(see 4.1.5). This raises the question what is the analogue of Kerov's central limit theorem (12) in random matrix context.

Recently, this question was studied by Erdős and Schröder [19]. Their result is both general (generalized Wigner matrices are considered), and strong (the $\approx$ sign in (15) below can be understood pointwise, with an explicit power-law estimate on the error term). In the special case of the Gaussian Unitary Ensemble, the result of [19] asserts that the fluctuations of the diagram corresponding to the eigenvalues of $\tilde{H}_n$ and of its principal submatrix are described by

$$\sum_{j=1}^{n+1} |x - \tilde{\lambda}_j| - \sum_{j=1}^{n} |x - \tilde{\lambda}_j^{**}| \approx \Omega_{LSVK}(x) + \frac{1}{\sqrt{n}} \Delta_1^{\mathfrak{M}}(x), \tag{15}$$

$$\Delta_1^{\mathfrak{M}}(x) = \frac{2g_1}{\pi} \arcsin \frac{x}{2} + \sum_{k \geq 1} \frac{2(g_k - g_{k+2})}{k+1} \frac{U_k(x/2)\sqrt{4-x^2}}{2\pi}, \tag{16}$$

where the argument of the arcsine is truncated at $\pm 1$ (see further 3.2.3 and 3.3.3). Parallel results for Jacobi $\beta$-ensembles were obtained by Gorin and Zhang in [22].

The right-hand side of (15) does not look similar to (12) for two reasons. First, a typical random partition of $n$ has $\asymp \sqrt{n}$ rows, therefore the normalization $1/\sqrt{n}$ in (15) would correspond to $1/\sqrt[4]{n}$, rather than $1/\sqrt{n}$ in (12). Second, the Gaussian process in (16) has a continuous modification (it is roughly a reparametrized Brownian motion), whereas the sum in (12) does not even converge in $L_2$.

These differences are natural in view of the Markov correspondence. As mentioned in Section 1.1 above, the Markov transform takes the left-hand side of (15) to the spectral measure $\tilde{\mu}_n$ (see 2.3.4). The fluctuations of the latter were studied, in the context of Gaussian ensembles, by Lytova and Pastur [41]. For the GUE, their result (see 3.2.3) asserts that

$$d\tilde{\mu}_n(x) \approx d\rho_{\text{s.c.}}(x) + \frac{1}{\sqrt{n}} \Delta_1(x)dx , \tag{17}$$

$$\Delta_1(x) \sim \sum_{k \geq 1} g_k U_k(x/2) \frac{\sqrt{4 - x^2}}{2\pi}. \qquad (18)$$

As noted in [41], the larger scale $1/\sqrt{n}$ reflects the Gaussian fluctuations of the eigenvectors of the random matrix. The result (17) was extended to other Wigner matrices by the same authors [42] and by Pizzo–Renfrew–Soshnikov [53]; in this more general setting, the limiting process may change and it is not necessarily Gaussian (see 3.3.3).

In 3.2.3, we feed (17) into the Markov correspondence and obtain a version of (15), albeit in much weaker topology than [19]. More general Wigner matrices are considered in 3.3.3. (Vice versa, it may be possible to deduce a strengthened version of the central limit theorems in [42, 53] from the result of Erdős–Schröder [19] in its full strength and generality; we do not pursue this direction here (See further 3.3.5).

The discussion above raises the expectation that a version of (15) with the critical points $\tilde{\lambda}_j^*$ in place of the submatrix eigenvalues $\tilde{\lambda}_j^{**}$ may bear more similarity to Kerov's theorem (12). Indeed, appealing to Johansson's central limit theorem (14), we show that

**Corollary 3.2.2.** *For the Gaussian Unitary Ensemble,*

$$\sum_{j=1}^{n} |x - \tilde{\lambda}_j| - \sum_{j=1}^{n-1} |x - \tilde{\lambda}_j^*| \approx \Omega_{LSVK}(x) + \frac{1}{n} \Delta_{\text{tr}}^{\mathfrak{M}}(x), \qquad (19)$$

$$\Delta_{\text{tr}}^{\mathfrak{M}}(x) \sim \frac{2}{\pi} g_1 \arcsin(x/2) + \sum_{k \geq 0} \frac{2\sqrt{k+2}}{k+1} g_{k+2} \frac{U_k(x/2)\sqrt{4 - x^2}}{2\pi}. \qquad (20)$$

A precise formulation is given in 3.2.2, a comparison between (19) and (15) — in Figure 2, and a generalization to other Wigner matrices, in 3.3.1.

Recently, Fyodorov asked what are the properties of the critical points $\tilde{\lambda}_j^*$ and, in particular, how to distinguish between them and $\tilde{\lambda}_j^{**}$. Differentiating the relations (19)–(20) and comparing to (15)–(16), we obtain an answer in the following form (stated here for GUE; the topology is as in Proposition 3.1.4):

$$\sum_{j=1}^{n} \delta_{\tilde{\lambda}_j} - \sum_{j=1}^{n-1} \delta_{\tilde{\lambda}_j^*} \approx \frac{dx}{\pi\sqrt{4 - x^2}} + \frac{1}{2n} \left[ \frac{d^2}{dx^2} \Delta_{\text{tr}}^{\mathfrak{M}}(x) \right] dx, \qquad (21)$$

$$\sum_{j=1}^{n} \delta_{\tilde{\lambda}_j} - \sum_{j=1}^{n-1} \delta_{\tilde{\lambda}_j^{**}} \approx \frac{dx}{\pi\sqrt{4 - x^2}} + \frac{1}{2\sqrt{n}} \left[ \frac{d^2}{dx^2} \Delta_1^{\mathfrak{M}}(x) \right] dx. \qquad (22)$$

Note the difference in scaling, and that in the former, the Gaussian process is a derivative of a log-correlated process, while in the latter, it is a derivative of
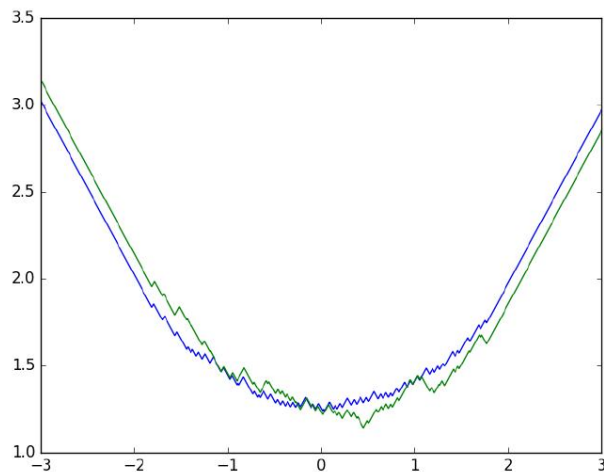
Fig. 2: The random continual diagrams (19) in blue (starts at $(-3, 3)$) and (15) in green (starts at $(-3, 3.1)$), for $\mathrm{GUE}_{n=50}$. The former fluctuates on scale $\sim 1/n$, while the latter — on scale $1/\sqrt{n}$.

(reparametrized) white noise (cf. (49)). The relation (22) was proved by Erdős and Schröder, in greater generality (Wigner matrices) and stronger topology (corresponding to test functions in the Sobolev space $H^2[-10, 10]$); they used it to prove (15). The relation (21) seems not to have been observed before. The formulæ (21) and (22) appear as (48) and (52), respectively, in the body of the paper.

The comparison between the statistical properties of the critical points and the eigenvalues in the local regime (i.e., on scales comparable to the mean spacing) is discussed in the companion paper [57].

In Section 4, we return to random partitions and to the theorems of Kerov and Ivanov–Olshanski stated above as (12) and (13). We describe the setting and use Proposition 3.1.4 to derive one from the other.

In Section 4.2, we outline a proof of (13) (and thus also of (12)) based on the combinatorial approach of Biane [10] and Okounkov [46] in the version of [26]. Our goal is to emphasize the similarity between the transition measure of a random diagram and the normalized eigenvalue counting measure of a random matrix. To this end, we compare the Jucys–Murphy elements (see 4.1.4) acting on a random representation of the symmetric group with Wigner random matrices chosen from an ensemble (see 4.1.6) for which a particularly clean version of the moment method is available (cf. [21, 56] and references therein). For this

ensemble,

$$d\tilde{\rho}_n(x) \approx d\rho_{\text{s.c.}}(x) + \frac{1}{n} \sum_{k \geq 3} \frac{\sqrt{k}}{2} g_k \frac{2T_k(x/2)dx}{\pi\sqrt{4-x^2}}, \tag{23}$$

and accordingly (Proposition 4.1.6),

$$\sum_{j=1}^{n} |x - \tilde{\lambda}_j| - \sum_{j=1}^{n-1} |x - \tilde{\lambda}_j^*|$$

$$\approx \Omega_{\text{LSVK}}(x) + \frac{1}{n} \sum_{k \geq 1} \frac{2\sqrt{k+2}}{k+1} g_{k+2} \frac{U_k(x/2)\sqrt{4-x^2}}{2\pi}. \tag{24}$$

In the combinatorial approach, the coefficient $k$ (the square root of which appears in (23)) acquires the interpretation as the number of ways to align two cycles of length $k$. The coefficient $k-1$ in (13) has exactly the same combinatorial meaning (with cycles of length $k - 1$).

Many of the results are probably familiar to experts. However, we hope to find a reader that would enjoy seeing them under a single cover, rephrased in the peculiar argot of spectral theory. We made no attempt to pursue the limits of the approach, and instead chose to illustrate the main ideas in the simplest setting.

## 2. Limit Shape

### 2.1. Jacobi matrices.

**2.1.1.** Let $J$ be a Jacobi matrix and let $J_n$ be its top-left $n \times n$ principal submatrix

$$J = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad J_n = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}. \tag{25}$$

Define a probability measure $\mu_n$ by

$$\int x^k d\mu_n(x) = (J_n^k)_{nn} = \langle J_n^k \delta_n, \delta_n \rangle, \quad k = 0, 1, 2, \ldots \tag{26}$$

It may be called the spectral measure of $J_n$ at $\delta_n$ or, following Kerov, the $n$-th transition measure of $J$. It is supported on the eigenvalues of $J_n$; the mass at an eigenvalue is equal to the squared $n$-th coordinate of the corresponding eigenvector.

Also define the normalized eigenvalue counting measure $\rho_n$ by

$$\int x^k d\rho_n(x) = \frac{1}{n} \operatorname{tr} J_n^k, \quad k = 0, 1, 2, \ldots \tag{27}$$

which has equal masses $1/n$ at the eigenvalues of $J_n$.

