

Problem of the heat radiative conductance in semitransparent media. The approximation of small reflection coefficient

A.V.Kolesnikov

Institute for Scintillation Materials, STC "Institute for Single Crystals", National Academy of Sciences of Ukraine, 60 Lenin Ave., 61001 Kharkiv, Ukraine

Received October 30, 2006

The stationary problem of the heat radiative conductance is solved at the so-called grey approximation in semitransparent media. Using the geometric optics approximation, the case of small coefficient θ of the ray reflection from the sample boundary is investigated. The problem is solved in frameworks of perturbation theory on the reflection coefficient powers. At first approximation, formulas of temperature stationary distribution are obtained with the asymptotic accuracy in the limit of large value of the material absorption coefficient.

Аналитически решена задача о стационарном распределении температуры в цилиндрическом образце полупрозрачной среды с учётом радиационно-кондуктивного теплообмена в условиях малого коэффициента отражения теплового излучения от границы и сильного поглощения излучения средой. Задача решается в приближении геометрической оптики и на основе закона серого излучения средой. Получены формулы в первом приближении для стационарного распределения температуры.

1. Introduction. This work is the continuation of the work [1] according to its idea sense. We develop the evaluation method for the heat radiative conductance which is based on the analytic solution of the irradiation conductance problem with the account of boundary conditions. Such a solution permits to formulate closed evolution equation for the temperature distribution in the sample. Further, the standard initial boundary problem of mathematical physics arises for the this equation. In this case, it is not necessary to solve the integral equation for the energy flux as it is done in [2]. The accurate explicit formula of the energy flux conducted by the irradiation in the layer of semitransparent media is obtained in [1]. But in the case of the cylindrical sample being studied in the work, we cannot give an accurate solution of such a problem in the form of the functional on arbitrary temperature distribution in the sample. Therefore, we use the expansion of the energy flux on reflection coefficient $\theta < 1$ powers. Each term of this expansion is obtained by the solving of the geometric problem of the ray moving from the irradiation point $\mathbf{x}'$ to observation point $\mathbf{x}$ and it takes place some fixed number of reflections from the sample boundary. This number is equal the reflection coefficient power. At the approximation under consideration, the contribution to the energy flux $P_\mu(\mathbf{x})$, $\mu = 1, 2, 3$ in the observation point $\mathbf{x}$ is represented as total energy flux of all rays possessing the property pointed out. The corresponding integral is very complex even in the case when reflections from the boundary are absent, i.e. for the zero order term of the reflection number expansion. Therefore, we obtain the explicit formula of the energy flux calculating all integrals on the basis of the saddle-point method using the supposition of the large media absorption of the irradiation.

At last, we point out that the expression of the energy flux which irradiating by small media volume is given by the so-called the *grey irradiation model*. It has the form of the functional on the temperature distribution.

2. The evaluation of the energy flux. Let we have the cylinderical sample of semitransparent media with the adsorption coefficient α. The cylinder has the radius R. We denote by $P_\mu(\mathbf{x})$ the energy flux of the irradiation in the observation point $\mathbf{x}$ in the cylinder. Correspondingly, $P_\nu^{(0)}(\mathbf{x}')$ is the flux of the energy irradiated by the media in the point $\mathbf{x}'$ (further, lower Greek indexes have values 1, 2, 3 and the convention relative to the summation on repeated indexes is set). Due to the linearity of electromagnetic field equations, the linear relation between these fluxes is postulated,

$$P_\mu(\mathbf{x}) = \frac{\alpha^3}{4\pi} \sum_\nu \int Q_{\mu\nu}(\mathbf{x}, \mathbf{x}') P_\nu^{(0)}(\mathbf{x}') d\mathbf{x}' , \tag{1}$$

