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Minimal Fermi model

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In this work, we consider the simple model for Fermi acceleration of a particle between two periodically oscillating walls. The law of wall movement is continuous but not smooth. Exact mapping for this system has been obtained. A fractal set of trajectories with infinitely increasing speed is shown to exist. The main characteristics of such trajectories are discussed. A comparison with high-energy approach has been carried out. Qualitative differences in behavior of exact and approximate description have been found. For example, in high-energy approach, there are no trajectories with unlimited speed increase.

В работе рассмотрена простая модель ускорения Ферми частицы между двумя периодически колеблющимися стенками. Закон движения стенок является непрерывным, но не гладким. Получено точное отображение для этой системы. Показано существование фрактального множества траекторий с неограничено возрастающей скоростью. Обсуждаются основные характеристики таких траекторий. Проведено сравнение с высокоэнергетическим приближением. Обнаружены качественные отличия в поведении точного и приближенного описания. Так в высокоэнергетическом приближении отсутствуют траектории с неограниченным ростом скорости.

To explain an origin of high-energy space beams, Fermi has proposed in 1949[1] a mechanism for acceleration of particles by moving magnetic clouds. In later works, the models realizing the Fermi acceleration mechanism by means of collisions of a particle with oscillating wall have been introduced. The most common of those is the model in which the particle moves freely between two walls. One of these walls moves under some law, usually a periodic one, the second wall is motionless [3-6]. The particle dynamics is described by means of mapping, which is usually achieved by neglecting the change of the wall position. Further, the particle speed is considered to exceed considerably the characteristic wall speed, that is why such approach is referred to as the high-energy one. This approach simplifies essentially the system mapping. Another popular model is the particle falling in gravitation field onto the oscillating wall [7]. These models, due to their simplicity, the possibility of numerical modeling of their behavior on extended times and to their practical applications, became standard models for the analysis of nonlinear Hamilton systems. The main subject to be studied in these models is the process of energy gain by the particle.

If the wall movement law is a random one, then the particle energy will increase as an average [1]. The case when the wall movement law is determined, e.g.,periodic, is less unequivocal. Numerical modeling for a case of smooth periodic movement of the wall carried out by Ulam [2] has shown that the particle movement is stochastic, but its energy is restricted by an upper limiting value. These Ulam results have been explained using analytical and numerical methods by Zaslavsky and Chirikov [3] and more completely by Brakhich [4], Lichtenberg and Liberman [5], Pustylnikov [8] and others. Those authors have proved that when the time dependence of wall speed is smooth enough, there is always some speed limit that the particle cannot exceeed. The speed limit value for particles with low initial energy is independent of initial conditions.

In this work, the case is studied when the wall movement follows a sawtooth law. This means that the collision phase is not random but the time dependence of the wall speed is not smooth. In [9] a single unbounded trajectory has been specified for such a system.

Let us consider a particle being between two mobile walls. Let walls move periodically first at a fixed speed from the system center, then at the same speed towards the system center. Thus,the walls move in an antiphase under the sawtooth law. This system is equivalent to that including one moving and one stationary wall. However, it is convenient to keep this insignificant technical complication assuming a further studying of inphase movement of walls. Collision of the particle with the wall will consider to occur instantly and absolutely elastically, in intervals between collisions the particle moves at constant speed and rectilinearly. Let denote the wall period movement as T and its speed as u.

It is convenient to describe the evolution of such system by means of mapping which connects the particle coordinates at $n+1$-th collision with the wall with the particle coordinates at n-th collision. Thus, knowing mapping and coordinates of a particle at the initial collision, it is possible to find those at the second collision, etc. As the coordinates, we shall choose the particle speed v_n prior to collision with the wall, expressed as a multiple of the wall movement speed, and the collision phase ξ_n. As the collision phase, we shall understand time counted from the start of the corresponding motion cycle of the wall, expressed in the wall movement periods. For the left wall, all times including phase of collision will be considered as negative. For the unit of distance, we shall accept the wall oscillation amplitude.