**2.1.2.**　　The asymptotics of $\mu_n$ (colloquially, the spectral measure at infinity) is determined by the asymptotics of $a_n$ and $b_n$. The following proposition is folklore (cf. Kerov [31]):

**Proposition.** *Let $J$ be a Jacobi matrix (25) such that*

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_{n-1} - a_n}{b_n} = 0. \tag{28}$$

*Then the sequence of measures $\widetilde{\mu}_n$ defined by*

$$\widetilde{\mu}_n(B) = \mu_n(b_{n-1}B + a_n), \quad B \in \mathscr{B}(\mathbb{R}),$$

*converges weakly to the semicircle measure $\rho_{\text{s.c.}}$,*

$$\rho_{\text{s.c.}}(B) = \int\limits_{B\cap(-2,2)} \frac{1}{2\pi}\sqrt{4 - x^2}\, dx.$$

*Proof.* The measure $\widetilde{\mu}_n$ is the spectral measure of $\widetilde{J}_n$ at $\delta_1$, where

$$(\widetilde{J}_n)_{ij} = b_{n-1}^{-1}\left((J_n)_{n+1-i\,n+1-j} - a_n\right) .$$

As $n \to \infty$,

$$b_{n-1}^{-1}(\widetilde{J}_n - a_n) \to T$$

in strong operator topology, where $T_{ij} = \delta_{i,j+1} + \delta_{i+1,j}$ is the adjacency matrix of $\mathbb{Z}_{>0}$. The spectral measure of $T$ at $\delta_1$ is exactly $\rho_{\text{s.c.}}$, as one can see, for example, from the relation

$$((T - z)^{-1})_{11} = (-z - ((T - z)^{-1})_{11})^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

which follows from the formula for the top-left matrix element of a matrix inverse. Therefore $\widetilde{\mu}_n \to \rho_{\text{s.c.}}$. ☐

**2.1.3.**　　The previous discussion remains valid for a sequence of Jacobi matrices which are not necessarily the sub-matrices of one infinite matrix. By the same argument as above, we have:

**Proposition.** *Let $(J_n)_{n\geq 0}$ be a sequence of finite Jacobi matrices*

$$J_n = \begin{pmatrix} a_{1,n} & b_{1,n} & 0 & 0 & \cdots \\ b_{1,n} & a_{2,n} & b_{2,n} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{n-1,n} & a_{n,n} \end{pmatrix}$$

*such that for any $k \geq 1$,*

$$\lim_{n\to\infty} \frac{b_{n-k-1,n}}{b_{n-1,n}} = 1, \quad \lim_{n\to\infty} \frac{a_{n-k,n} - a_{n,n}}{b_{n-1,n}} = 0. \tag{29}$$

*Then the sequence of measures $\widetilde{\mu}_n(B) = \mu_n(b_{n-1,n}B + a_{n,n})$ converges weakly to the semicircle measure $\rho_{\text{s.c.}}$.*

**2.1.4.** The asymptotics of $\rho_n$ is also determined by the asymptotics of $a_n$ and $b_n$. However, $\mu_n$ is insensitive to multiplication of the coefficients by a sequence $c_n$ such that $\lim \frac{c_n}{c_{n-1}} = 1$, whereas $\rho_n$ depends on the growth of the coefficients. For example, we have

**Proposition.** *Suppose*

$$a_n = o(\sqrt{n}), \quad b_n = \sqrt{n}(1 + o(1)). \tag{30}$$

*Then the rescaled measures $\widetilde{\rho}_n(B) = \rho_n(\sqrt{n}B)$ converge weakly to $\rho_{\mathrm{s.c.}}$.*

*Proof.* Let $a_n^{\mathrm{H}} = 0$, $b_n^{\mathrm{H}} = \sqrt{n}$. Here H stands for Hermite, and $\rho_n^{\mathrm{H}}$ is the normalized zero counting measure of Hermite polynomials. It is known [59] that $\widetilde{\rho}_n^{\mathrm{H}} \to \rho_{\mathrm{s.c.}}$. This can also be proved directly from the Jacobi matrix as in 2.1.2.

To justify the approximation $\rho_n \approx \rho_n^{\mathrm{H}}$, let $\epsilon > 0$. Choose $n_0$ such that for $n > n_0$

$$|a_n| < \epsilon\sqrt{n}, \quad |b_n - \sqrt{n}| < \epsilon\sqrt{n}. \tag{31}$$

Let

$$a_n^1 = \begin{cases} 0, & n \le n_0 \\ a_n, & n > n_0 \end{cases}, \quad \begin{cases} b_n^1 = \sqrt{n}, & n \le n_0 \\ b_n, & n > n_0 \end{cases}.$$

Then, for any segment $[a, b]$,

$$\left| \widetilde{\rho}_n[a, b] - \widetilde{\rho}_n^1[a, b] \right| \le \frac{n_0}{n}$$

by the interlacing property of rank-one perturbation, and

$$\widetilde{\rho}_n^{\mathrm{H}}[a + \epsilon, b - \epsilon] \le \widetilde{\rho}_n^1[a, b] \le \widetilde{\rho}_n^{\mathrm{H}}[a - \epsilon, b + \epsilon]$$

by (31). It remains to let $n \to \infty$ and then $\epsilon \to +0$. $\square$

**2.1.5.** In the case of Proposition 2.1.4, the sequences $\rho_n$ and $\mu_n$ share the same asymptotics. As emphasized by Kerov, this is an exception rather than a rule. The limit of the former is a kind of integrated density of states, while the latter describes the spectral properties at infinity. Neither is directly related to the usual spectral measure at $\delta_1$.

**Example.** *For $J = T$, $\rho_n \to \rho_{\mathrm{arcsin}}$, where*

$$\rho_{\mathrm{arcsin}}(B) = \int_{B \cap (-2,2)} \frac{1}{\pi} \frac{dx}{\sqrt{4 - x^2}}.$$

*The same conclusion holds for any Jacobi matrix with*

$$a_n = o(1), \quad b_n = 1 + o(1).$$

### 2.2. Continual diagrams.

**2.2.1.** A continual diagram is a function $\omega : \mathbb{R} \to \mathbb{R}$ such that for some $a \in \mathbb{R}$

$$|\omega(x) - \omega(y)| \leq |x - y|, \tag{32}$$

$$\omega(x) = |x - a| \quad \text{for sufficiently large } x. \tag{33}$$

The collection of diagrams is equipped with the topology of uniform convergence. A diagram $\omega$ is said to be supported in a (closed) segment $I$ (denoted: $\omega \in \mathscr{D}(I)$) if (33) holds for all $x \notin I$.

**2.2.2.** Denote by $\mathscr{M}(I)$ the collection of Borel probability measures on $I$, equipped with weak topology.

**Theorem** (Markov [43]; Akhiezer–Krein [3]; Kerov [30]). *For any segment $I$, the relation*

$$\exp\left\{-\frac{1}{2}\int \log(1 - zx) d\omega'(x)\right\} = \int \frac{d\mu(x)}{1 - zx}$$

*defines a homeomorphism* $\mathfrak{M} : \mathscr{D}(I) \longleftrightarrow \mathscr{M}(I)$.

The homeomorphism $\mathfrak{M}$ is called the Markov transform. In the language of what is now called the Markov moment problem, the construction of the bijection $\mathfrak{M}$ for the case of a segment goes back to Markov [43]. It was developed and generalized by Akhiezer and Krein in the 1930s, who published a series of papers [1–4] and a book [5] on this subject. The formulation in the language of continual diagrams is due to Kerov [30], who also observed that $\mathfrak{M}$ is a homeomorphism; see further [34].

**Example.** *The Logan–Shepp–Vershik–Kerov diagram*

$$\Omega_{\text{LSVK}}(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin\frac{x}{2} + \sqrt{4 - x^2}\right), & |x| \leq 2, \\ |x|, & |x| > 2, \end{cases}$$

*corresponds to the semicircle:* $\mathfrak{M}\,\Omega_{\text{LSVK}} = \rho_{\text{s.c.}}$.

**2.2.3.** Let $J$ be a Jacobi matrix. Denote by $\lambda_j^{(n)}$ the eigenvalues of $J_n$, and define the diagram $\omega_n$ corresponding to the interlacing sequences $(\lambda_j^{(n)})$ and $(\lambda_j^{(n-1)})$ via (4), i.e., as a continuous function such that

$$\begin{cases} \omega_n'(x) &= -1 + 2\,\#\left\{j, \lambda_j^{(n)} \leq x\right\} - 2\,\#\left\{j, \lambda_j^{(n-1)} \leq x\right\}, \\ \omega_n(x) &= \left|x - \sum_{j=1}^{n} \lambda_j^{(n)} + \sum_{j=1}^{n-1} \lambda_j^{(n-1)}\right| \quad \text{for sufficiently large } x. \end{cases}$$

Equivalently, if $P_n(z) = \det(z - J_n)$,

$$\omega'_n(x) = \text{sign}\,\frac{P_{n-1}(x)}{P_n(x)}.$$

Also define a diagram $\varpi_n$ such that

$$\varpi'_n(x) = \text{sign}\,\frac{P'_n(x)}{P_n(x)} = \text{sign}\,\frac{1}{n}\frac{P'_n(x)}{P_n(x)}.$$

**Lemma.** $\mathfrak{M}\,\omega_n = \mu_n$ *and* $\mathfrak{M}\,\varpi_n = \rho_n$.

*Proof.* Representing $P_{n-1}/P_n$ and $P'_n/P_n$ as a sum of simple fractions, we obtain:

$$\frac{P_{n-1}(z)}{P_n(z)} = \int \frac{d\mu_n(x)}{z - x}, \quad \frac{1}{n}\frac{P'_n(z)}{P_n(z)} = \int \frac{d\rho_n(x)}{z - x}$$

and then use the bijection $\mathfrak{M}$ from Theorem 2.2.2. $\qquad\qquad\square$

As noted by Kerov (e.g., [33, Section 6]), this lemma is a finite-dimensional trace formula (the study of trace formulæ goes back to the works of Lifshits [39] and Krein [36], see further the survey of Birman and Yafaev [11]).