where the *transfer matrix* $Q_{\mu\nu}(\mathbf{x}, \mathbf{x}')$ is calculated by the approximation of geometric optics. It is represented by μ-th component of the energy flux at the point $\mathbf{x}$ transported by one ray that is irradiated with the unit intensity by the small area being at the point $\mathbf{x}'$ and oriented to the ν-th direction. The coefficient α^3 before the integral points out that the irradiation occurs of duration αds along the each dimension among three of them. The multiply $(4\pi)^{-1}$ corresponds to the averaging on all irradiation directions. The integration in Eq.(1) is fulfilled on the whole sample with the account of contributions to the total energy irradiation flux in the point $\mathbf{x}$ of all rays which are irradiated by all sample points. We find the transfer matrix $Q_{\mu\nu}(\mathbf{x}, \mathbf{x}')$ in the form of the expansion

$$Q_{\mu\nu}(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty} Q_{\mu\nu}^{(l)}(\mathbf{x}, \mathbf{x}') , \tag{2}$$

where transfer matrices $Q_{\mu\nu}^{(l)}(\mathbf{x}, \mathbf{x}')$, $l = 0, 1, 2, ...$ are defined as contributions into the common transfer matrix to the point $\mathbf{x}$ of all rays irradiated from the point $\mathbf{x}'$ which have had strictly l reflections from the sample boundary before they arrive to $\mathbf{x}$ and, therefore, they are proportional to θ^l where θ is the reflection coefficient of the ray from the boundary.

In this work, we calculate the term with $l = 0$ only. It is taken into account only such a motion of each ray when reflections from the boundary do not occur. Since the vector of the energy flux $P_\mu(\mathbf{y})$ transferring by the ray irradiated from the point $\mathbf{x}'$ is decreased on the value $\alpha P(\mathbf{y}) ds$, $P(\mathbf{y}) = [P_\mu(\mathbf{y}) P_\mu(\mathbf{y})]^{1/2}$ after it has passed the distance ds from a point $\mathbf{y}$ to the point $\mathbf{y} + \mathbf{n} ds$ ($\mathbf{n}$ is the unit vector to the ray direction) and its direction $\mathbf{n}$ does not change, then, after the attaining of the point $\mathbf{x}$, this energy flux is equal to $\exp(-\alpha|\mathbf{x} - \mathbf{x}'|) P_\mu^{(0)}(\mathbf{x}')$. Consequently,

$$Q_{\mu\nu}^{(0)}(\mathbf{x}, \mathbf{x}') = \delta_{\mu\nu} \exp(-\alpha|\mathbf{x} - \mathbf{x}'|) . \tag{3}$$

The direction of the ray spreading is defined by the vector

$$\mathbf{n} = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} . \tag{4}$$

We consider that the irradiation is isotropic at each sample point $\mathbf{x}'$, i.e.

$$P_\nu^{(0)}(\mathbf{x}') = n_\nu P^{(0)}(\mathbf{x}') \tag{5}$$

where $P^{(0)}(\mathbf{x}')$ is the absolute value of the energy flux from the point $\mathbf{x}'$. Thus, on the basis of Eqs.(1),(5), at the above mentioned approximation, we have

$$P_\mu(\mathbf{x}) = \frac{\alpha^3}{4\pi} \int_V \exp(-\alpha|\mathbf{x} - \mathbf{x}'|) P^{(0)}(\mathbf{x}') \frac{x_\mu - x'_\mu}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' . \tag{6}$$

Our further problem is the calculation of this integral using the supposition of the large value of the absorption coefficient. We determine it at the main approximation order evaluating the asymptotics being proportional to inverse power α^{-1}. This asymptotics has the universal form. It is not connected with the concrete sample geometry. Let us change the integration variable in Eq.(5) $\mathbf{x}-\mathbf{x}' \Rightarrow \mathbf{y}$. In a result, we obtain

$$P_\mu(\mathbf{x}) = -\frac{\alpha^3}{4\pi}\int\limits_{V-\mathbf{x}} \exp(-\alpha|\mathbf{y}|)P^{(0)}(\mathbf{y}+\mathbf{x})\frac{y_\mu}{|\mathbf{y}|}d\mathbf{y} \tag{7}$$