The introduced coordinates are optimal for the specified system, since they allow to determine (along with speed with sign and phase of collision) with which of the walls a collision is occurred.

The particle changes its speed when colliding with the wall. If the particle and the wall moved towards each other before the collision, then the particle will change its movement direction after collision, its speed module will be increased by the doubled wall speed. If before the collision the particle and the wall moved in the same direction, then the particle speed will be reduced by decrease for the doubled wall speed. After collision, the particle may either change the movement direction or will continue to move in the former direction, depending on its speed before collision; in both cases, its speed module will decrease. If the particle speed before collision with the wall exceeds the doubled wall speed, then after collision the particle will change the movement direction to opposite. If the particle speed is intermediate between the doubled and single speed of the wall, then the particle movement in the former direction will continue, and if the particle speed is less than the wall speed, the particle cannot collide with the wall. In this case, the second collision will occur after the wall will change its speed to the opposite following the law of its movement.

Thus, the wall can repulse a particle by one or two collisions. Two collisions occur, if at collision the particle runs down the wall, but its speed is less than doubled wall speed. In this case, the particle changes its movement direction at the second collision, and its speed module increases by the doubled wall speed. Two collisions occur also in the case when at collision the particle runs down the wall, its speed exceeded doubled wall speed (but is less than Triple one), but after the first collision the wall, having changed its speed according the movement law, has drived to the particle moving away from it (that has lost its speed at the first collision down to a lower value than the wall speed). This case is to distinguish from that when the particle speed before the first collision was less than the wall one. The difference is that in the case of two collisions, the wall runs down the particle, while in case of one collision, the wall and the particle move towards each other. Though the particle speed module before collision in these two cases may be the same, it will be different after collision. To distinguish these cases, the particle speed less than the wall speed before the first collision we shall consider positive, while the particle speed less than the wall one in case of two collisions, negative. Therefore, the particle speed before collision can take values from -1 to infinity.

Thus, if before collision with the wall the particle speed module is less than the wall one, it is just the collision phase sign that defines, with which of the walls a collision occurs; the speed sign, the first or the second collision occurred, and, accordingly, either the particle moves towards the wall or escapes therefrom.

It is easy to see, that necessary (but not sufficient) condition for the case of two collisions is: $|\xi_n| < \frac{1}{2}$ and $1 < v_n < 3$, i.e. the particle before collision should have a speed exceeding the wall one to drive to it, while after the collision, it should be less than the wall one, so that the wall could drive to the particle. If $1 < v_n < 2$, the particle after the first collision will not change its movement direction, while if $2 < v_n < 3$, it will change (only in this case, under condition of two collisions, the particle speed before collision is negative).

The mapping considering all cases in the compact form has the following form:

$$v_{n+1} = \begin{cases} -(v_n - 2), \ \ |\xi_n| < \frac{1}{2}, v_n < 2 + \frac{\xi_n}{sign\xi_n - \xi_n}, \\ v_n + 2\,sign(|\xi_n| - \frac{1}{2}), \ \ |\xi_n| > \frac{1}{2} \quad \text{or} \quad (|\xi_n| < \frac{1}{2}, \quad v_n > 2 + \frac{\xi_n}{sign\xi_n - \xi_n}) \end{cases} \tag{1}$$

$$\xi_{n+1} = \begin{cases} \xi_n + \frac{sign\xi_n - 2\xi_n}{3 - v_n}, \ \ |\xi_n| < \frac{1}{2}, v_n < 2 + \frac{\xi_n}{sign\xi_n - \xi_n} \\ \zeta^{(n)} - \frac{\frac{sign\xi_n}{2} + sign\left(\left|\zeta^{(n)} - \frac{sign\xi_n}{2v_{n+1}}\right| - \frac{1}{2}\right)\left(\zeta^{(n)} + \frac{sign\xi_n}{2}\right)}{v_{n+1} + sign\left(\left|\zeta^{(n)} - \frac{sign(\xi_n)}{2v_{n+1}}\right| - \frac{1}{2}\right)}, \ |\xi_n| > \frac{1}{2} \text{ or } \left(|\xi_n| < \frac{1}{2},\ v_n > 2 + \frac{\xi_n}{sign(\xi_n) - \xi_n}\right) \end{cases} \tag{2}$$