**2.2.4.**

**Corollary** (Kerov [31]). *Let $J$ be a Jacobi matrix (25) satisfying (28). Then*

$$b_{n-1}^{-1}\omega_n(b_{n-1}x + a_n) \to \Omega_{\text{LSVK}}(x)$$

*uniformly in $x$.*

*Proof.* Follows from Proposition 2.1.2, Lemma 2.2.3 and Theorem 2.2.2. $\quad\square$

**2.2.5.** The corresponding statement for $\varpi_n = \mathfrak{M}^{-1}\rho_n$ is much less general.

**Corollary** (Kerov [31]). *Let $J$ be a Jacobi matrix (25) satisfying (30). Then*

$$\frac{1}{\sqrt{n}}\varpi_n(\sqrt{n}x) \to \Omega_{\text{LSVK}}(x)$$

*uniformly in $x$.*

*Proof.* Follows from Proposition 2.1.4, Lemma 2.2.3 and Theorem 2.2.2. $\quad\square$

**2.2.6.**　　Introduce the rescaling operators

$$(\mathscr{R}_L\,\mu)(B) = \mu(LB) \quad (B \in \mathscr{B}(\mathbb{R})), \quad (\mathscr{R}_L\,\omega)(x) = \frac{1}{L}\omega(Lx).$$

Consider the following random Jacobi matrix, constructed by Dumitriu and Edelman [16]. Fix $\beta > 0$, and let $J^\beta$ be such that

$$a_n^\beta \sim N(0, 2/\beta), \quad \beta\, b_n^\beta \sim \chi_{n\beta},$$

and these random variables are jointly independent. Recall that the $\chi_a$ distribution is defined by the density

$$p_a(x) = \frac{2^{1-a/2}x^{a-1}e^{-x^2/2}}{\Gamma(a/2)}.$$

We have almost surely:

$$a_n^\beta = O(\sqrt{\log n}), \; b_n^\beta = \sqrt{n}(1 + o(1)).$$

Therefore, by Propositions 2.1.2 and 2.1.4,

$$\mathscr{R}_{\sqrt{n}}\,\mu_n^\beta \to \rho_{\text{s.c.}} \;, \quad \mathscr{R}_{\sqrt{n}}\,\rho_n^\beta \to \rho_{\text{s.c.}} \quad \text{weakly},$$

and by Corollaries 2.2.4 and 2.2.5,

$$\mathscr{R}_{\sqrt{n}}\,\omega_n^\beta \to \Omega_{\text{LSVK}}(x) \;, \quad \mathscr{R}_{\sqrt{n}}\,\varpi_n^\beta \to \Omega_{\text{LSVK}}(x) \quad \text{uniformly},$$

almost surely.

**Remark.** *These statements are not directly related to the spectral properties of $J^\beta$, see [12, 13] for the properties of the latter and a discussion.*

### 2.3.　Random matrices.

**2.3.1.**　　The Gaussian Unitary Ensemble (GUE) is the ensemble of semi-infinite Hermitian matrices $H = (H(i,j))_{i,j\geq 1}$ such that for any $n$ the top-left $n \times n$ submatrix $H_n$ has the probability density

$$Z_n^{-1} \exp\left\{ -\frac{1}{2}\operatorname{tr} H_n^2 \right\}$$

with respect to the Lebesgue measure on the space of $n \times n$ Hermitian matrices.

Let $\lambda_j^{(n)}$ be the eigenvalues of $H_n$, and let $P_n(z) = \det(z - H_n)$. Define the spectral measure $\mu_n^{\text{GUE}}$ and the normalized eigenvalue distribution $\rho_n^{\text{GUE}}$ by the formulæ

$$\int x^k d\mu_n^{\text{GUE}} = (H_n^k)_{nn}\;, \quad \int x^k d\rho_n^{\text{GUE}} = \frac{1}{n}\operatorname{tr} H_n^k\;, \quad k = 0, 1, 2, \ldots$$

Also define the diagrams $\omega_n^{\text{GUE}}$ and $\varpi_n^{\text{GUE}}$ by

$$\frac{d}{dx}\omega_n^{\text{GUE}}(x) = \operatorname{sign}\frac{P_{n-1}(x)}{P_n(x)}, \quad \frac{d}{dx}\varpi_n^{\text{GUE}}(x) = \operatorname{sign}\frac{P_n'(x)}{P_n(x)}.$$

**2.3.2.**

**Theorem** (Dumitriu–Edelman [16]). *The joint distribution of*

$$(\lambda_j^{(n)})_{j=1}^n \quad and \quad (\lambda_j^{(n-1)})_{j=1}^{n-1}$$

*for the GUE coincides with the joint distribution of the same quantities for $J^{\beta=2}$ from 2.2.6.*

In other words, $J^{\beta=2}$ is obtained from the GUE by tridiagonalization.

**Corollary.** *The distribution of $\mu_n^{\mathrm{GUE}}, \rho_n^{\mathrm{GUE}}, \omega_n^{\mathrm{GUE}}, \varpi_n^{\mathrm{GUE}}$ from 2.3.1 coincides with the distribution of $\mu_n^{\beta=2}, \rho_n^{\beta=2}, \omega_n^{\beta=2}, \varpi_n^{\beta=2}$ associated with the random Jacobi matrix $J^{\beta=2}$ of 2.2.6.*

**2.3.3.** Combining Corollary 2.3.2 with the conclusion of 2.2.6, we obtain:

**Corollary** (Kerov [31], special case). *Almost surely*

$$\mathscr{R}_{\sqrt{n}}\, \mu_n^{\mathrm{GUE}}, \mathscr{R}_{\sqrt{n}}\, \rho_n^{\mathrm{GUE}} \to \rho_{\mathrm{s.c.}} \qquad\qquad weakly,$$
$$\mathscr{R}_{\sqrt{n}}\, \omega_n^{\mathrm{GUE}}, \mathscr{R}_{\sqrt{n}}\, \varpi_n^{\mathrm{GUE}} \to \Omega_{\mathrm{LSVK}} \qquad\qquad uniformly.$$

**2.3.4.** Corollary 2.3.3 can be extended to the class of Wigner matrices. Let $H = (H(i,j))_{i,j\geq 1}$ be an arbitrary semi-infinite Hermitian random matrix such that

$$\{H(i,j),\, i \leq j\}$$

are jointly independent with $\mathbb{E}H(i,j) = 0$, $(H(i,i))$ are identically distributed, and $(H(i,j))_{i<j}$ are identically distributed with $\mathbb{E}|H(i,j)|^2 = 1$. Let $H_n$ be the top-left principal submatrix of $H$, and let $\mu_n, \rho_n, \omega_n, \varpi_n$ be defined as in 2.3.1.

Let us quote Wigner's law [62]. In the current generality it was proved by Pastur [51].

**Proposition** (Wigner; Pastur). *Almost surely $\mathscr{R}_{\sqrt{n}}\, \rho_n \to \rho_{\mathrm{s.c.}}$.*

**Corollary** (Kerov [31]). *Almost surely $\mathscr{R}_{\sqrt{n}}\, \varpi_n \to \Omega_{\mathrm{LSVK}}$.*

**2.3.5.** Proposition 2.3.4 and Corollary 2.3.4 have a counterpart for $\mu_n$ and $\omega_n$. The former is the following version of Wigner's law (proved similarly to [51]):

**Proposition.** *In the setting of 2.3.4, $\mathscr{R}_{\sqrt{n}}\, \mu_n \to \rho_{\mathrm{s.c.}}$ almost surely.*

**Corollary** (Kerov [31]; Bufetov [14] /general case/). *$\mathscr{R}_{\sqrt{n}}\, \omega_n \to \Omega_{\mathrm{LSVK}}$ almost surely.*

## 3.    Corrections to the Limit Shape

### 3.1.    Deterministic corrections.

**3.1.1.**    Let $\omega_n$ be a sequence of continual diagrams such that $\omega_n \to \Omega$ in uniform topology. Then

$$\mu_n = \mathfrak{M}\,\omega_n \to \rho = \mathfrak{M}\Omega$$

in weak topology. Our goal is to relate the corrections $\omega_n - \Omega$ to $\mu_n - \rho$. We start with a lemma.

**Lemma.** *Let $\epsilon_n \to +0$. Suppose $\omega_n$ and $\Omega$ are continual diagrams such that the corresponding measures $\mu_n = \mathfrak{M}\,\omega_n$ and $\rho = \mathfrak{M}\Omega$ satisfy*

$$\int (x-z)^{-1}\, d\mu_n(x) = \int (x-z)^{-1} d\rho(x) + \epsilon_n R(z) + o(\epsilon_n), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (34)$$

*Then*

$$\int (x-z)^{-1}\, d(\omega_n(x) - \Omega(x)) = -2\epsilon_n \frac{R(z)}{w_\rho(z)} + o(\epsilon_n), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (35)$$

*where we defined the Stieltjes transform*

$$w_\rho(z) = \int (x-z)^{-1}\, d\rho(x). \quad (36)$$

As one can see from the proof below, if the convergence in the assumption is uniform on compact sets, it is so also in the conclusion.

*Proof.* By Theorem 2.2.2 applied with $z^{-1}$ in place of $z$,

$$\int \log(1 - z^{-1}x)\, d\omega_n'(x) = -2\log \int (1 - z^{-1}x)^{-1}\, d\mu_n(x)$$

$$= -2\log z + 2\log \int (x-z)^{-1}\, d\mu_n(x).$$

Using the assumption (34), we deduce

$$\int \log(1 - z^{-1}x)\, d\omega_n'(x)$$

$$= -2\log z + 2\log \int (x-z)^{-1}\, d\rho(x) + 2\epsilon_n \frac{R(z)}{w_\rho(x)} + o(\epsilon_n).$$

Similarly, $\Omega$ and $\rho$ are related by

$$\int \log(1 - z^{-1}x)\, d\Omega'(x) = -2\log z + 2\log \int (x-z)^{-1}\, d\rho(x),$$

hence

$$\int \log(1 - z^{-1}x) \, d(\omega'_n(x) - \Omega'(x)) = \frac{2\epsilon_n R(z)}{w_\rho(x)} + o(\epsilon_n). \tag{37}$$

Integrating by parts, we rewrite the left-hand side of (37) as

$$\int \log(1 - z^{-1}x) \, d(\omega'_n(x) - \Omega'(x)) = - \int (x - z)^{-1} \, d(\omega_n(x) - \Omega(x))$$

and this concludes the proof. □

**3.1.2.** The conclusion of Lemma 3.1.1 can be reformulated in the following form, from which one can see that the asymptotics of $\omega_n - \Omega$ (and not just of the derivative) is determined.

