Journal of Mathematical Physics, Analysis, Geometry
2017, vol. 13, No. 3, pp. 242–253
doi:10.15407/mag13.03.242

Notes on Ricci Solitons in f-Cosymplectic Manifolds

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Received November 18, 2015

The purpose of the article is to study an f-cosymplectic manifold M admitting Ricci solitons. Here we consider mainly two classes of Ricci solitons on f-cosymplectic manifolds. One is the class of contact Ricci solitons. The other is the class of gradient Ricci solitons, for which we give the local classifications of M. We also give some properties of f-cosymplectic manifolds.

Key words: contact Ricci soliton, gradient Ricci soliton, f-cosymplectic manifold, cosymplectic manifold, Einstein manifold.

Mathematical Subject Classification 2010: 53C25, 53D10.

1. Introduction

In contact geometry, an important class of almost contact manifolds that are almost Kenmotsu manifolds, was first introduced by Kenmotsu in [11]. Given an almost Kenmotsu structure (ϕ, ξ, η, g), we can get an almost α-Kenmotsu structure by a homothetic deformation

$$\phi' = \phi, \quad \eta' = \frac{1}{\alpha}\eta, \quad \xi' = \alpha\xi, \quad g' = \frac{1}{\alpha^2}$$

for some non-zero real constant α. Note that almost α-Kenmotsu structures are related to some special conformal deformations of almost cosymplectic structures [18].

The notion of almost cosymplectic manifolds was first given by Goldberg and Yano in [6]. Later Kim and Pak in [12] defined a new class of manifolds

The author is supported by the Science Foundation of China University of Petroleum-Beijing (No.2462015YQ0604) and partially supported by the Personnel Training and Academic Development Fund (2462015QZDX02).

called as almost α-cosymplectic manifolds by combining almost cosymplectic and almost α-Kenmotsu manifolds, where α is a real number. Recently, basing on Kim and Pak's work, Aktan et al. [1] considered a wide subclass of almost contact manifolds, which are called *almost f-cosymplectic manifolds*, defined by choosing a smooth function f in the conception of almost α-cosymplectic manifolds instead of any real number α.

In the following, we recall that a *Ricci soliton* (g, V) is a Riemannian metric g together with a vector field V that satisfies the equation

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} - \lambda g = 0, \tag{1.1}$$

where λ is a constant and V is a potential vector field. The Ricci soliton is said to be *shrinking, steady or expanding* depending on whether λ is positive, zero or negative, respectively. When the potential vector V is taken as the Reeb vector field on an almost contact metric manifold, it is called a *contact Ricci soliton*, and if $V = DF$, the gradient vector field of some function F on M, the Ricci soliton is called a *gradient Ricci soliton*. The Ricci soliton is important not only for studying the topology of manifolds, but also in the string theory. Compact Ricci solitons are the fixed points of the Ricci flow, $\frac{\partial}{\partial t}g = -2\,\mathrm{Ric}$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flows on compact manifolds.

The study of Ricci solitons has a long history, and a lot of conclusions were acquired, see [3,5,8,9,13,14,16] etc. In particular, we should note that Ghosh [7] studied a three-dimensional Kenmotsu manifold admitting a Ricci soliton and proved it to be of constant curvature -1.

As the generalization of Kenmotsu manifolds, in this paper, we study a normal almost f-cosymplectic manifold, which is said to be an *f-cosymplectic manifold*, and get the classifications of f-cosymplectic manifolds whose metrics are contact Ricci solitons and gradient Ricci solitons, respectively. In order to prove our theorems we need some basic concepts which are given in Section 2. Section 3 contains the main results and proofs.

2. Some Basic Concepts and Related Results

Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold. An *almost contact structure* on M is a triple (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a unit vector field, η is a one-form dual to ξ, satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0. \tag{2.1}$$

A smooth manifold with such a structure is called an *almost contact manifold*. It is well known that there exists a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$. From (2.1) and (2.2), it is easy to get

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X). \tag{2.3}$$

An almost contact structure (ϕ, ξ, η) is said to be *normal* if the Nijenhuis torsion

$$N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2d\eta(X, Y)\xi$$

vanishes for any vector fields X, Y on M.