where $V-\mathbf{x}$ is the domain which is occupied by the sample shifted on the vector $-\mathbf{x}$. For the determining of the asymptotic expansion on the α inverse powers, we change the integration variable due to the rule $\alpha\mathbf{y} \Rightarrow \mathbf{y}$,

$$P_\mu(\mathbf{x}) = -\frac{1}{4\pi}\sum_\nu \int\limits_{\alpha V-\mathbf{x}} \exp(-|\mathbf{y}|)P^{(0)}(\mathbf{y}/\alpha+\mathbf{x})\frac{y_\mu}{|\mathbf{y}|}d\mathbf{y}\,. \tag{8}$$

After that, we expand the integrand on powers α^{-1},

$$P^{(0)}(\mathbf{y}/\alpha+\mathbf{x}) = P^{(0)}(\mathbf{x}) + \frac{y_\mu}{\alpha}\nabla_\mu P^{(0)}(\mathbf{x}) + \frac{1}{2}\frac{y_{\mu_1}y_{\mu_2}}{\alpha^2}\nabla_{\mu_1}\nabla_{\mu_2}P^{(0)}(\mathbf{x}) + \dots .$$

After the substitution of this expansion into Eq.(8), we obtain the expression

$$P_\mu(\mathbf{x}) = -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \exp(-|\mathbf{y}|)\times$$

$$\times\left(P^{(0)}(\mathbf{x}) + \frac{y_{\mu_1}}{\alpha}\nabla_{\mu_1}P^{(0)}(\mathbf{x}) + \frac{1}{2}\frac{y_{\mu_1}y_{\mu_2}}{\alpha^2}\nabla_{\mu_1}\nabla_{\mu_2}P^{(0)}(\mathbf{x}) + \dots\right)\frac{y_\mu}{|\mathbf{y}|}d\mathbf{y}$$

at $\alpha \to \infty$ with the accuracy up to exponentially small terms $\sim \exp(-\alpha L)$ where L is the characteristic sample linear size. In this expansion, the terms with even α^{-1} powers equal to zero,

$$\int\limits_{\mathbb{R}^3} e^{-|\mathbf{y}|}\frac{y_\mu}{|\mathbf{y}|}d\mathbf{y} = 0\,, \quad \int\limits_{\mathbb{R}^3} e^{-|\mathbf{y}|}y_{\mu_1}y_{\mu_2}\frac{y_\mu}{|\mathbf{y}|}d\mathbf{y} = 0$$

since these integrals presents some tensors of the odd range and they are invariant relative to arbitrary rotations but they change the sign at the space reflection. Consequently, at the main approximation, the flux $P_\mu(\mathbf{x})$ is defined by the term proportional to α^{-1},

$$P_\mu(\mathbf{x}) = -\frac{1}{4\pi\alpha}\left(\int\limits_{\mathbb{R}^3} \exp(-|\mathbf{y}|)\frac{y_\nu y_\mu}{|\mathbf{y}|}d\mathbf{y}\right)\nabla_\nu P^{(0)}(\mathbf{x})\,. \tag{9}$$

The integral represents the tensor being invariant relative to rotations and the reflection. Therefore, it is proportional to the Kronecker symbol,

$$\int\limits_{\mathbb{R}^3} e^{-|\mathbf{y}|}y_\nu\frac{y_\mu}{|\mathbf{y}|}d\mathbf{y} = A\delta_{\mu\nu}\,. \tag{10}$$

Calculating the trace of two equality sides, we obtain

$$3A = \int\limits_{\mathbb{R}} e^{-|\mathbf{y}|}|\mathbf{y}|d\mathbf{y} = 4\pi\int\limits_0^\infty e^{-|\mathbf{y}|}|\mathbf{y}|^3 d|\mathbf{y}| = 4\cdot 3!\cdot\pi\,.$$

Consequently, $A = 8\pi$. Together with Eq.(10), it gives the desired expression of the flux

$$P_\mu(\mathbf{x}) = -\frac{2}{\alpha}\nabla_\mu P^{(0)}(\mathbf{x}) \tag{11}$$

at the stated approximation.