**Corollary.** *In the setting of Lemma 3.1.1, suppose that the convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. Then*

$$\int_I (x - z)^{-1}(\omega_n(x) - \Omega(x)) \, dx = 2\epsilon_n \int^z \frac{R(\zeta) \, d\zeta}{w_\rho(\zeta)} + o(\epsilon_n), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{38}$$

*where $I$ is an interval in which $\Omega$ is supported, the integral is from $\pm i\infty$ along a path avoiding the real axis, and the convergence is also uniform on compact subsets of $\mathbb{C} \setminus \mathbb{R}$.*

*Proof.* Let $I$ be an interval in which $\Omega$ is supported. The assumption of uniform convergence on compact subsets implies that, for real $x$ outside $I$,

$$\omega_n(x) - \Omega(x) = o(\epsilon_n). \tag{39}$$

Therefore we can choose $\delta_n = o(\epsilon_n)$ such that the shifted diagram

$$\omega_n^{\rightarrow}(x) = \omega_n(x - \delta_n)$$

and $\Omega(x)$ have the same center, i.e., coincide for sufficiently large $|x|$. Note that $\omega_n^{\rightarrow}(x) - \omega_n(x) = o(\epsilon_n)$ uniformly in $x$. By Lemma 3.1.1,

$$\int (x - z)^{-1} \, d(\omega_n^{\rightarrow}(x) - \Omega(x)) = -2\epsilon_n \frac{R(z)}{w_\rho(z)} + o(\epsilon_n), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

whence, integrating by parts and replacing $z$ with $\zeta$,

$$\frac{d}{d\zeta} \int (x - \zeta)^{-1}(\omega_n^{\rightarrow}(x) - \Omega(x)) \, dx = 2\epsilon_n \frac{R(\zeta)}{w_\rho(\zeta)} + o(\epsilon_n), \quad \zeta \in \mathbb{C} \setminus \mathbb{R}. \tag{40}$$

Integrating (40), we obtain

$$\int (x - z)^{-1}(\omega_n^{\rightarrow}(x) - \Omega(x)) \, dx = 2\epsilon_n \int^z \frac{R(\zeta) \, d\zeta}{w_\rho(\zeta)} + o(\epsilon_n), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where the integral can be taken over any interval containing the support of $\Omega$, and this implies (38). □

**3.1.3.** Corollary 3.1.2 allows us to drag the corrections to the limiting shape through the Markov correspondence. Denote by $\mathfrak{B}_{[a,b]}$ the space of analytic test functions $\phi : [a,b] \to \mathbb{C}$. The space $\mathfrak{B}_{[a,b]}$ is topologized as the projective limit of the spaces of analytic functions in shrinking neighborhoods of $[a,b]$. Also consider the space of continuous functionals $\mathfrak{B}'_{[a,b]}$, and observe that the topology on this space coincides with the minimal topology in which the functionals

$$F \mapsto \langle F, (\cdot - z)^{-1} \rangle, \quad z \notin [a,b]$$

are continuous. In the setting of 3.1.2, define $F, F^{\mathfrak{M}} \in \mathfrak{B}'_{[a,b]}$ by

$$\langle F, \phi \rangle = \oint_\Gamma \phi(z) R(z) \frac{dz}{2\pi i}, \quad \langle F^{\mathfrak{M}}, \phi \rangle = -\oint_\Gamma \Phi(z) \frac{R(z)}{w_\rho(z)} \frac{dz}{\pi i} \ ,$$

where $\Phi(z) = \frac{1}{2}(\int_{-2}^z - \int_z^2)\phi(x)\, dx$, and $\Gamma$ encircles $I$ counterclockwise within the domain of analyticity of $\phi$.

**Proposition.** *Let $\mu_n, \rho$ be probability measures such that $\operatorname{supp} \rho \subset [a,b]$, and let $\omega_n = \mathfrak{M}^{-1}\mu_n$ and $\Omega = \mathfrak{M}^{-1}\rho$. If, for some $\epsilon_n \to +0$,*

$$\epsilon_n^{-1}(d\mu_n(x) - d\rho(x)) \to F \quad in \ \mathfrak{B}'_{[a,b]}, \tag{41}$$

*then*

$$\epsilon_n^{-1}(\omega_n(x)\, dx - \Omega(x)\, dx) \to F^{\mathfrak{M}} \quad in \ \mathfrak{B}'_{[a,b]}. \tag{42}$$

*Proof.* Use Corollary 3.1.2 and the Cauchy theorem. $\qquad\qquad\square$

By the construction of $\mathfrak{B}'_{[-2,2]}$, the left-hand side of (42) is implicitly multiplied by the indicator $\mathbb{1}_{[a,b]}(x)$. The indicator can be dropped if

$$\int x\, d\rho_n(x) = \int x\, d\rho(x) + o(\epsilon_n).$$

Otherwise, it is necessary, as observed in [19] and as one can see from Figure 2. Second, the implication in the proposition is in fact an equivalence: (42) implies (41), as one can see by tracing the arguments.

**3.1.4.** Consider the following example. Assume that $\rho = \rho_{\mathrm{s.c.}}$ is the semi-circle measure, the Stieltjes transform of which is given by

$$w_\rho(z) = \int_{-2}^2 \frac{1}{2\pi} \sqrt{4 - x^2} \frac{dx}{x - z} = \frac{-z + \sqrt{z^2 - 4}}{2}.$$

Recall the definition of Chebyshev polynomials

$$T_n(\cos\theta) = \cos(n\theta), \quad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \tag{43}$$

or explicitly,

$$T_0(x/2) = 1, \qquad T_1(x/2) = \frac{x}{2}, \qquad T_2(x/2) = \frac{x^2}{2} - 1, \qquad \cdots$$

$$U_0(x/2) = 1, \qquad U_1(x/2) = x, \qquad U_2(x/2) = x^2 - 1, \qquad \cdots$$

They satisfy the orthogonality relations

$$\int T_k(x/2)T_l(x/2)\,d\rho_{\mathrm{arcsin}}(x) = \frac{1 + \delta_{k0}}{2}\delta_{kl}, \quad \int U_k(x/2)U_l(x/2)\,d\rho_{\mathrm{s.c.}}(x) = \delta_{kl}.$$

Assume that

$$\frac{1}{\epsilon_n}d(\mu_n(x) - \rho_{\mathrm{s.c.}}(x)) \to \left\{\sum_{k\geq 1}\frac{2c_k T_k(x/2)\,dx}{\pi\sqrt{4-x^2}}\right\} \quad \text{in } \mathcal{B}'_{[-2,2]}, \tag{44}$$

i.e., for any $z \in \mathbb{C} \setminus [-2,2]$,

$$\int\frac{d(\mu_n(x) - \rho_{\mathrm{s.c.}}(x))}{x-z} = \epsilon_n\left\{\sum_{k\geq 1}c_k\int_{-2}^{2}\frac{2T_k(x/2)\,dx}{\pi\sqrt{4-x^2}(x-z)}\right\} + o(\epsilon_n). \tag{45}$$

Observing that

$$w_{\rho_{\mathrm{arcsin}},k}(z) = \int_{-2}^{2}\frac{T_k(x/2)\,dx}{\pi\sqrt{4-x^2}(x-z)} = \frac{-1}{\sqrt{z^2-4}}T_k(z/2) + \frac{1}{2}U_{k-1}(z/2)$$

$$= -\frac{1}{\sqrt{z^2-4}}\left\{\frac{z - \sqrt{z^2-4}}{2}\right\}^k,$$

we deduce that for $k \geq 2$,

$$-2\frac{w_{\rho_{\mathrm{arcsin}},k}(z)}{w_{\rho_{\mathrm{s.c.}}}(z)} = 2w_{\rho_{\mathrm{arcsin}},k-1}(z) = \frac{2}{k-1}\int_{-2}^{2}\frac{\frac{d}{dx}\left[U_{k-2}(x/2)\frac{\sqrt{4-x^2}}{2\pi}\right]}{x-z}\,dx,$$

whereas

$$-2\frac{w_{\rho_{\mathrm{arcsin}},1}(z)}{w_{\rho_{\mathrm{s.c.}}}(z)} = 2w_{\rho_{\mathrm{arcsin}}}(z) = -\frac{2}{\pi}\int_{-2}^{2}\frac{\frac{d}{dx}\arcsin(x/2)}{x-z}\,dx,$$

and finally

$$\epsilon_n^{-1}(\omega_n(x) - \Omega_{\mathrm{LSVK}}(x))dx \longrightarrow$$

$$\left\{-\frac{4c_1}{\pi}\arcsin(x/2) + \sum_{k\geq 2}\frac{4c_k}{k-1}U_{k-2}(x/2)\frac{\sqrt{4-x^2}}{2\pi}\right\}dx \quad \text{in } \mathcal{B}'_{[-2,2]}. \tag{46}$$

We proved:

**Proposition.** *Let $\{c_k\}$ be a sequence with $\limsup |c_k|^{1/k} = 1$. If $\mu_n = \mathfrak{M} \omega_n$ satisfy (44), then (46) holds.*

As we noted above, the arguments go in both directions, and (44) is actually equivalent to (46).

### 3.2. Fluctuations about the limit shape.

**3.2.1.** Although Proposition 3.1.4 was proved in the deterministic setting, standard arguments allow us to apply it in the stochastic setting, i.e., for random measures $\mu_n$, after proper modifications. For example, if the assumptions holds almost surely, then so does the conclusion; if the assumption holds in distribution, then so does the conclusion. The former version follows directly from the deterministic statement, while the latter version follows from the former one using Skorokhod's representation theorem. In the sequel we will work with the convergence in distribution.

