Functional Materials 13, No.1 (2006)

Investigation of the material fragmentation model with the uniform scale subdivision energy distribution

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Received April 20, 2005

Using the probabilistic Kolmogorov approach for the description of the material fragmentation process, it is shown that in the typical physical situation when the intensity energy inflow in the system is constant the Kolmogorov equation must be temporally inhomogeneous. In this case, for the special model with the uniform scale energy distribution expended for the subdivision, the limit Kolmogorov distribution law for fragment sizes is proved.

Используя вероятностный подход А.Н.Колмогорова для описания процесса фрагментации материала, показано, что в типичной физической ситуации, при постоянной интенсивности накачки энергии в систему уравнение Колмогорова должно быть неоднородным по времени. Для этого случая, в специальной модели с равномерным по масштабам распределением энергии, расходуемой на дробление, доказан предельный закон Колмогорова для распределения вероятностей размеров фрагментов.

In the classical paper [1], it have been proposed the probabilistic approach to investigation of som complicated processes which have got later the common name the *fragmentation processes*. In genera case, these temporal physical processes connected with subdivision of large system to some single-typ weakly connected subsystems having sizes less than the initial system size. This subdivision is fulfille due to the energy pumping from outside. Usually, such processes consist of subdivisions of large soli sample to pieces with more and more small sizes with different random geometric forms and values o parameters characterizing them. In connection with the complication of mathematical description of suc processes, it have been propose by A.N.Kolmogorov to describe the fragmentation processes by only on parameter r which characterizes the size of each fragment in average. Its value is random. Of course, suc an approach is not suitable in all cases. However, in many cases, it succeed to describe correctly, from th physical point of view, the time evolution of the distribution function $F(r,t)$ on sizes r. In the work [1], i frameworks of some assumptions, it has been obtained also the limit logarithmically normal distributio at $t \to \infty$ for the function $F(r,t)$. At the obtaining of this limit distribution, it is important that i have been assumed that the conditional probability of breakdown to parts with different sizes at th subdivision of one fragment during one evolution step does not depend on the time moment (it has bee assumed that the time is discrete). In the work [2] it was pointed out that full volume of all fragments i usually conserved in natural fragmentation processes. Moreover, it is important that the energy pumpin into the fragmentation system is fulfilled with an intensity which determines really the time dependenc of the conditional transfer probability. From this point of view, the case analyzed by Kolmogorov is no physical since, for its realization, the energy pumping is done with the exponential intensity increasing

However, using the constant energy pumping intensity and another problem formulation, it has been obtained the limit distribution in the work [3], which is differed of the Kolmogorov law.

In this work, we demonstrate in frameworks of the Kolmogorov approach [1] that, at first, the fragmentation models have the time depending conditional probability distribution for the fragment subdivision at one evolution step, if one takes into account the constancy of the energy pumping intensity and, at second, we demonstrate that there exists the physical situation when it is possible the appearance of the logarithmically normal limit distribution if one takes into account the mentioned special time dependence.

Basic equation. As in the paper [1], we assume that the time is discrete. Let us designate by $N(r,t)$ the average number of fragments in the time moment t such that their sizes are less then r and the number $N(t)$ is the average value of the corresponding total number of fragments at the same moment. Further, let $S(r,r';t) = S(r/r';t)$ be the mathematical expectation of the fragment number with sizes which are not exceed the value r and formed at subdivision at the time moment t by one fixed fragment with the size r'. Thus, it is assumed that the fragmentation process is characterized by the scale invariance.

The number of fragments with sizes being less than r at the time $t+1$ is equal to the sum of the total number of splinters of all fragments having sizes which are less then r at the previous evolution step and those splinters with sizes being less than r which are formed by the subdivision of large fragments. Let us assume that, for such average physical variables, the next balance relation takes place

$$N(r,t+1) = \int_0^{r-0} S(1;t)\,dN(r',t) + \int_{r-0}^{\infty} S(r/r';t)\,dN(r',t)\ , \tag{1}$$

from the general physical concepts and the definition of functions $N(r,t)$, $S(r,r';t)$. Here, the inequality $S(1+0,t) < 1$ is fulfilled at any physical reasonable choice of the function $S(k,t)$. It is possible to consider the relation (1) as the equation that connects the average values of above described random physical variables in the "average field"approximation. Thus, on its plan, the form of the equation (1) does not depend strongly, in definite frames, on the concrete statistical model using for the description of the fragmentation process. Nevertheless, it is necessary to point out that there exists the more important fact using for the postulation of this equation. It is its Markov property, i.e. the size distribution at each next moment depends directly only on the distribution at the previous one and, moreover, it depends by linear way.