Denote by ω the fundamental 2-form on M defined by $\omega(X, Y) := g(\phi X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. If η and ω are closed, then an almost contact structure is called *almost cosymplectic*, and it is said to be *cosymplectic* if in addition the almost contact structure is normal. An almost contact structure is said to be almost *α-Kenmotsu* if $d\eta = 0$ and $d\omega = 2\alpha\eta \wedge \omega$ for a non-zero constant α. More generally, if the constant α is any real number, then an almost contact structure is said to be *almost α-cosymplectic* [15]. Moreover, Aktan et al. [1] generalized the real number α to a smooth function f on M and defined an *almost f-cosymplectic manifold*, which is an almost contact metric manifold (M, ϕ, ξ, η, g) such that $d\omega = 2f\eta \wedge \omega$ and $d\eta = 0$ for a smooth function f satisfying $df \wedge \eta = 0$.

In addition, if the almost f-cosymplectic structure on M is normal, we say that M is an *f-cosymplectic manifold*. Obviously, if f is constant, then an f-cosymplectic manifold is either cosymplectic under condition $f = 0$, or α-Kenmotsu ($\alpha = f \neq 0$). Furthermore, there exists a distribution $\mathcal{D}$ of an f-cosymplectic manifold defined by $\mathcal{D} = \ker \eta$, which is integrable since $d\eta = 0$.

Besides, for an almost contact manifold (M, ϕ, ξ, η, g), we denote $h := \frac{1}{2}\mathcal{L}_\xi \phi$, which is a self-dual operator. Since an f-cosymplectic manifold is normal, $h = 0$. Therefore, in virtue of [1, Proposition 9, Proposition 10], we know that for a $(2n+1)$-dimensional f-cosymplectic manifold the following identities are valid:

$$\nabla_X \xi = -f\phi^2 X, \tag{2.4}$$

$$Q\xi = -2n\tilde{f}\xi, \tag{2.5}$$

$$R(X, Y)\xi = \tilde{f}[\eta(X)Y - \eta(Y)X], \tag{2.6}$$

where ∇ and Q denote respectively the Levi–Civita connection and the Ricci operator of M, and $\tilde{f} \triangleq \xi(f) + f^2$.

Proposition 2.1. *For any f-cosymplectic manifold, if $\xi(\tilde{f}) = 0$, then $\tilde{f} =$* const.

Proof. Differentiating (2.6) along any vector field Z, we have

$$\begin{aligned}(\nabla_Z R)(X, Y)\xi &= \nabla_Z(R(X, Y)\xi) - R(\nabla_Z X, Y)\xi - R(X, \nabla_Z Y)\xi - R(X, Y)\nabla_Z \xi \\ &= Z(\tilde{f})[\eta(X)Y - \eta(Y)X] + \tilde{f}f[g(X, Z)Y - g(Y, Z)X]\end{aligned}$$

$$- fR(X,Y)Z.$$

Then, using the second Bianchi identity

$$(\nabla_Z R)(X,Y)\xi + (\nabla_X R)(Y,Z)\xi + (\nabla_Y R)(Z,X)\xi = 0,$$

we have

$$\begin{aligned}&[Y(\tilde{f})\eta(Z) - Z(\tilde{f})\eta(Y)]X + [Z(\tilde{f})\eta(X) - X(\tilde{f})\eta(Z)]Y \\ &\quad + [X(\tilde{f})\eta(Y) - Y(\tilde{f})\eta(X)]Z - f[R(X,Y)Z + R(Y,Z)X + R(Z,X)Y] = 0.\end{aligned}$$

By taking $Z = \xi$ and using (2.6), we know

$$\xi(\tilde{f})[\eta(Y)X - \eta(X)Y] - X(\tilde{f})\phi^2 Y + Y(\tilde{f})\phi^2 X = 0. \tag{2.7}$$

If we assume $\xi(\tilde{f}) = 0$, then we can obtain $X(\tilde{f}) = 0$ for every vector field X by taking the inner product of (2.7) with Y, putting $Y = e_i$ and summing over i in the resulting equation (where $\{e_i\}$ is the local orthonormal frame of M). □

Obviously, taking into account $df \wedge \eta = 0$, it deduces immediately the following corollary.