**3.2.2.** Let $g_k$ be independent, identically distributed standard Gaussian random variables, and let

$$\Delta_{\mathrm{tr}}(x) = \sum_{k \geq 1} \frac{\sqrt{k}}{2} g_k \frac{2T_k(x/2)}{\pi \sqrt{4 - x^2}},$$

$$\Delta_{\mathrm{tr}}^{\mathfrak{M}}(x) = -\frac{2}{\pi} g_1 \arcsin(x/2) - \sum_{k \geq 0} \frac{2\sqrt{k+2}}{k+1} g_{k+2} \frac{U_k(x/2)\sqrt{4 - x^2}}{2\pi},$$

where the series are understood in $\mathfrak{B}'_{[-2,2]}$. Following [25], we note the similarity between $\Delta_{\mathrm{tr}}(x)$ and

$$\frac{d}{dx}\Delta_{\mathrm{tr}}^{\mathfrak{M}}(x) \sim -\frac{g_1}{\pi \sqrt{4 - x^2}} - \sum_{k \geq 1} \sqrt{k} g_{k+1} \frac{2T_k(x/2)}{\pi \sqrt{4 - x^2}}.$$

**Theorem** (Johansson [27]). *For the Gaussian Unitary Ensemble,*

$$n \, d(\mathscr{R}_{\sqrt{n}}[\rho_n^{\mathrm{GUE}}](x) - \rho_{\mathrm{s.c.}}(x)) \to \Delta_{\mathrm{tr}}(x) \, dx$$

*as random functionals on $\mathfrak{B}_{[-2,2]}$.*

In the original work [27], the convergence was established in slightly weaker topology; now the result is available in the topology corresponding to $\mathfrak{B}_{[-2,2]}$ and even a much stronger one (cf. 3.3.2 below). Appealing to (a stochastic version of) Proposition 3.1.4, we obtain:

**Corollary.** *For the GUE,*

$$n\left(\mathscr{R}_{\sqrt{n}}[\varpi_n^{\mathrm{GUE}}](x) - \Omega_{\mathrm{LSVK}}(x)\right)dx \to \Delta_{\mathrm{tr}}^{\mathfrak{M}}(x)\,dx, \tag{47}$$

$$n\,d\mathscr{R}_{\sqrt{n}}[\rho_n^{\mathrm{GUE}} - \rho_{n-1}^{\mathrm{GUE},*}](x) \to \frac{1}{2}\frac{d^2}{dx^2}\Delta_{\mathrm{tr}}^{\mathfrak{M}}(x)\,dx, \tag{48}$$

*where $\rho_{n-1}^*$ is the normalized counting measure of the zeros of $P_n'$.*

**3.2.3.** Now let

$$\Delta_1(x) = \sum_{k\geq 1} g_k U_k(x/2)\frac{\sqrt{4-x^2}}{2\pi} = \sum_{k\geq 1}\frac{g_k}{2}\frac{2T_k(x) - 2T_{k+2}(x)}{\pi\sqrt{4-x^2}},$$

$$\Delta_1^{\mathfrak{M}}(x) = -\frac{2g_1}{\pi}\arcsin\frac{x}{2} - \sum_{k\geq 1}\frac{2(g_k - g_{k+2})}{k+1}U_k(x/2)\frac{\sqrt{4-x^2}}{2\pi}.$$

We mention that $\Delta_1$ can also be described as follows:

$$\Delta_1(x)\,dx = \sqrt{\rho_{\mathrm{s.c.}}'(x)}dB(x) - \left\{\int_{-2}^{2}\sqrt{\rho_{\mathrm{s.c.}}'(y)}dB(y)\right\}\rho_{\mathrm{s.c.}}'(x)\,dx, \tag{49}$$

where $\rho_{\mathrm{s.c.}}'(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$, and $B(x)$ is the Brownian motion. While $\Delta_1(x)dx$ is a generalized Gaussian process, its integral

$$\Delta_1^{\int}(x) = \int_{-2}^{\min(x,2)}\Delta_1(y)dy$$

has a continuous modification, and so does $\Delta_1^{\mathfrak{M}}(x)$.

The fluctuations of the measure $\mu_n$ are described by the following result of Lytova–Pastur [41] (where a stronger topology defined by test functions in $C^1$ is used, see further 3.3.4).

**Theorem** (Lytova–Pastur [41] /weak form/)**.** *For the GUE,*

$$\sqrt{n}\,d(\mathscr{R}_{\sqrt{n}}[\mu_n^{\mathrm{GUE}}](x) - \rho_{\mathrm{s.c.}}(x)) \xrightarrow{\mathrm{distr}} \Delta_1(x)\,dx \tag{50}$$

*as random functionals on $\mathfrak{B}_{[-2,2]}$.*

This theorem and Proposition 3.1.4 imply:

**Corollary** (Erdős and Schröder [19] /special case, weak form/)**.**

$$\sqrt{n}(\mathscr{R}_{\sqrt{n}}[\omega_n^{\mathrm{GUE}}](x) - \Omega_{\mathrm{LSVK}}(x))\,dx \xrightarrow{\mathrm{distr}} \Delta_1^{\mathfrak{M}}(x)\,dx, \tag{51}$$

$$\sqrt{n}\,d\mathscr{R}_{\sqrt{n}}[\rho_n^{\mathrm{GUE}} - \rho_{n-1}^{\mathrm{GUE}}](x) \xrightarrow{\mathrm{distr}} \frac{1}{2}\frac{d^2}{dx^2}\Delta_1^{\mathfrak{M}}(x)\,dx. \tag{52}$$

### 3.3. On generalizations.

**3.3.1.** Johansson's Theorem 3.2.2 has been extended beyond the Gaussian Unitary Ensemble, see [52, Section 18.4], [6, Section 2.1.7] and references therein. We quote (53) below from the work of Bai and Yao [8]; related results were proved earlier by Khorunzhy, Khoruzhenko, and Pastur [35].

Let $H$ be a Wigner matrix as in 2.3.4; let us assume that the matrix entries have finite fourth moments, and that $\mathbb{E}H(1,2)^2 = 0$ (the second condition can be omitted at the expense of making the formulæ more cumbersome). Then

$$n \, d(\mathscr{R}_{\sqrt{n}}[\rho_n^{\mathrm{GUE}}](x) - \rho_{\mathrm{s.c.}}(x)) \xrightarrow{\mathrm{distr}} \sum_{k \geq 1} \frac{a_k g_k + b_k}{2} \frac{2T_k(x/2)}{\pi\sqrt{4-x^2}} \qquad (53)$$

as random functionals on $\mathfrak{B}'_{[-2,2]}$, where

$$a_1 = \sqrt{\mathbb{E}H(1,1)^2}, \quad a_2 = \sqrt{2\mathbb{E}(|H(1,2)|^2 - 1)^2}, \quad a_3 = \sqrt{3}, \quad a_4 = \sqrt{4}, \quad \dots$$

$$b_2 = \mathbb{E}H(1,1)^2 - 1, \quad b_4 = \mathbb{E}(|H(1,2)|^2 - 1)^2 - 1, \quad b_1 = b_3 = b_5 = b_6 = \cdots = 0.$$

Using (53) in place of Theorem 3.2.2, Corollary 3.2.2 is extended as follows:

$$n \, (\mathscr{R}_{\sqrt{n}}[\varpi_n^{\mathrm{GUE}}](x) - \Omega_{\mathrm{LSVK}}(x)) \, dx \xrightarrow{\mathrm{distr}} -\frac{2}{\pi} a_1 g_1 \arcsin(x/2) dx$$

$$- \sum_{k \geq 0} \frac{2\sqrt{k+2}}{k+1}(a_{k+2}g_{k+2} + b_{k+2}) \frac{U_k(x/2)\sqrt{4-x^2}}{2\pi} \, dx. \qquad (54)$$

**3.3.2.** It may also be possible to strengthen the topology in Corollary 3.2.2, using as an input a topologically stronger central limit theorem for linear statistics such as the one proved by Shcherbina [55] (see further Sosoe and Wong [58]). To follow this strategy, one needs a version of Proposition 3.1.4 for non-analytic test functions; this can be obtained, for example, by the method of pseudoanalytic extension [17, 18, 23].

**3.3.3.** As for the fluctuations of the spectral measure, Theorem 3.2.3 of Lytova–Pastur was extended beyond the Gaussian ensembles by Lytova–Pastur [42] and Pizzo–Renfrew–Soshnikov [53]. We quote one of the results in [53]: for Wigner matrices as in 2.3.4 having finite fourth moments,

$$\sqrt{n} \, d(\mathscr{R}_{\sqrt{n}}[\mu_n^{\mathrm{GUE}}](x) - \rho_{\mathrm{s.c.}}(x)) \xrightarrow{\mathrm{distr}} \sum_{k \geq 1} h_k U_k(x/2) \, d\rho_{\mathrm{s.c.}}(x), \qquad (55)$$

where $h_k$ are independent random variables: $h_1 = -H(n,n)$, $h_2$ is centered Gaussian with variance depending on the moments of $H(1,2)$, and $(h_k)_{k \geq 3}$ are centered Gaussian with variance depending on the symmetry class of $H$ (cf. (58)

below for explicit formulæ). The topology is stronger than that of $\mathfrak{B}'_{[-2,2]}$, see 3.3.4 below.

Applying Proposition 3.1.4, we deduce that for $H$ as above,

$$
\sqrt{n}(\mathscr{R}_{\sqrt{n}}[\omega_n^{\mathrm{GUE}}](x) - \Omega_{\mathrm{LSVK}}(x))\,dx
$$
$$
\xrightarrow{\mathrm{distr}} -\frac{2h_1}{\pi}\arcsin\frac{x}{2}\,dx - \sum_{k\geq 1}\frac{2(h_k - h_{k+1})}{k+1}U_k(x/2)\,d\rho_{\mathrm{s.c.}}(x) \qquad (56)
$$

in the topology of random functionals on $\mathfrak{B}_{[-2,2]}$.

**3.3.4.** The result of Erdős and Schröder [19, Theorem 3] is topologically much stronger than our statements in (51) and (56), and is applied, for example, to test functions which are indicators of intervals. This raises the question whether the topology can be strengthened in the central limit theorems (50), (55) for the spectral measure.

In the case of the GUE, one can use the convergence to the semicircle on scale $n^{-1/2}$ and a functional central limit theorem for sums of independent exponentially distributed random variables to obtain the following version of Theorem 3.2.3:
$$
\sqrt{n}\left(\mu_n^{\mathrm{GUE}}(x\sqrt{n}) - \rho_{\mathrm{s.c.}}(x)\right) \xrightarrow{\mathrm{distr}} \Delta_1^{\int}(x) \qquad (57)
$$

as random continuous functions (here we identify measures with their cumulative distribution functions).

In the case of general Wigner matrices with finite fourth moments, the results of [53] imply that (55) holds in the topology of functionals on test functions $\phi \in C^7[-2-\delta, 2+\delta]$. It is also proved in [53] that for Wigner matrices the entries of which satisfy the Poincaré inequality the limit theorem (55) holds for Lipschitz test functions. This topology is still (most probably) insufficient to recover the results of [19] in their full strength.

**3.3.5.** In a remark in Section 1.2 (following (17)), we suggested that the results of [19] could be pulled through the Markov correspondence to obtain a version of (55) in stronger topology. However, Erdős and Schröder found [20] a direct approach to the latter problem.