where

$$\zeta^{(n)} = -\left(\xi_n + \frac{\frac{sign\xi_n}{2} + sign(|\xi_n| - \frac{1}{2})(\xi_n - \frac{sign\xi_n}{2})}{v_{n+1}} + sign\xi_n \frac{2(\frac{1}{\delta} - 1)}{v_{n+1}}\right) \pmod 1$$

Here δ is a system parameter defined as the ratio of the oscillation amplitude of walls to average distance between them. Using this mapping, various properties of trajectories of a particle moving between walls will be investigated. This mapping is exact.

When studying properties of exact mapping, the main point of interest is the process of energy gain by the particle. In exact mapping (1), (2) trajectories with unlimited speed increase are observed. In principle, all trajectories in phase space of exact display, can be subdivide into two types. The trajectories of the first type have a boundary speed, which particle cannot exceed. In particular, all periodic orbits get into this class. Trajectories of the second type have no such boundary and eventually energy increases beyond all bounds. This class of trajectories is observed at all possible values of parameter $\delta < 1$.

A certain difficulty in studying of these trajectory types is due to low informativity of the exact mapping phase portrait. The typical phase portrait of unlimitedly growing trajectory is shown in Figure 5. The cause is the mapping discontinuity for speed. In dimensionless Variables, the speed changes by ± 2 at each step of discrete time. This complicates the qualitative analysis of a phase flow (cascade) and requires to make use of less traditional analytical methods

Let us consider as an example a trajectory belonging to unlimitedly growing speed type. In Fig. 1, dependence of the particle speed on number of iteration is shown. The important conclusion consists in observation of a significant speed reduction at the initial stage of evolution with its subsequent unlimited increase.

Thus, the particle speed can decrease considerably at the initial stage of evolution. In other words, during evolution, the applicability limits of high-energy approximation are overrunned. Any arbitrary high initial particle speed cannot guarantee that the particle will not descend to low speed region. For example, we may take an unlimitedly growing trajectory and change the particle and the wall movement directions to the opposite ones. Then the particle will start to descend along the same trajectory along which it ascended. This is equivalent to consideration of a growing trajectory in back-going time. In this way, the particle can reduce its speed down to a minimal possible one. Then, it will be «reflected» and will start to gain energy again.

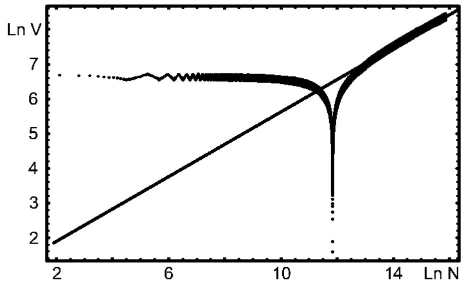

Fig. 1. Dependence of the particle speed on number of collision in log-log scale for trajectory $\xi_0 = 0.04$, $v_0 = 931.5$, $\delta = 0.1$, $N = 7500000$. The linear section at late evolution stage is clearly visible. For convenience, the straight line with slope 1/2 is included.

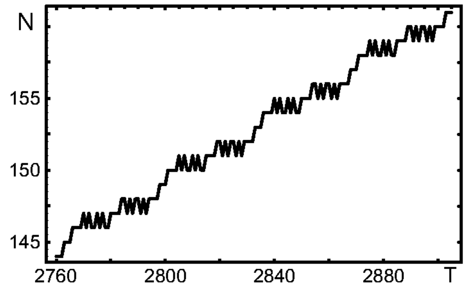

Fig. 2. Dependence of number of collisions of the particle during one wall movement period onthe periods number for trajectory $\xi_0 = 0.04$, $v_0 = 931.5$ and parameter $\delta = 0.1$.