Corollary 2.2. *An f-cosymplectic manifold is an α-cosymplectic manifold if f vanishes along ξ.*

Proposition 2.3. *A compact f-cosymplectic manifold M^{2n+1} with $\xi(\tilde{f}) = 0$ is α-cosymplectic. In particular, if $\tilde{f} = 0$, then M is cosymplecitc.*

Proof. As $\xi(\tilde{f}) = \xi(\xi(f)) + 2f\xi(f) = 0$, we obtain $\xi(\xi(f)) = -2f\xi(f)$. On the other hand, we know that f satisfies $df \wedge \eta = 0$, which means that $Df = \xi(f)\xi$, where D is the gradient operator with respect to g. For every field X, it follows from (2.4) that

$$\nabla_X Df = X(\xi(f))\xi + \xi(f)\nabla_X \xi = X(\xi(f))\xi - f\xi(f)\phi^2 X.$$

Since $\nabla_\xi \xi = 0$, for every point $p \in M$, we can take a locally orthonormal basis $\{e_i\}$ of T_pM such that $e_{2n+1} = \xi$ and $\nabla_{e_i} e_i = 0$. Therefore,

$$\Delta f = \sum_i g\left(\nabla_{e_i} Df, e_i\right) = \xi(\xi(f)) + 2nf\xi(f) = (1-n)\xi(\xi(f)), \tag{2.8}$$

where Δ is the Laplace operator. Since $e_i(f) = g(Df, e_i) = 0$ for $i = 1, \ldots, 2n$, we find $\Delta f = \sum_i e_i(e_i(f)) = \xi(\xi(f))$. Hence, it follows from (2.8) that $\xi(\xi(f)) = 0$, which shows that $\Delta f = 0$, i.e., f is constant. If $\tilde{f} = 0$, i.e., $0 = \xi(f) + f^2 = f^2$, then $f = 0$. □

Remark 2.4. In [2], Blair proved that a cosymplectic manifold is locally the product of a Kähler manifold and an interval or unit circle S^1.

Moreover, for the three-dimensional case, we have

Lemma 2.5. *For a three-dimensional f-cosymplectic manifold M^3, we have*

$$QY = \left(-3\tilde{f} - \frac{R}{2}\right)\eta(Y)\xi + \left(\tilde{f} + \frac{R}{2}\right)Y, \tag{2.9}$$

where R is the scalar curvature of M.

Proof. It is well known that the curvature tensor of any three-dimensional Riemannian manifold is written as

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + \mathrm{Ric}(Y,Z)X \\ - \mathrm{Ric}(X,Z)Y - \frac{R}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{2.10}$$

Putting $Z = \xi$ and using (2.5), (2.6), we have

$$\left(\tilde{f} + \frac{R}{2}\right)(\eta(Y)X - \eta(X)Y) = \eta(Y)QX - \eta(X)QY.$$

Moreover, by taking $X = \xi$ and using (2.6) again, we obtain (2.9). □

3. Main Results and Proofs

In this section, we mainly discuss two classes of Ricci solitons, i.e., contact Ricci solitons and gradient Ricci solitons in f-cosymplectic manifolds. At first, for a general Ricci soliton we have the following lemma, which was shown by Cho.

Lemma 3.1 ([4, Lemma 3.1])**.** *If (g, V) is a Ricci soliton of a Riemannian manifold, then we have*

$$\frac{1}{2}||\mathcal{L}_V g||^2 = V(R) + 2\mathrm{div}(\lambda V - QV),$$

where R denotes the scalar curvature.

Theorem 3.2. *If an f-cosymplectic manifold M^{2n+1} admits a contact Ricci soliton, then M^{2n+1} is locally isometric to the product of a line and a Ricci-flat Kähler (Calabi–Yau) manifold.*

Proof. In view of (2.4), we have

$$(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2f[g(X,Y) - \eta(X)\eta(Y)].$$

Therefore it implies from the Ricci equation (1.1) with $V = \xi$ that

$$\mathrm{Ric}(X,Y) = (\lambda - f)g(X,Y) + f\eta(X)\eta(Y). \tag{3.1}$$

By (3.1), the Ricci operator Q is provided

$$QX = (\lambda - f)X + f\eta(X)\xi$$

for any vector field X on M. Thus,

$$Q\xi = \lambda\xi, \tag{3.2}$$
$$R = (2n+1)\lambda - 2nf. \tag{3.3}$$

By Lemma 3.1, (3.2) and (3.3), we find that

$$\frac{1}{2}||\mathcal{L}_\xi g||^2 = -2n\xi(f). \tag{3.4}$$

Since $(\mathcal{L}_\xi g)(X,Y) = 2fg(\phi X, \phi Y)$ and $\tilde{f} = -\frac{\lambda}{2n}$ is constant that follows from a comparison of (2.5) and (3.2), a straightforward computation implies $f^2 =$ const, i.e., f is constant. Hence ξ is Killing by (3.4). Moreover, we get $f = 0$ from $(\mathcal{L}_\xi g)(X,Y) = 2fg(\phi X, \phi Y)$. Namely, M is cosymplectic. Further we have $\mathrm{Ric} = 0$ since $\lambda = -2n\tilde{f} = 0$. Thus we complete the proof of our result. □

In view of the above proof, we get immediately the following corollary.