Thus, from the general evolution equation of the temporally changed temperature distribution in the sample,

$$c\rho\frac{\partial T(\mathbf{x},t)}{\partial t}=\kappa\Delta T(\mathbf{x},t)-\nabla_\mu P_\mu(\mathbf{x},t)$$

(c is the media specific heat, ρ is the density and κ is thermal conductivity), we find, on the basis of Eq.(11) that

$$c\rho\frac{\partial T(\mathbf{x},t)}{\partial t}=\kappa\Delta T(\mathbf{x},t)+\frac{2}{\alpha}\Delta P^{(0)}(\mathbf{x},t)\,.$$

It is supposed that the flux $P^{(0)}(\mathbf{x},t)$ is temporally changed also since it is the functional on the temperature distribution.

As it is mentioned in the introduction, we consider that the value of the irradiated energy flux at each point $\mathbf{x}$ is determined at the grey approximation, i.e. we put

$$P^{(0)}(\mathbf{x})=\sigma T^4(\mathbf{x})$$

where σ is the Stefan-Boltzmann constant. Then, we obtain finally the following evolution equation

$$c\rho\frac{\partial T(\mathbf{x},t)}{\partial t}=\kappa\Delta T(\mathbf{x},t)+\frac{2\sigma}{\alpha}\Delta T^4(\mathbf{x},t)\,. \tag{12}$$

3. The problem of the stationary distribution in the cylinder. In this part, we solve the problem connected with the determination of the equilibrium temperature distribution in the cylinder sample, when the temperature difference at its end-walls is constant and the temperature on its lateral area is defined by the condition of the gas environment. It follows from Eq.(12) that the stationary distribution of the temperature $T(\mathbf{x})$ is determined by the equation

$$\kappa\Delta T=-\frac{2\sigma}{\alpha}\Delta T^4 \tag{13}$$

at the approximation under consideration. Introducing the function

$$W=T+\frac{2\sigma}{\kappa\alpha}T^4\,, \tag{14}$$

the problem solution of the distribution $T(\mathbf{x})$ determination is reduced to the solution of the equation

$$\Delta W=0\,. \tag{15}$$

Let us formulate boundary conditions for the concrete problem under consideration. We introduce cylindrical coordinates $\langle r,\varphi,z\rangle$ in the sample by the natural way. Further, we put on the cylinder end-walls $z=0, z=L$

$$T(r,\varphi,0)=T_-\,,\quad T(r,\varphi,L)=T_+\,,\quad T_+>T_- \tag{16}$$

where L is the cylinder length. At the cylinder lateral area at $r=R$ (R is the cylinder radius), we put

$$T(R,\varphi,z)=T_-+\frac{z}{L}(T_+-T_-) \tag{17}$$

since we consider that it is in the gas environment and there is the equilibrium temperature difference with $T|_{z=0}=T_-$, $T|_{z=L}=T_+$. But we take into account that the gas media does not irradiate and this distribution satisfy to the equation $\Delta T=0$ where it depends on z only. In the consequence with the boundary conditions (16),(17), the function W satisfy to conditions

$$W(r,\varphi,0)\equiv W_-=T_-+\frac{2\sigma}{\alpha\kappa}T_-^4\,,\quad W(r,\varphi,L)\equiv W_+=T_++\frac{2\sigma}{\alpha\kappa}T_+^4\,, \tag{18}$$

$$W(R,\varphi,z)\equiv V(z)=T^{(0)}(z)+\frac{2\sigma}{\alpha\kappa}T^{(0)}(z)\,, \tag{19}$$