Integrating by parts in Eq.(1), we obtain

$$N(r,t+1) = S(1;t)\,N(r',t)\big|_0^{r-0} + S(r/r';t)\,N(r',t)\big|_{r-0}^{\infty} -$$
$$- \int_{r-0}^{\infty} N(r',t)\,dS(r/r';t)\ .$$

Here, the terms being outside of the integral disappear since it is necessary to assume naturally that $S(0,t) = 0$, $N(0,t) = 0$. Introducing the integration variable $k = r/r'$, we obtain the fundamental evolution equation in Kolmogorov's form

$$N(r,t+1) = \int_0^{1+0} N(r/k,t)\,dS(k,t)\ . \tag{2}$$

Difference of this equation from the analogous one in the cited work consists of the possibility of nonzero probability for the fact that the fragment breakdown is absent during one evolution step. Correspondingly, the function $S(k;t)$ may have the jump at the point $k=1$ which is necessary to account in the integral on all system states that, in turn, brings to the Stiltjes integral in Eq.(2) with the jump on the upper bound of integration.

Conservation laws. Since $N(t) = N(\infty, t)$, the changing of the total fragment number at one evolution step is defined by the recurrent relation

$$N(t+1) = \int_0^{1+0} N(\infty, t)\, dS(k;t) = N(t)\, S(1+0, t)\,. \qquad (3)$$

More important difference of our buildings, in comparing with the classical work [1], together the pointed out evolution equation modification, consists of the evident account of the conservation law of total fragment volume and also the fact that the energy pumping in the system is expended, in general, to the formation of new surfaces of their separation. Usually, in the fragmentation processes, there exists the limitation on the energy inflow to the system. This brings to the slowing of the fragmentation process when the fragment number grows. Consequently, the function $S(k,t)$ defining the system evolution depends on time t. This may lead to the appearance of the final probability distribution for fragment sizes which is not connected in any way with the central limit theorem in the probability theory (this fact is signed in the work [1]). We use further the assumption that the intensity of energy pumping is constant. Together with the volume conservation law, it gives us the additional limitation for the function $S(k,t)$. We take into account the volume conservation law, in frameworks of using approach, by the following relation which has been proposed in [2],

$$V = \int_0^\infty r^3 dN(r,t) = \text{const}\,.$$

Than, using the equation (2), we obtain

$$V = \int_0^\infty r^3 dN(r, t+1) = \int_0^\infty r^3 d\left(\int_0^{1+0} N(r/k, t)\, dS(k;t)\right) =$$

$$= \int_0^{1+0} dS(k;t) \int_0^\infty r^3 dN(r/k, t) = \int_0^{1+0} k^3 dS(k;t) \int_0^\infty r^3 dN(r,t) =$$

$$= V \int_0^{1+0} k^3 dS(k;t)\,.$$

Thus, the requirement of volume conservation brings to the limitation

$$\int_0^{1+0} k^3 dS(k;t) = 1\,. \qquad (4)$$

At the obtaining of limitation connected with the constancy of the energy pumping intensity, we consider as in the work [2] that all of energy is expended for the increasing of the total surface of all fragments. Thus, the constancy of energy pumping intensity brings to the fact that the total surface area $\Sigma(t)$ defined by analogous way as the total volume

$$\Sigma(t) = \int_0^\infty r^2 dN(r,t)\,,$$

is changed in time by linear way

$$\Sigma(t) = \Sigma_0 + \sigma t\,, \quad \sigma = \text{const}\,.$$

Consequently, using the equation (2) again, we find

$$\Sigma(t+1) = \int_0^\infty r^2 dN(r, t+1) = \int_0^\infty r^2 d\left(\int_0^{1+0} N(r/k, t)\, dS(k,t)\right) =$$

$$= \Sigma(t) \int_0^{1+0} k^2 dS(k,t)$$

that is

$$\Sigma_0 + \sigma(t+1) = (\Sigma_0 + \sigma t) \int_0^{1+0} k^2 dS(k,t) \,,$$

$$(\Sigma_0 + \sigma t) \left(\int_0^{1+0} k^2 dS(k,t) - 1 \right) = \sigma \,.$$