Corollary 3.3. *A contact Ricci soliton in an f-cosymplectic manifold is steady.*

In the following, we will assume that an f-cosymplectic manifold M^{2n+1} admits a gradient Ricci soliton and the function f satisfies $\xi(\tilde{f}) = 0$.

Theorem 3.4. *Let M^{2n+1} be an f-cosymplectic manifold with a gradient Ricci soliton. If $\xi(\tilde{f}) = 0$, then one of the following statements holds:*

1. *M is locally the product of a Kähler manifold and an interval or unit circle S^1.*
2. *M is Einstein.*

In order to prove the theorem, we first prove

Lemma 3.5. *Let M^3 be a three-dimensional f-cosymplectic manifold with a Ricci soliton. Then the following equation holds:*

$$2\xi(\tilde{f}) + \frac{\xi(R)}{2} + 2\left(3\tilde{f} + \frac{R}{2}\right)f = 0. \tag{3.5}$$

Proof. By Lemma 2.5, we compute

$$\begin{aligned}(\nabla_X \operatorname{Ric})(Y,Z) &= \left[-3X(\tilde{f}) - \frac{X(R)}{2}\right]\eta(Y)\eta(Z) + \left[-3\tilde{f} - \frac{R}{2}\right](\nabla_X\eta)(Y)\eta(Z) \\ &\quad + \left[-3\tilde{f} - \frac{R}{2}\right]\eta(Y)(\nabla_X\eta)(Z) + \left[X(\tilde{f}) + \frac{X(R)}{2}\right]g(Y,Z) \\ &= \left[-3X(\tilde{f}) - \frac{X(R)}{2}\right]\eta(Y)\eta(Z) + \left[-3\tilde{f} - \frac{R}{2}\right]fg(\phi X,\phi Y)\eta(Z) \\ &\quad + \left[-3\tilde{f} - \frac{R}{2}\right]f\eta(Y)g(\phi X,\phi Z) + \left[X(\tilde{f}) + \frac{X(R)}{2}\right]g(Y,Z). \end{aligned} \tag{3.6}$$

Notice that for every vector Z, the following relation holds:

$$\sum_{i=1}^{3}[(\nabla_Z \operatorname{Ric})(e_i,e_i) - 2(\nabla_{e_i} \operatorname{Ric})(e_i,Z)] = 0, \tag{3.7}$$

which follows from formulas (8) and (9) of [7], where $\{e_1,,e_2,e_3=\xi\}$ is a local orthonormal frame of M.

Making use of (3.6), we obtain from (3.7) that

$$\left[-3\xi(\tilde{f}) - \frac{\xi(R)}{2}\right]\eta(Z) + 2\left(-3\tilde{f} - \frac{R}{2}\right)f\eta(Z) + Z(\tilde{f}) = 0.$$

Putting $Z=\xi$ in the above formula gives (3.5). □

Proof of Theorem 3.4. By Proposition 2.1, $\tilde{f} = \text{const}$. It is clear that the Ricci soliton equation (1.1) with $V = DF$ for some smooth function F implies

$$\nabla_Y DF = -QY + \lambda Y. \tag{3.8}$$

Therefore we have $R(X,Y)DF = (\nabla_Y Q)X - (\nabla_X Q)Y$. Putting $Y=\xi$ further gives

$$R(X,\xi)DF = (\nabla_\xi Q)X - (\nabla_X Q)\xi. \tag{3.9}$$

On the other hand, from (2.6) and the Bianchi identity, we have

$$R(X,\xi)Y = \tilde{f}[g(X,Y)\xi - \eta(Y)X]. \tag{3.10}$$

Replacing Y by DF in (3.10) and comparing with (3.9), we get

$$(\nabla_\xi Q)X - (\nabla_X Q)\xi = \tilde{f}[X(F)\xi - \xi(F)X]. \tag{3.11}$$

Taking the inner product of the previous equation with ξ and using (2.5), we arrive at

$$\tilde{f}[X(F) - \xi(F)\eta(X)] = 0. \tag{3.12}$$

Then we divide the proof into the following cases.