For the special case of Wigner matrices as in 2.3.4, their result asserts the following (the more general setting of [20] applies to matrices the entries are not necessarily identically distributed): if $H(1,1)$ and $H(1,2)$ have moments of arbitrary order, then

$$
\sqrt{n}(d\,\mathscr{R}_{\sqrt{n}}\,\mu_n^H(x) - d\rho_{\mathrm{s.c.}}(x)) \xrightarrow{\mathrm{distr}} d\rho_{\mathrm{s.c.}}(x)\Bigg[ - H(n,n)U_1(x/2)
$$

$$+ \sqrt{\mathbb{E}|H(1,2)|^4 - 1}\, g_2 U_2(x/2) + \sum_{k \geq 3} \sqrt{1 + (\mathbb{E}H(1,2)^2)^k}\, g_k U_k(x/2) \Bigg], \quad (58)$$

where $g_k$ are independent standard Gaussian, and the topology is defined by test functions $\phi : \mathbb{R} \to \mathbb{R}$ of bounded variation in $[-3, 3]$. For the convenience of the reader, we have not modified the phrasing of the remark in Section 1.2.

## 4. Random Partitions and Random Matrices

### 4.1. Plancherel growth.

**4.1.1.** A partition $\lambda^n$ of $n \geq 0$ (denoted $\lambda^n \vdash n$) is a non-increasing sequence

$$\lambda^n = (\lambda_1^n \geq \lambda_2^n \geq \lambda_3^n \geq \cdots)$$

of non-negative integers adding up to $n$. We identify a partition $\lambda^n$ with the Young diagram with rows of lengths $\lambda_1^n, \lambda_2^n, \cdots$, i.e., the union of unit squares (boxes) with a corner at

$$\left\{ (j,k) \mid 1 \leq j,\ 1 \leq k \leq \lambda_j^n \right\}.$$

For example, the partition $19 = 7 + 4 + 4 + 3 + 1$ of 19 corresponds to the Young diagram in Figure 1a. The content of a box $\square = (j,k)$ is by definition $\mathrm{ct}(\square) = k - j$. For two diagrams $\lambda^n \vdash n$ and $\lambda^{n+1} \vdash n+1$, we write $\lambda^n \nearrow \lambda^{n+1}$ if $\lambda^{n+1}$ is obtained by adding a single box $\lambda^{n+1} \setminus \lambda^n$ to $\lambda^n$. A box $\square \in \lambda^n$ is called an inner corner of $\lambda^n$ if $\lambda^n \setminus \square$ is a partition (Young diagram). A box $\square \notin \lambda^n$ is called an outer corner if $\lambda^n \cup \square$ is a partition. The continual diagram $\omega[\lambda^n]$ associated to a partition $\lambda^n$ is defined by

$$\omega'(x) = -1 + 2 \,\#\, \{\text{outer corners } \square \text{ with } \mathrm{ct}(\square) < x\}$$
$$- 2 \,\#\, \{\text{inner corners } \square \text{ with } \mathrm{ct}(\square) < x\}$$
$$\omega(x) = |x| \quad \text{for sufficiently large } x.$$

See Figure 1c.

**4.1.2.** A standard Young tableau is a chain

$$\mathbf{T} = (\lambda^0 = \varnothing \nearrow \lambda^1 \nearrow \lambda^2 \nearrow \cdots \nearrow \lambda^n). \quad (59)$$

The dimension $\dim \widetilde{\lambda}^n$ of a Young diagram $\widetilde{\lambda}^n$ is the number of chains (59) with $\lambda^n = \widetilde{\lambda}^n$. In the representation theory of the symmetric group, the irreducible representations are in one-to-one correspondence with the partitions of $n$, and $\dim \lambda$ equals the dimension of the irreducible representation corresponding to $\lambda$. One has (see 4.1.4 below):

$$\dim \lambda^n = \sum_{\lambda^{n-1} \nearrow \lambda^n} \dim \lambda^{n-1}, \quad \sum_{\lambda^n \vdash n} \dim^2 \lambda^n = n!. \quad (60)$$

**4.1.3.** Consider the space $\mathrm{Tab}_\infty$ of infinite sequences

$$\lambda^0 = \varnothing \nearrow \lambda^1 \nearrow \lambda^2 \nearrow \cdots \nearrow \lambda^n \nearrow \cdots \tag{61}$$

equipped with product topology. Define a probability distribution on $\mathrm{Tab}_\infty$ by

$$\mathbb{P}\left\{\lambda^1 = \widetilde{\lambda}^1, \cdots, \lambda^n = \widetilde{\lambda}^n\right\} = \frac{\dim \widetilde{\lambda}^n}{n!}$$

for any $n$ and any chain $\widetilde{\lambda}^0 \nearrow \cdots \nearrow \widetilde{\lambda}^n$. The corresponding process is called Plancherel growth, and the distribution

$$\mathbb{P}\left\{\lambda^n = \widetilde{\lambda}^n\right\} = \frac{\dim^2 \widetilde{\lambda}^n}{n!}$$

of $\lambda^n$ — the Plancherel measure. The transition measure of a diagram $\widetilde{\lambda}^n \vdash n$ is the probability measure

$$\mu[\widetilde{\lambda}^n] = \sum_{\text{outer corners } \square} \frac{\mathbb{P}\left\{\lambda^{n+1} = \widetilde{\lambda}^n \cup \square,\ \lambda^n = \widetilde{\lambda}^n\right\}}{\mathbb{P}\left\{\lambda^n = \widetilde{\lambda}^n\right\}} \delta_{\mathrm{ct}(\square)}. \tag{62}$$

This measure was introduced by Kerov [30], who proved that $\mathfrak{M}\omega[\lambda^n] = \mu[\lambda^n]$.

**Theorem** (Logan–Shepp [40], Vershik–Kerov [60,61]; Kerov [31])**.**

$$\mathscr{R}_{\sqrt{n}}\left\{\omega[\lambda^n]\right\} \to \Omega_{\mathrm{LSVK}} \quad and \quad \mathscr{R}_{\sqrt{n}}\,\mu[\lambda^n] \to \rho_{\mathrm{s.c.}}$$

*(in distribution and almost surely).*

The first part was proved by Logan–Shepp [40] and Vershik–Kerov [60, 61], while the second part was deduced by Kerov [31] using the Markov correspondence. Vice versa, one can first prove the second part (see Biane [10] and 4.2.4, 4.2.5 below), and then deduce the first part.

**4.1.4.** The construction above arises in the representation theory of the symmetric group. Let us explain the minimum that we need below, and refer to [34] for more details. The left regular representation of the symmetric group $S_n$ is the space $\mathbb{C}[S_n]$ of linear combinations $\sum_{\pi \in S_n} a_\pi \pi$, equipped with the action of $S_n$ by left multiplication. It can be decomposed into a sum of irreducible representations so that the multiplicity of the representation corresponding to $\lambda^n$ is equal to $\dim \lambda^n$. This implies the second equality in (60).

The decomposition of an irreducible representation corresponding to $\lambda^n \vdash n$ into irreducible representations of $S_{n-1}$ contains exactly the representations corresponding to $\lambda^{n-1} \nearrow \lambda^n$, and the multiplicity of each of these is exactly one.

This implies the first equality in (60). Furthermore, it follows that an irreducible representation corresponding to $\lambda^n \vdash n$ is decomposed into one-dimensional subspaces corresponding to chains $\mathbf{T}$ of the form (59).

The arguments below also rely on the properties of the Jucys–Murphy elements $X_m \in \mathbb{C}[S_n]$, which are defined as sums of transpositions

$$X_m = (1\,m) + (2\,m) + \cdots + (m-1\,m).$$

Every $\mathbf{T}$ is invariant under all $X_m$, and $X_m \mid_{\mathbf{T}} = \mathrm{ct}(\lambda^m \setminus \lambda^{m-1})$ (see [47], where the Jucys–Murphy elements are used to reconstruct the representation theory of the symmetric group, and [10, 46]). This implies

$$\frac{1}{n!} \operatorname{tr} P(X_n) = \mathbb{E} \int P \, d\mu[\lambda^{n-1}]$$

for any polynomial $P$. The use of Jucys–Murphy elements to compute moments of the transition measure goes back to the work of Biane [10].

**4.1.5.** Define two Gaussian processes

$$\Delta_{\mathrm{part}}(x) = \sum_{k \geq 3} \frac{\sqrt{k-1}}{2} g_k \frac{2 T_k(x/2)}{\pi \sqrt{4 - x^2}}, \tag{63}$$

$$(-)\Delta_{\mathrm{part}}^{\mathfrak{M}}(x) = \sum_{k \geq 1} \frac{2 g_{k+2}}{\sqrt{k+1}} \frac{U_k(x/2)\sqrt{4 - x^2}}{2\pi}. \tag{64}$$

**Theorem** (Kerov [32]; Ivanov–Olshanski [25])**.**

$$\sqrt{n} \left\{ \mathscr{R}_{\sqrt{n}} \, \omega[\lambda^n] - \Omega_{\mathrm{LSVK}} \right\} dx \to \Delta_{\mathrm{part}}^{\mathfrak{M}}(x) dx$$

*and*

$$\sqrt{n} \left\{ d\mathscr{R}_{\sqrt{n}} \, \mu[\lambda^n](x) - d\rho_{\mathrm{s.c.}}(x) \right\} \to \Delta_{\mathrm{part}}(x) dx$$

*in distribution, as random functionals on* $\mathfrak{B}_{[-2,2]}$.

The first part was proved by Kerov [32] in 1993. A simplified proof, based on Kerov's notes, was published by Ivanov and Olshanski [25]. The second part was proved in [25], independently of Theorem 4.1.5 (although by a similar method). In view of Proposition 3.2.1, the second part implies the first part (and vice versa). In Section 4.2, we sketch a proof of the second part using a combinatorial approach which was used by Biane [10] and by Okounkov [46], and recently developed in [26] As pointed out in [25], $\Delta_{\mathrm{part}}$ is similar to $\Delta_{\mathrm{tr}}$ from 3.2.2. It is even more similar to the process from Proposition 4.1.6. The argument will highlight this similarity, and also explain the appearance of the factor $\sqrt{k-1}$ in (63) in place of $\sqrt{k}$ in (14).