One may notice that the equation (4) gives the inequality

$$\int_0^{1+0} k^2 dS(k,t) > 1 \,.$$

Then, from the obtained equation, it follows the condition of the energy pumping constancy in the next form

$$\int_0^{1+0} k^2 dS(k,t) = 1 + (t + \Sigma_0/\sigma)^{-1} \,. \tag{5}$$

General analysis of fragmentation process. As one can viewed from Eq.(2), the function $N(r,t)$ at $t > 0$ is determined from the initial distribution $N(r,0)$ by consecutive iterations,

$$N(r,t) = \int_0^1 \left(\int_0^1 \left(\ldots \left(\int_0^1 N(r/k_0 k_1 \ldots k_{t-1}\,, 0)\, dS(k_0, 0) \right) \times \right.\right.$$

$$\left.\left. \times \ldots \right) dS(k_{t-2}, t-2) \right) dS(k_{t-1}, t-1) \,. \tag{6}$$

Let us introduce the normalized functions

$$F(r,t) = N(r,t)/N(t) \,, \quad P(k,t) = S(k,t)/S(1,t) \,.$$

They satisfy the conditions $F(\infty,t) = 1$, $F(0,t) = 0$, $P(1+0,t) = 1$, $P(0,t) = 0$. Therefore, one may consider them as the probability distribution functions of some "effective"random values which we designate $\tilde{r}_t$, $\tilde{k}_t$, $t = 0, 1, ...$ correspondingly. In terms of these functions, the equation (2) is represented by the following way

$$F(r,t+1) = \int_0^{1+0} F(r/k,t)\, dP(k,t) \,. \tag{7}$$

Further, following the work [1], we go to logarithmic variable $y = \ln\, k$. Let us designate the distribution functions

$$G(x,t) = F(r_0 e^x, t) \,, \quad Q(y,t) = P(e^y, t)$$

expressed by this new variable. In terms of these functions, the equation (7) turns into the equation with the difference kernel,

$$G(x,t+1) = \int_{-\infty}^{+0} G(x-y,t)\, dQ(y,t) \,. \tag{8}$$

Correspondingly, the time changing of the function $G(x,t)$ is represented by the convolution sequence,

$$G(x,t) = \int_{-\infty}^{+0} \left(\int_{-\infty}^{+0} \left(\dots \left(\int_{-\infty}^{+0} G\left(x - (y_0 + y_1 + \dots + y_{t-1}), 0\right) dQ\left(y_0, 0\right) \right) \times \right.\right.$$

$$\left.\left. \times \dots \right) dQ\left(y_{t-2}, t-2\right) \right) dQ\left(y_{t-1}, t-1\right) ,$$

i.e. $G(x,t)$ is the distribution function of the sum

$$\tilde{x}_t = \tilde{x}_0 + \tilde{y}_0 + \tilde{y}_1 + \dots + \tilde{y}_{t-1} , \quad (9)$$

where $\tilde{x}_t = \ln(\tilde{r}_t/r_0)$. Characteristic functions $\bar{G}$, $\bar{Q}$ of the distribution functions G and Q are defined by formulas

$$\bar{G}(\xi,t) = \int_{-\infty}^{\infty} G(x,t) e^{ix\xi} dx , \quad \bar{Q}(\xi,t) = \int_{-\infty}^{\infty} Q(x,t) e^{ix\xi} dx .$$

According to the equation (8), connection between functions $\bar{G}(\xi,t)$ and $\bar{Q}(\xi,t)$ is represented by the relation

$$\bar{G}(\xi,t) = \bar{G}(\xi,0) \prod_{s=0}^{t-1} \bar{Q}(\xi,s) , \quad (10)$$

i.e. Eq.(9) represents the sum of statistically independent random values.