$$T^{(0)}(z) = T_- + \frac{z}{L}(T_+ - T_-)\,. \tag{20}$$

At given boundary conditions (18),(19), the solution of Eq.(15) does not depend on φ, $W(r,\varphi,z) = W(r,z)$. We represent the solution of Eq.(15) in the form

$$W(r,z) = V(z) + \frac{2\sigma}{\alpha\kappa}U(r,z)\,, \tag{21}$$

such as

$$\Delta V = 0\,, \quad \Delta U = 0\,. \tag{22}$$

On the basis of boundary conditions (18),(19), we find

$$U(r,0) = 0\,, \quad U(r,L) = 0 \tag{23}$$

at the cylinder end-walls and

$$U(R,z) = T^{(0)4}(z) - T_-^4 - \frac{z}{L}(T_+^4 - T_-^4) \tag{24}$$

at the lateral area.

Let us notice that the function $U(R,z)$ is positive since $T^{(0)4}(z)$ is the convex function at $z \geq 0$ and it is equal to the linear function $(T_+^4 - T_-^4)z/L + T_-^4$ at $z = 0, L$. Thus, it is necessary to solve the problem in the cylinder for the function $U(r,z)$, $\Delta U = 0$ with the boundary conditions (23),(24). We find the solution in the form

$$U(r,z) = \sum_{n=1}^{\infty} \alpha_n(r) \sin\frac{\pi z}{L}n \tag{25}$$

satisfying to boundary condition (23). Then, we obtain the system of ordinary second order equations

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\alpha_n(r) - \left(\frac{\pi n}{L}\right)^2 \alpha_n(r) = 0 \tag{26}$$

for the coefficients $\alpha_n(r)$, $n \in \mathbb{N}$. The boundary conditions $\alpha_n(R)$, $n \in \mathbb{N}$ are determined by the expansion

$$U(R,z) = \sum_{n=1}^{\infty} \alpha_n(R) \sin\frac{\pi z}{L}n\,, \quad \alpha_n(R) = \frac{2}{L}\int_0^L U(R,z)\sin\frac{\pi z}{L}n\,dz\,. \tag{27}$$

Using Eq.(24), we evaluate coefficients $\alpha_n(R)$ of the expansion. Calculating the integral in Eq.(27), we have

$$\alpha_n(R) = -\frac{24}{\pi n}\left(\frac{T_+ - T_-}{\pi n}\right)^3 \left[(T_-^2 - (-1)^n T_+^2) - 2\left(\frac{T_+ - T_-}{\pi n}\right)^2 (1 - (-1)^n)\right]. \tag{28}$$

Now, we may find the formula for the calculation of the function $W(r,z)$. Solutions of Eq.(26) are expressed by modified Bessel functions. We take into account of boundedness property of the solution inside the disk $\{\langle r,\varphi\rangle : r = R\}$. It is equal to the boundedness of coefficients $\alpha_n(r)$, $n \in \mathbb{N}$. Therefore, we obtain

$$\alpha_n(r) = \alpha_n I_0\left(\frac{\pi n}{L}r\right)\,. \tag{29}$$

Substituting the boundary condition at $r = R$, we have

$$\alpha_n = \frac{\alpha_n(R)}{I_0\left(\frac{\pi n}{L}R\right)}\,.$$

Consequently,

$$U(r,z) = \sum_{n=1}^{\infty} \alpha_n(R)\frac{I_0\left(\frac{\pi n}{L}r\right)}{I_0\left(\frac{\pi n}{L}R\right)}\sin\frac{\pi n}{L}z \tag{30}$$

where the function $I_0(x)$ is defined by the series

$$I_0(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(n!)^2} \,.$$

Thus, for the obtaining of numerical values, it is necessary to calculate the relation $I_0\left(\frac{\pi n}{L} r\right)/I_0\left(\frac{\pi n}{L} R\right) < 1$ with the given accuracy. Since

$$\left| \frac{\alpha_n(R)}{(T_-^2 - T_+^2)(T_-^2 + T_+^2)} \right| \sim \frac{24}{(\pi n)^3} \,,$$

the term number in the series (30) that is sufficient for the security of the given evaluation accuracy is simply determined. For example, at the accuracy 10^{-2}, since $\sum_{m=n+1}^{\infty} m^{-3} \sim (n+1)^{-2}$, it is sufficient to put $n = 9$, $(n+1)^{-2} < 10^{-2}$ in the series (30) for the accuracy security.