Case I: $\tilde{f} = 0$ and $n > 1$. That is, if $\xi(f) = -f^2$, then $Df = -f^2\xi$. If $f \not\equiv 0$, then there is an open neighborhood $\mathcal{U}$ such $f|_{\mathcal{U}} \neq 0$. Thus in this case $\xi = -\frac{Df}{f^2} = D\left(\frac{1}{f}\right)$. Since $\Delta f = 0$ (see the proof of Proposition 2.3),

$$0 = \Delta\left(f\frac{1}{f}\right) = \frac{1}{f}\Delta f + 2g\left(Df, D\left(\frac{1}{f}\right)\right) + f\Delta\frac{1}{f} = 2\xi(f) + f\operatorname{div}\xi.$$

From (2.4), we know that $\operatorname{div}\xi = 2nf$. When $n > 1$, substituting this into the previous equation implies $f = 0$, which leads to a contradiction. Hence $f \equiv 0$, that is, M is cosymplectic.

Case II: $\tilde{f} \neq 0$. By (3.12), the following identity is obvious:

$$DF = \xi(F)\xi. \tag{3.13}$$

Substituting this into (3.8) and using (2.4), we get

$$Y(\xi(F))\xi - f\xi(F)\phi^2 Y = -QY + \lambda Y. \tag{3.14}$$

By taking an inner product with ξ and using (2.5), we further find

$$Y(\xi(F)) = (2n\tilde{f} + \lambda)\eta(Y). \tag{3.15}$$

Now taking (3.15) into (3.14) implies that for every vector X,

$$\lambda g(X,Y) - \operatorname{Ric}(X,Y) = (2n\tilde{f} + \lambda)\eta(X)\eta(Y) + f\xi(F)g(\phi X, \phi Y). \tag{3.16}$$

Moreover, we derive from (3.16) that the scalar curvature

$$R = 2n(-\tilde{f} + \lambda - f\xi(F)). \tag{3.17}$$

On the other hand, using (3.13) and (2.5), we have

$$\operatorname{Ric}(X, DF) = \xi(F)g(QX, \xi) = -2n\tilde{f}\eta(X)\xi(F). \tag{3.18}$$

It is well known that for any vector field X on M,

$$g(DR, X) = 2\operatorname{Ric}(DF, X), \tag{3.19}$$

which can be found in [10]. Applying (3.18) and (3.17) to this identity, we have

$$X(f)\xi(F) + (2n\tilde{f} + \lambda)\eta(X) = 2f\xi(F)\eta(X). \tag{3.20}$$

Substituting $X = \xi$ into (3.20), we get

$$(\xi(f) - 2\tilde{f})\xi(F) + f(2n\tilde{f} + \lambda) = 0. \tag{3.21}$$

Differentiating (3.21) along ξ, we obtain from (3.15) that

$$\xi(\xi(f))\xi(F) + 2(\xi(f) - \tilde{f})(2n\tilde{f} + \lambda) = 0. \tag{3.22}$$

Since $\xi(\tilde{f}) = 0$, we have $\xi(\xi(f)) = -2f\xi(f)$. Substituting this into (3.22) yields

$$f\xi(f)\xi(F) + f^2(2n\tilde{f} + \lambda) = 0. \tag{3.23}$$

Differentiating the above formula again along ξ, we obtain

$$\left(\xi(f)^2 - 2f^2\xi(f)\right)\xi(F) + 3f\xi(f)(2n\tilde{f} + \lambda) = 0.$$

Applying (3.21) to this equation implies

$$\left(\xi(f) + 4f^2\right)\xi(f)\xi(F) = 0.$$

Now, if $\xi(f) + 4f^2 = 0$ on some neighborhood $\mathcal{O}$ of $p \in M$, then $3f^2 = -\tilde{f}$ is constant, i.e., f is constant on $\mathcal{O}$. Further, we know $f = 0$, which implies $\tilde{f} = 0$ on $\mathcal{O}$. It is a contradiction with the assumption $\tilde{f} \neq 0$. Therefore $\xi(f)\xi(F) = 0$, and it follows from (3.23) that

$$f^2(2n\tilde{f} + \lambda) = 0.$$

If $2n\tilde{f} + \lambda = 0$, then it deduces from (3.21) that $(\xi(f) - 2\tilde{f})\xi(F) = 0$, i.e., $(\xi(f) + 2f^2)\xi(F) = 0$. As before, we know $\xi(F) = 0$. It shows that DF is identically zero because of (3.13). Thus M is Einstein. Moreover, from (3.17) we get $R = 2n(\lambda - \tilde{f})$.

If $2n\tilde{f} + \lambda \neq 0$, we have $f \equiv 0$, that is, M is cosymplectic.