To obtain the final probability distribution corresponding the distribution $G(x,t)$, we pass from the random values $\tilde{x}_t$, $t = 0, 1, \dots$ to random values obtaining from them by centering and normalizing over the average squared deviation for each t, i.e. $\tilde{x}_t \Rightarrow (\tilde{x}_t - M(t))/D^{1/2}(t)$, where $M(t) = \langle \tilde{x}_t \rangle$ and $D(t) = \langle \tilde{x}_t^2 \rangle - M^2(t)$. Here, the angle brackets designate the averaging on the probability distribution $G(x,t)$. Probability distribution function of the new random value is

$$H(x,t) = \Pr\{(\tilde{x}_t - M(t))/D^{1/2}(t) < x\} = G\left(D^{1/2}(t)x + M(t), t\right) .$$

Correspondingly, the characteristic function $\bar{H}(\xi,t)$ of this distribution is represented by the following transformation of the characteristic function $\bar{G}(\xi,t)$,

$$\bar{G}(\xi,t) \Rightarrow \bar{G}(\xi/D^{1/2}(t),t) \exp(-i\xi M(t)/D^{1/2}(t)) = \bar{H}(\xi,t) .$$

Then, on the basis of Eq.(10), this function is represented in the form

$$\bar{H}(\xi,t) = \bar{G}\left(\frac{\xi}{D^{1/2}(t)}, 0\right) \exp\left(-i\xi \frac{M(t)}{D^{1/2}(t)}\right) \prod_{s=0}^{t-1} \bar{Q}\left(\frac{\xi}{D^{1/2}(t)}, s\right) .$$

If $D(t)$ increases in time, then, due to $\bar{G}(0,0) = 1$, the next asymptotic relation is realized

$$\ln \bar{H}(\xi,t) = \sum_{s=0}^{t-1} \ln \bar{Q}\left(\xi/D^{1/2}(t), s\right) - i\xi M(t)/D^{1/2}(t) + o(1) .$$

From other hand, taking into account $\bar{Q}(0,s) = 1$ and

$$iM(t) = \sum_{s=0}^{t-1} \left(\frac{\partial \bar{Q}(\xi,s)}{\partial \xi} \right)_{\xi=0} ,$$

the decomposition $\ln \bar{Q}(\xi,t)$ on powers of ξ gives

$$\ln \bar{H}(\xi,t) = -\frac{\xi^2}{2D(t)} \sum_{s=0}^{t-1} \delta(s) + o(1)$$

where $\delta(s) = \langle \tilde{y}_s^2 \rangle - \langle \tilde{y}_s \rangle^2$. If the remainder in this asymptotic formula is small uniformly on $\xi \in \mathbb{R}$ that it is supported by realization of the so-called *Lindenberg conditions* [4], then the central limit theorem for sums of independent random values (9) takes place since in this case

$$\lim_{t\to\infty} \bar{H}(\xi,t) = \exp(-\xi^2/2)$$

and, therefore, it takes place

$$\lim_{t\to\infty} H(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)dy \qquad (11)$$

uniformly on $x \in \mathbb{R}$ [4].

In the next section we show that such a circumstance takes place certainly for the special model of temporal dependence of the function $S(k,t)$.

Model with changing uniformed on scales. In this section we analyze the special model which has the uniformed on scales temporal changing of the fragment breakdown probability. To construct such a model, we put

$$S(k,t) = \lambda(t)\,\theta(k-1) + \mu(t)\,S(k) \qquad (12)$$

where $\theta(\cdot)$ is Heaviside's function being right-continuous and, besides, we put $S(k) = S(k-0,0)$. Here, as in the work [1], we assume that the following condition takes place

$$\int_0^1 (\ln k)^2 dS(k) < \infty\,.$$

It means that the energy deficit at the creation of new surfaces affects equally on all fragments not depending on their sizes. For introduced model, the volume conservation law of the form (4) connects the functions $\lambda(t)$ and $\mu(t)$,