**4.1.6.** To emphasize the similarity between random matrices and random partitions, we consider the following special Wigner matrix: $H = H^{\mathrm{unif}} = (H(i,j))$ with

$$H(i,i) = 0 , \quad H(i,j) \sim \mathrm{Unif}(S^1) \quad (i < j) .$$

This ensemble is particularly convenient due to the identities (66), see [56] for a survey of applications of relations of this kind and historical remarks. (For other ensembles of Wigner matrices (66) is no longer an identity, however, the difference between the left-hand side and the right-hand side is a small quantity, cf. [21].) For the ensemble $H^{\mathrm{unif}}$, the central limit theorem is stated as follows.

**Proposition.** *For* $H^{unif}$,

$$n \, d(\mathscr{R}_{\sqrt{n}}[\rho_n^{\mathrm{unif}}](x) - \rho_{\mathrm{s.c.}}(x)) \to \sum_{k \geq 3} \frac{\sqrt{k}}{2} g_k \frac{2 T_k(x/2) \, dx}{\pi \sqrt{4 - x^2}},$$

$$n \, (\mathscr{R}_{\sqrt{n}}[\varpi_n^{\mathrm{unif}}](x) - \Omega_{\mathrm{LSVK}}(x)) dx \to \sum_{k \geq 1} \frac{2\sqrt{k+2}}{k+1} g_{k+2} \frac{U_k(x/2)\sqrt{4 - x^2} \, dx}{2\pi}.$$

In the next section, we prove this proposition in parallel with Theorem 4.1.5.

## 4.2. Asymptotics of moments.

**4.2.1.** Consider the following sequence of polynomials:

$$P_{l,m}(x) = (m - 1)^{\frac{l}{2}} U_l \left( \frac{x}{2\sqrt{m-1}} \right) - (m - 1)^{\frac{l-2}{2}} U_{l-2} \left( \frac{x}{2\sqrt{m-1}} \right) \tag{65}$$

with the convention $U_{-2} \equiv U_{-1} \equiv 0$. For $p = (u_0, u_1, \dots, u_l)$, set

$$\widehat{\pi}(p; H) = H(u_0, u_1) H(u_1, u_2) \dots H(u_{l-1}, u_l).$$

Also let

$$\widehat{\mathscr{P}}_{l,n}(u, v) = \Big\{ (u_0, u_1, \dots, u_{l-1}, u_l) \in \{1, \dots, n\}^l \mid$$

$$u_0 = u, \, u_l = v, \, u_r \neq u_{r-1}, u_{r-2} \Big\}.$$

Graphically, we may represent a tuple in $\widehat{\mathscr{P}}_{l,n}(u, v)$ as a path of length $l$ from $u$ to $v$ which does not backtrack.

**Lemma** (e.g. [56]). *For any Hermitian $H$ with $|H(u,v)| = 1 - \delta_{uv}$ and any $u, v$,*

$$P_{l,n-1}(H_n)(u, v) = \sum_{p \in \widehat{\mathscr{P}}_{l,n}(u,v)} \widehat{\pi}(p; H), \tag{66}$$

*and consequently*

$$\mathbb{E} \prod_r P_{l_r,n-1}(H)(u_r, v_r) = \left\{ \underline{p} \in \prod_r \widehat{\mathscr{P}}_{l_r,n} \mid \prod_r \widehat{\pi}(p_r; H) \equiv 1 \right\} .$$

Graphically, the product is identically one if every edge is traversed forward the same number of times as backward.

**4.2.2.** Now we state (following [26]) the counterpart of (65) for random partitions. Let $p = (j_1, j_2, \ldots, j_l)$ be a sequence of numbers. Denote:

$$\pi_m(p) = (j_1\, m)(j_2\, m) \cdots (j_l\, m) .$$

Also let

$$\mathscr{P}_{l,m} = \left\{ p \in \{1, \cdots, m-1\}^l \mid \forall 1 \le r \le l-1\, j_r \ne j_{r+1} \right\}. \tag{67}$$

The following lemma can be checked by induction:

**Lemma** ([26, Lemma 4.1]).

$$P_{l,m-1}(X_m) = \sum_{p \in \mathscr{P}_{l,m}} \pi_m(p),$$

*and consequently*

$$\operatorname{tr} \prod_r P_{l_r,m_r-1}(X_{m_r}) = \# \left\{ \underline{p} \in \prod_r \mathscr{P}_{l_r,m_r} \mid \prod_r \pi_{m_r}(p_r) = 1 (= \operatorname{id}_{S_n}) \right\}. \tag{68}$$

**4.2.3.** Let us prove that Wigner's law holds in the mean:

$$\mathbb{E} \mathscr{R}_{\sqrt{n}}\, \rho_n^{\mathrm{unif}} = \mathbb{E} \mathscr{R}_{\sqrt{n}}\, \mu_n^{\mathrm{unif}} \to \rho_{\mathrm{s.c.}} . \tag{69}$$

We omit many details which can be found, for example, in [56, 2.4.1].

*Proof of Wigner's law in the mean.* The polynomials $U_k(x/2)$ are orthogonal with respect to $\rho_{\mathrm{s.c.}}$, therefore to prove (69) it suffices to show that

$$\lim_{n \to \infty} \mathbb{E} \int U_k(x/2) d\mathscr{R}_{\sqrt{n}}\, \rho_n^{\mathrm{unif}}(x)$$

$$= \lim_{n \to \infty} \mathbb{E} \int U_k(x/2) d\mathscr{R}_{\sqrt{n}}\, \mu_n^{\mathrm{unif}}(x) = 0, \quad k = 1, 2, \cdots,$$

which, by (65), is equivalent to

$$\lim_{n \to \infty} n^{-\frac{k}{2}} \mathbb{E} P_{k,n-1}(H)(n, n) = 0, \quad k \ge 1, \tag{70}$$

and we do this using Wigner's power-counting argument, as follows.

The quantity $\mathbb{E}P_{k,n}(H)(n,n)$ counts the number of paths in $\mathscr{P}_{k,n-1}(n,n)$ which satisfy the parity condition

$$\forall a \neq b \quad \#\{j,\, u_j = a,\, u_{j+1} = b\} = \#\{j,\, u_j = b,\, u_{j+1} = a\},$$

where we set $u_0 = u_k = n$. First, we divide $\mathscr{P}_{k,n-1}(n,n)$ into isomorphism classes: $p$ is isomorphic to $p'$ if $p = \sigma \circ p'$ for some permutation $\sigma \in S_n$. The number of isomorphism classes is bounded as $n \to \infty$, for every $k$. Second, to every class we associate the characteristic $\chi = V - E + 1$, where $V$ is the number of vertices and $E$ is the number of edges, and observe that $E \leq k/2$ and hence the contribution of a class is bounded by $C_k n^{\frac{k}{2}+\chi-1}$. Finally, a path can not be tree-like (such as $(u_0, \cdots, u_{10}) = (j_1, j_2, j_3, j_4, j_3, j_5, j_6, j_5, j_3, j_2, j_1))$, since trees have leaves at which the path backtracks (in the example in the previous parentheses, $u_2 = u_4 = j_3$, et cet.), hence $\chi \leq 1$. A somewhat more careful argument shows that $\chi \leq 0$ with equality for paths isomorphic to the one on [26, Figure 4, right], [56, Figure 3.1, middle], but this is not needed at the moment. Finally,

$$0 \leq n^{-\frac{k}{2}}\mathbb{E}P_{k,n-1}(H)(n,n) \leq C'_k n^{-1}.$$

This proves (69) and (70). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

**4.2.4.** In the same way, 4.2.2 can be used to prove that Theorem 4.1.3 holds in the mean:

$$\mathscr{R}_{\sqrt{n}}\,\mathbb{E}\mu[\lambda^n] \to \rho_c.$$

*Proof of Kerov's law in the mean.* It suffices to show that

$$\lim_{n\to\infty} \frac{1}{n^{k/2}}\frac{1}{n!}\,\operatorname{tr} P_{k,n-1}(X_n) = 0, \quad k = 1, 2, 3, \cdots \tag{71}$$

The argument, essentially due Biane (see [10]), is similar to 4.2.3. We write

$$\operatorname{tr} P_{k,n-1}(X_n) = \#\{p \in \mathscr{P}_{k,n} \mid \pi_n(p) = 1\}$$

and divide the solutions to the equation $\pi_n(p) = 1$ into isomorphism classes (by $S_{n-1}$-conjugation). Then we assign to every class the characteristic

$$\chi = \#\{\text{distinct indices in } p\} - \frac{k}{2} + 2.$$

See [26, 4.2.1] for a graphical interpretation. The irreducibility condition (67) rules out tree-like solutions such as

$$(j_1\, n)\,(j_2\, n)\,(j_3\, n)\,(j_3\, n)\,(j_2\, n)\,(j_4\, n)\,(j_4\, n)\,(j_1\, n) = 1$$

with $\chi = 2$, therefore $\chi \leq 1$ (in fact, $\chi \leq 0$, but this is not required at the moment), and this implies (71). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

**4.2.5.** To prove that Wigner's law and Kerov's law hold in distribution, we need to show that also the variance of

$$\int U_k(x/2)\, d\mathscr{R}_{\sqrt{n}}\rho_n^{\text{unif}} \quad \text{and} \quad \int U_k(x/2)\, d\mathscr{R}_{\sqrt{n}}\mu[\lambda^n]$$

tends to zero. This is accomplished via a combinatorial argument which is similar to that in 4.2.3 and 4.2.4: one shows that, for any isomorphism class of pairs contributing to the variance, the number of distinct vertices / indices is at most $k/2$. We omit the details.