In particular, when $n = 1$, we know that $\lambda + 2\tilde{f} = 0$ and $R = 2(\lambda - \tilde{f})$. Moreover, using Lemma 2.5, we obtain

$$QY = (-2\tilde{f} - \lambda)\eta(Y)\xi + \lambda Y = -2\tilde{f}Y.$$

Case III: $\tilde{f} = 0$ and $n = 1$. If $f \equiv 0$, then M is cosymplectic. Further we will assume $f \neq 0$ on some neighborhood. By (2.5), we have $Q\xi = 0$ when $\tilde{f} = 0$. Because $\tilde{f}$ is constant, we obtain $\xi(R) = 0$ from (3.19). That means $R = 0$ by (3.5). Moreover, in view of Lemma 2.5, we get $Q = 0$.

Summarizing the above discussion, we have proved that either $f \equiv 0$ or $QY = -2\tilde{f}Y$. Thus, by Remark 2.4, we complete the proof of the theorem. □

Since an α-cosymplectic manifold is actually an f-cosymplectic manifold such that f is constant, we obtain the corollary below from Theorem 3.4.

Corollary 3.6. *Let M^{2n+1} be an α-cosymplectic manifold with a gradient Ricci soliton. Then M is either locally the product of a Kähler manifold and an interval or unit circle S^1, or Einstein.*

We note that Perelman in [14] proved that on a compact Riemannian manifold a Ricci soliton is always a gradient Ricci soliton. Thus the following corollary is clear from Theorem 3.4.

Corollary 3.7. *Let M^{2n+1} be a compact f-cosymplectic manifold with a Ricci soliton. If $\xi(\tilde{f}) = 0$, then M is either locally the product of a Kähler manifold and an interval or unit circle S^1, or Einstein.*

When $n = 1$, we have

Corollary 3.8. *Let M^3 be a three-dimensional α-cosymplectic manifold with a Ricci soliton. If $\xi(R) = 0$, then M is either locally the product of a Kähler manifold and an interval or unit circle S^1, or Einstein.*

Proof. Since an α-cosymplectic manifold is an f-cosymplectic manifold, where $f = \alpha$ is constant, we have $\tilde{f} = \alpha^2$. Since $\xi(R) = 0$, making use of (3.5), we obtain

$$\left(3\alpha^2 + \frac{R}{2}\right)\alpha = 0.$$

Therefore, $\alpha = 0$ or $R = -6\alpha^2$. We complete the proof of Lemma 2.5. □

Finally we give an example of an f-cosymplectic manifold satisfying $\xi(\tilde{f}) = 0$.

Example 3.9. As in the example of three dimension in [1], we also consider a three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where x, y, z are the standard coordinates in $\mathbb{R}^3$. On M, we define the Riemannian metric

$$g = \frac{1}{e^{2\theta(z)}}(dx \otimes dx + dy \otimes dy) + dz \otimes dz,$$

where $\theta(z)$ is a smooth function on M.

Clearly, the vector fields

$$e_1 = e^{\theta(z)}\frac{\partial}{\partial x}, \quad e_2 = e^{\theta(z)}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent with respect to g at each point of M. Also, we see that $g(e_i, e_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for every field X and ϕ be the $(1, 1)$-tensor field defined by $\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Hence, it is easy to get that $\eta = dz$, $\omega(e_1, e_2) = g(\phi(e_1), e_2) = 1$ and $\omega(e_1, e_3) = \omega(e_2, e_3) = 0$.

Furthermore, a straightforward computation gives the brackets of the vector fields e_1, e_2, e_3:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\theta'(z)e_1, \quad [e_2, e_3] = -\theta'(z)e_2.$$

Consequently, the the Nijenhuis torsion of ϕ is zero, i.e., M is normal.

On the other hand, as in [1], it easily follows

$$\omega = \frac{1}{e^{2\theta(z)}} dx \wedge dy$$

and

$$d\omega = -2\theta'(z)e^{-2\theta(z)} dx \wedge dy \wedge dz = 2\theta'(z)\omega \wedge \eta.$$

Therefore, M is an f-cosymplectic manifold with $f(x, y, z) = \theta'(z)$.

In order that $\xi(\tilde{f}) = 0$, i.e.,

$$\theta'''(z) + 2\theta'(z)\theta''(z) = 0,$$

we need $\theta''(z) + [\theta'(z)]^2 = c$ for a constant c. In view of theory of ODE, the above equation is solvable.

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