4. The stationary distribution. The stationary temperature distribution in the sample is constructed by such a way as it is done in the work [1] when the function $W(r,z)$ is known. It is necessary to solve the algebraic equation

$$V + \frac{2\sigma}{\alpha\kappa} U = T + \frac{2\sigma}{\alpha\kappa} T^4 \tag{31}$$

relative to the function $T(\mathbf{x})$. It follows from Eq.(14) and Eq.(21). Since the initial equation (13) for the stationary temperature distribution is obtained up to the accuracy $o(\alpha^{-3})$ when the parameter α^{-1} is small, it is necessary to find the desired solution expanding it by the following way

$$T = T^{(0)} + T^{(1)} + T^{(2)} + ... \tag{32}$$

up to the same accuracy and the term $T^{(0)}$ is given by Eq.(20). We note that the function V depends on α and it has the expansion

$$V = V_0 + \frac{2\sigma}{\alpha\kappa} V_1 \tag{33}$$

where $V_0(z) = T^{(0)}(z)$ and

$$V_1 = T_-^4 + \frac{z}{L}(T_+^4 - T_-^4) \,. \tag{34}$$

From Eq.(31) we obtain equations

$$T^{(1)} + \frac{2\sigma}{\alpha\kappa} T^{(0)4} = \frac{2\sigma}{\alpha\kappa}(V_1 + U) \,,$$

$$T^{(2)} + \frac{8\sigma}{\alpha\kappa} T^{(0)3} T^{(1)} = 0$$

for the approximations $T^{(1)}$, $T^{(2)}$. Therefore, the corresponding terms of the expansion (32) of the desired stationary temperature distribution are defined by the formulas

$$T^{(1)} = \frac{2\sigma}{\alpha\kappa}(V_1 + U - V_0^4) \,.$$

$$T^{(2)} = -16 \left(\frac{\sigma}{\alpha\kappa}\right)^2 V_0^3 (V_1 + U - V_0^4) \,.$$

They give the total solution of the problem at the approximation under consideration.

5. Conclusion. We notice in the conclusion that the construction proposed in this work in the case of the grey irradiation permits to calculate the stationary temperature distribution in the cylinder sample for arbitrary approximation order on the reflection coefficient when the absorption coefficient is large. It is done by the expansion being the double series on powers θ and α^{-1}. Moreover, such an expansion may be always built for the sample of arbitrary form as the Dirichlet boundary problem for the function W is solved in the domain of $\mathbb{R}^3$ which presents this sample.

References

1. A.V.Kolesnikov , Yu.P.Virchenko, *Functional Materials*, **13**, 372 (2006).
2. E.M.Sparrow, R.D.Cess, Radiation Heat Transfer. Brooks/Cole Publishing Company, Belmont, California [1996].

Проблема радіаційно-кондуктивного теплообміну у напівпрозорих середовищах. Наближення малого коефіцієнта відбиття

О.В.Колесников

Аналітично розв'язано проблему про стаціонарний розподіл температури у циліндричному зразку напівпрозорого середовища з обліком радіаційно-кондуктивного теплообміну в умовах малого коефіцієнта відбиття теплового випромінювання від границі та сильного поглинання випромінювання середовищем. Проблема розв'язується у наближенні геометричної оптики та на основі закону сірого випромінювання середовищем. Одержано формули у першому наближенні для стаціонароного розподілу температури.