**4.2.6.** In order to consider fluctuations, we introduce another family of polynomials

$$Q_{k,m}(x) = 2(m-1)^{\frac{k}{2}}T_k\left(\frac{x}{2\sqrt{m-1}}\right) - 2(m-1)^{\frac{k-2}{2}}T_{k-2}\left(\frac{x}{2\sqrt{m-1}}\right). \quad (72)$$

Then we have, for $k \geq 3$,

$$\operatorname{tr} Q_{k,n-1}(H) = \sum_{p\in\widehat{\mathscr{P}}^{\circ}_{k,n}} \widehat{\pi}(p,H) - \sum_{p\in\widehat{\mathscr{P}}^{\circ}_{k-2,n}} \widehat{\pi}(p,H), \quad (73)$$

where

$$\widehat{\mathscr{P}}^{\circ}_{k,n} = \left\{ p \in \widehat{\mathscr{P}}_{k,n} \mid u_{k-1} \neq u_1 \right\}$$

is the collection of cyclically non-backtracking paths. The relation (72) is derived from (65) using the identity

$$U_k - U_{k-2} = 2T_k.$$

For $k = 1, 2$, we have:

$$\operatorname{tr} 2\sqrt{n-2}\,T_1\left(\frac{H}{2\sqrt{n-2}}\right) = 0, \quad \operatorname{tr} 2(n-2)T_2\left(\frac{H}{2\sqrt{n-2}}\right) = -n(n-3).$$

Consequently, we obtain that

$$\operatorname{tr} 2(n-2)^{\frac{k}{2}}T_k\left(\frac{H}{2\sqrt{n-2}}\right) = \sum_{p\in\widehat{\mathscr{P}}^{\circ}_{k,n}} \widehat{\pi}(p,H) - n(n-3)\mathbb{1}_{n \text{ is even}}. \quad (74)$$

The relation (74) was first derived by Oren, Godel and Smilansky [48], using a trace formula (see further [49, 50] and 4.3.3 below). Related combinatorial interpretations of the traces of Chebyshev polynomials of the first kind appeared in the works of Anderson and Zeitouni [7] (who used generating functions), and of Kusalik, Mingo and Speicher [38] and Schenker and Schulz-Baldes [54].

*Proof of Proposition 4.1.6.* The proposition is derived from (74) using an approximation argument and the following two claims.

**Claim A.** *Let $\widehat{\mathscr{C}}_{k,n}$ be the sub-collection of paths in $\widehat{\mathscr{P}}^{\circ}_{k,n}$ in which all the vertices are distinct. Then*

$$(n-2)^{-\frac{k}{2}} \sum_{p \in \widehat{\mathscr{P}}^{\circ}_{k,n} \setminus \widehat{\mathscr{C}}_{k,n}} \widehat{\pi}(p, H) \to 0 \tag{75}$$

*in distribution.*

**Claim B.** *Let $g_3, g_4, \cdots$ be independent standard Gaussian variables. Then*

$$\left\{ (n-2)^{-\frac{k}{2}} \sum_{p \in \widehat{\mathscr{C}}_{k,n}} \widehat{\pi}(p, H) \right\}_{k \geq 3} \to \left\{ \sqrt{k} g_k \right\}_{k \geq 3} \tag{76}$$

*in distribution.*

Claim A is proved by power-counting as in the 4.2.3 above: the elements of $\widehat{\mathscr{C}}_{k,n}$ have $V = E$, whereas for all the other paths in $\widehat{\mathscr{P}}^{\circ}_{k,n}$ $V < E$. This implies that $\mathbb{E}[\text{LHS of}\,(75)]^2 \to 0$.

To prove Claim B, denote the expression inside the braces on the left-hand side of (76) by $S_{k,n}$; then (and this is the main combinatorial step in the proof)

$$\lim_{n \to \infty} \mathbb{E} \prod_{j=1}^{r} S_{k_j,n}$$

$$= \sum_{\text{pairings of } \{1,\cdots,r\}} \prod_{\text{pair } (j,j')} \# \left\{ \text{alignments of a } k_j\text{-cycle with a } k_{j'}\text{-cycle} \right\}$$

$$= \sum_{\text{pairings of } \{1,\cdots,r\}} \prod_{\text{pair } (j,j')} k_j \, \delta_{k_j,k_{j'}},$$

where the first equality is a consequence of power-counting. Claim B follows by the Wick rule. $\qquad\square$

**4.2.7.**

*Proof of Proposition 4.1.5.* Let us denote by $\text{move}_m$ any linear combination of permutations for which $m$ is not a fixed point. Then (for $k \geq 3$)

$$Q_{k,m-1}(X_m) = \sum_{p \in \mathscr{P}^{\circ}_{k,m}} \pi_m(p) + \text{move}_m,$$

where

$$\mathscr{P}_{k,m}^{\circ} = \left\{ p \in \mathscr{P}_{k,m} \mid u_k = u_1 \,,\, u_{k-1} \neq u_2 \right\}$$

is the collection of cyclically non-backtracking paths. In particular,

$$\mathrm{tr} \prod_{j=1}^{r} Q_{k_j, m_j - 1}(X_{m_j}) = \mathrm{tr} \prod_{j=1}^{r} \sum_{p \in \mathscr{P}_{k_j, m_j}^{\circ}} \pi_{m_j}(p),$$

if $m_1 < \cdots < m_r$. Consider the sub-sub-collection

$$\mathscr{C}_{k,m} = \{ (j_1, \cdots, j_k < m \mid j_k = j_1 \text{ and } j_1, \cdots, j_{k-1} \text{ are pairwise distinct} \}.$$

(Note that the cycles have $k-1$ vertices, as opposed to the $k$-long cycles of 4.2.6.) Then Claim 4.2.6.A takes the form

$$n^{-\frac{k}{2}} \left[ Q_{k,n-1}(X_n)|_{\mathbf{T}} - \sum_{p \in \mathscr{C}_{k,n}} \pi_n(p)|_{\mathbf{T}} \right] \longrightarrow 0,$$

where $\mathbf{T}$ is sampled from the Plancherel growth process, whereas in place of Claim 4.2.6.B one has

$$\left\{ \sum_{p \in \mathscr{C}_{k,n}} \pi_n(p)|_{\mathbf{T}} \right\}_{k \geq 3} \longrightarrow \left\{ \sqrt{k-1} \, g_k \right\}_{k \geq 3}.$$

Both claims are proved by evaluating moments, where we take $m_j = n - j + 1$. Then Proposition 4.1.5 follows from the claims by an approximation argument. $\square$

### 4.3. Some comments.

**4.3.1.** The counterpart of Theorem 3.2.3 (i.e., the special case of (55)) for the ensemble $H^{\mathrm{unif}}$ is stated as follows:

$$\sqrt{n} \, d(\mathscr{R}_{\sqrt{n}}[\mu_n^{\mathrm{unif}}](x) - \rho_{\mathrm{s.c.}}(x)) \xrightarrow{\mathrm{distr}} \sum_{k \geq 3} g_k U_k(x/2) \frac{\sqrt{4 - x^2}}{2\pi} \, dx \,, \qquad (77)$$

while (56) takes the form

$$\sqrt{n}(\mathscr{R}_{\sqrt{n}}[\omega_n^{\mathrm{unif}}](x) - \Omega_{\mathrm{LSVK}}(x)) \, dx \to$$

$$\sum_{k \geq 3} 2 g_k \left\{ \frac{U_{k-2}(x/2)}{k-1} - \frac{U_k(x/2)}{k+1} \right\} \frac{\sqrt{4 - x^2}}{2\pi} \, dx.$$

The coefficient "1" in front of $g_k$ in (77) is combinatorially interpreted as the number of ways to align two cycles of length $k$ with a marked vertex.

**4.3.2.** The parallelism between random matrices and random Young diagrams is somewhat more transparent if described via the matrix of transpositions

$$
\Gamma_n = \begin{pmatrix}
0 & (12) & (13) & \cdots & (1n) \\
(12) & 0 & (23) & \cdots & (2n) \\
& & \ddots & & \\
(1n) & (2n) & (3n) & \cdots & 0
\end{pmatrix}
$$

introduced (in a different gauge) by Biane [10]. The spectral properties of the restriction $\Gamma_n|_{\lambda^n}$ to $\lambda^n \otimes \mathbb{C}^n$, where $\lambda^n$ is an irreducible representation, are similar to those of the Jucys–Murphy elements; if $\lambda^n$ is randomly sampled from the Plancherel measure, the corresponding matrix $\Gamma_n|_{\lambda^n}$ can be considered as a counterpart to the random matrix $H_n$.

In particular, the use of $\Gamma_n$ in place of $X_n$ makes the argument in 4.2.7 even more similar to that in 4.2.6, and also introduces a conceptual simplification by allowing to work with central elements only.

**4.3.3.** The combinatorial constructions of Section 4 can be recast in terms of Ihara-type zeta-functions [9, 24]. For a Hermitian matrix $H$ such that $|H(u,v)| = 1 - \delta_{uv}$ (or, more suggestively, $H(u,v)H(v,u) = 1 - \delta_{uv}$), consider the zeta functions

$$
\zeta_{H_n}(u) = \prod_{l \geq 0} \prod_{p \in \widehat{\mathscr{P}}_{l,n}^{\circ}} (1 - \widehat{\pi}(p;H)u^l)^{-1}.
$$

A variant of the Bass determinantal formula [9] asserts that

$$
\zeta_{H_n}(u)^{-1} = (1 - u^2)^{\binom{n-1}{2}-1} \det(\mathbb{1} - uH_n + (n-1)u^2\mathbb{1}). \tag{78}
$$

Similarly, consider the $\mathbb{C}[S_n]$-valued function

$$
\zeta_n(u) = \prod_{l \geq 0} \prod_{p \in \mathscr{P}_{l,n}^{\circ}} (1 - \pi_n(p)u^l)^{-1},
$$

for which one has

$$
\zeta_n(u)^{-1} = (1 - u^2)^{\binom{n-1}{2}-1} \det_{n \times n}(\mathbb{1} - u\,\Gamma_n + (n-1)\,u^2\mathbb{1}).
$$

Both sides of this identity are central, and hence act as scalars on each irreducible representation $\lambda^n$ of $S_n$; thus we obtain:

$$
[\zeta_n(u)|_{\lambda^n}]^{-1} = (1 - u^2)^{\binom{n-1}{2}-1} \det_{n \times n}(\mathbb{1} - u\,\Gamma_n|_{\lambda^n} + (n-1)\,u^2\mathbb{1}). \tag{79}
$$

The logarithms of the relations (78) and (79) are trace formulæ relating the spectra of $H_n$ and $\Gamma_n$ with the quantities $\widehat{\pi}_n$ and $\pi_n$, respectively.

The trace formulæ thus obtained do not explicitly separate between the semi-circle and the corrections to it. Improved trace formulæ, in which such a separation is explicit, were obtained in the setting of $d$-regular graphs by Oren, Godel and Smilansky, see [48, 49] and further [50]. Their approach can be applied to the problems discussed here.

**4.3.4.** Chebyshev polynomials appear naturally in the combinatorial approach described in Section 4.2, and also in the trace formulæ mentioned in 4.3.3. Another approach in which their rôle is apparent was developed by Joyner and Smilansky [28, 29]; it relies on the study of the Fokker–Planck equation describing the evolution of the ensemble under random walk. We refer in particular to [29], where Gaussian fluctuations are analyzed.

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