$$\int_0^1 k^3 d\left(\lambda(t)\,\theta(k-1) + \mu(t)\,S(k)\right) = 1\,,$$

$$\lambda(t) + \mu(t) = 1\,. \qquad (13)$$

The requirement of the energy pumping uniformity brings to the equation

$$\int_0^1 k^2 d\left(\lambda(t)\,\theta(k-1) + \mu(t)\,S(k)\right) = 1 + (t + \Sigma_0/\sigma)^{-1}\,,$$

$$\lambda(t) + \mu(t)\,(1 + \sigma/\Sigma_0) = 1 + (t + \Sigma_0/\sigma)^{-1}$$

since it follows from Eq.(5) that

$$\int_0^1 k^2 dS(k) = 1 + \sigma/\Sigma_0\,.$$

From obtained relation, it follows taking into account Eq.(13) that

$$\mu(t) = (\sigma t/\Sigma_0 + 1)^{-1}\,. \qquad (14)$$

Thus,

$$S(k,t) = \left(1 - (\sigma t/\Sigma_0 + 1)^{-1}\right)\theta(k-1) + \frac{S(k)}{\sigma t/\Sigma_0 + 1} =$$

$$= \frac{(\sigma t/\Sigma_0)\,\theta(k-1) + S(k)}{\sigma t/\Sigma_0 + 1}$$

and, consequently,

$$P(k,t) = \frac{S(k,t)}{S(1,t)} = \frac{(\sigma t/\Sigma_0)\,\theta(k-1) + S(k)}{\sigma t/\Sigma_0 + S(1)}\,. \tag{15}$$

Let us find the mathematical expectation and the dispersion at $t \to \infty$,

$$\langle \tilde{y}_t \rangle = \int_0^1 \ln k\, dP(k,t) = (\sigma t/\Sigma_0 + S(1))^{-1} \int_0^1 \ln\, k\, dS(k) \sim \frac{m_1 \Sigma_0}{\sigma t}\,,$$

$$\langle \tilde{y}_t^2 \rangle = \int_0^1 (\ln\, k)^2\, dP(k,t) = (\sigma t/\Sigma_0 + S(1))^{-1} \int_0^1 (\ln\, k)^2\, dS(k) \;\sim \frac{m_2 \Sigma_0}{\sigma t}\,.$$

We have at $t \to \infty$ the following asymptotic expression of the sum dispersion,

$$D(t) = \sum_{s=0}^{t-1} \delta(s) \sim \sum_{s=0}^{t-1} \frac{m_2}{\sigma s/\Sigma_0 + S(1)}\,, \tag{16}$$

that is the sum

$$\sum_{s=0}^{t-1} \frac{m_1^2}{(\sigma s/\Sigma_0 + S(1))^2}$$

tends to finite limit. That is why, the dispersion $D(t)$ is proportional to $\ln\, t$ asymptotically at $t \to \infty$ since

$$D(t) \sim \frac{1}{S(1)} \sum_{s=0}^{t-1} \frac{m_2}{\sigma s/S(1)\Sigma_0 + 1} \sim \frac{m_2}{S(1)} \int_0^t \frac{ds}{1 + \dfrac{\sigma s}{S(1)\Sigma_0}} \sim \frac{m_2 \Sigma_0}{\sigma} \ln t\,.$$

Now, it is necessary to solve the problem of applicability of the central limit theorem to study the introduced model. It is needed to inspect the possibility of the Lindenberg condition realization. To do this, we formulate this condition in the form of the zero equality of the following limit value

$$\lim_{t\to\infty} \frac{1}{D(t)} \sum_{s=0}^{t-1} \int\limits_{|y - \langle \tilde{y}_s \rangle| > \eta D^{1/2}(t)} (y - \langle \tilde{y}_s \rangle)^2 dQ(y,s) = 0\,. \tag{17}$$

The value being under the limit sign in the last equation is the ratio of the conditional dispersion of the sum consisting of $\tilde{y}_s$, $s = 0, 1, ..., t-1$ to their unconditional dispersion $D(t)$ at any $\eta > 0$ where the condition is represented by the inequality for all deviations $|\tilde{y}_s - \langle \tilde{y}_s \rangle| > \eta D^{1/2}(t)$. If Eq.(17) is realized, the following limit equality takes place [4]

$$\lim_{t\to\infty} \Pr\left\{ D^{-1/2}(t) \sum_{s=0}^{t-1} (\tilde{y}_s - \langle \tilde{y}_s \rangle)^2 < x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-y^2/2\right) dy\,,$$

which brings to Eq.(11) taking into account that the term $(\tilde{x}_0 - \langle \tilde{x}_0 \rangle)^2 / D^{1/2}(t)$ gives only a negligibly small contribution to the sum (9).

To verify the condition (17), we notice that

$$Q(y,t) = P(e^y, t) = \frac{(\sigma t/\Sigma_0)\,\theta(y) + S(e^y)}{\sigma t/\Sigma_0 + S(1)}$$

according to Eq.(15). Since, for the integral in Lindenberg's condition, the following upper estimation is true,

$$\int\limits_{|y - \langle \tilde{y}_s \rangle| > \eta D^{1/2}(t)} (y - \langle \tilde{y}_s \rangle)^2 dQ(y,s) < 2 \int\limits_{|y| > \eta D^{1/2}(t) - C} (y^2 + C^2) dQ(y,s)$$

where we take into account that all averages $\langle y_s \rangle$ are uniformly bounded on their absolute value and the corresponding bounding constant C has the order $m_1\Sigma_0/\sigma$, then the sum expressing the conditional dispersion in Lindenberg's condition is upper estimated by the following way

$$\sum_{s=0}^{t-1} \int\limits_{|y-\langle\tilde{y}_s\rangle|>\eta D^{1/2}(t)} (y-\langle\tilde{y}_s\rangle)^2 dQ(y,s) <$$

$$< 2\left(\sum_{s=0}^{t-1}(S(1)+\sigma s/\Sigma_0)^{-1}\right) \int\limits_{|y|>\eta D^{1/2}(t)-C} (y^2+C^2)dS\,(e^y)\ .$$

In view of the fact that $D(t) \to \infty$, the last integral tends to zero at $t \to \infty$. And the expression in the brackets coincides asymptotically with $D(t)/m_2$ according to (16). That is why, we conclude that Lindenberg's condition for the considering model under consideration is realized.

Let us formulate the main result of the work. We have investigated, in frameworks of Kolmogorov's approach to the fragmentation processes study, the special breakdown model in which the subdivision probability of each fixed fragment to fragments with less sizes depends on time. This temporal dependence arises due to the conservation laws at the fragmentation, i.e. we formulate the conservation of the fragment volume and the energy spending during the breakdown process in frameworks of our model. These facts are not take into account in the work [1]. The introducing model is characterized by the uniform on scales energy distribution spent for subdivision. The obtained final distribution for the function $H(x,t)$ shows that in the model under consideration is realized also the logarithmically normal law for fragment sizes (for this, it is needed to return to initial physical variable, namely, the fragment size r). Indeed, taking into account the connections between the distribution functions F, G and H,

$$F(r,t) = G(\ln(r/r_0),t) = H\left(\frac{\ln(r/r_0)-M(t)}{D^{1/2}(t)}, t\right)\ ,$$

it follows from Eq.(11) that the following asymptotic at $t \to \infty$ equality takes place

$$F(r,t) \sim \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\frac{\langle \ln (r/\tilde{r}_t)\rangle}{D^{1/2}(t)}} e^{-y^2/2}dy\,.$$

Thus, we have shown that there are some physical situations when Kolmogorov's law is realized even if we take into account the pointed out natural physical requirements. In what follows, it is important to come from frameworks of investigated model and to extend the applicability field of the obtained result. It is also important to explain how to combine the results of works [3], [2] in frameworks of the used approach in this work.

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Дослідження моделі фрагментації матеріалу з рівноважним за масштабом розподіленням енергії дроблення

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Застосовуючи імовірносний підхід Колмогорова до опису процесу фрагментації матеріалу, показано, що у типовій фізичній ситуації, при постійній інтенсивності накачування енергії до системи рівняння Колмогорова має бути неоднорідним за часом. Для цього випадку, у спеціальній моделі з рівноважним за масштабами розподіленням енергії, що витрачається на дроблення, доведено граничний закон Колмогорова для розподілу імовірностей розмірів фрагментів.