Let us discuss the asymptotic rate of speed increase after its decrease stage. For convenience, the speed dependence on number of collisions n is shown in Fig. 1 in log-log scale. The Figure evidences a linear dependence with proportionality factor of 1/2. This means the square root type of speed increase $v_n \sim \sqrt{n}$. However, it is important to transform this dependence to that on «real» time. For this purpose, it is necessary to consider how the number of collisions with walls varies in time. It is found that the number of collisions per unit time increases in such a manner that the particle speed eventually grows linearly, i.e. for each period of the wall movement, the particle speed increases by the same value.

That is possible to see from dependence of number of collisions per one period on the period number brought in Fig. 2. It is of interest to note that this dependence shows periodic oscillations against the background of monotonous increase. In a certain sense, this is an additional reason for preservation of the speed increase mode for unlimited time. Thus, the particle speed in this mode increases in proportion to the time. It is a rather unusual mode of speed increase. Usually, for example at chaotic wandering, speed increases more slowly, proportional to $\sqrt{t}$. In this sense, it is possible to state abnormal increase of the particle speed. Phenomenologically, this means presence of average effective force acting on the particle. At the fixed parameter δ, the value of this force may not be the same for different infinite trajectories. Its dependence on parameter δ is rather complex.

Thus, either the particle energy is limited, or the particle eventually unlimitedly increases its energy. We shall establish a fixed system parameter δ, place the particle near the bottom speed ($v_0 < 20$) and determine at which initial data the particle can ascend to the certain energy ($v_n \geq 100$) for the certain number of steps ($n \leq 10^5$) and with which it will be limited. Choosing initial data in some area of phase space, it is possible to construct a pool of trajectories with unlimited speed increase.

It appears that the set of points belonging to the pool ($v_n \to \infty$ with $n \to \infty$) has fractal structure. The fractal dimension of this set is close to two. Informational and correlation dimensions have also been calculated and are also essentially equal to two. Thus, infinitely growing trajectories are distributed at a high homogeneity over the whole phase space.This does not mean, however, that all trajectories are infinitely growing. At the parameter value $\delta = 0.1$, on the average only 37 trajectories from 100 are infinitely growing. The number of infinitely growing trajectories depends on the system parameter. With increase of the latter, the percentage of growing trajectories increases. So, at $\delta = 0.33$, on the average 81 of 100 trajectories are growing.

There is also another way to make sure of sensitivity of trajectory type. For example, choosing the same initial conditions and studying influence of the mapping parameter δ variation on the trajectory type. Let a particle be placed near the bottom speed, fixing its initial speed ($v_0 < 20$) and phase. Now let us determine, what values of parameter results in particle having energy higher than ($v_n \geq 100$) after

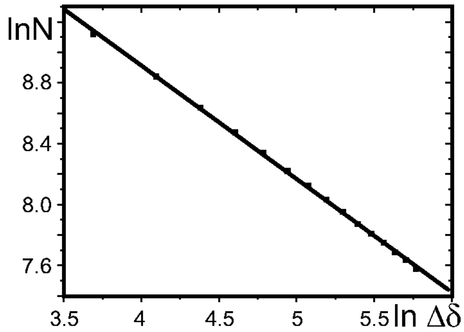

Fig. 3. To determination of the fractal dimension of set of parameters at which the particle with the fixed initial data reaches the predetermined energy. The fractal dimension is $D_F = 0.745$.

the specified number of steps ($n \leq 10^5$) and what corresponds to bounded particles. The initial speed of the particle and the collision phase do not vary, the parameter of system changes only. Therefore, the actually obtained data show how properties of a single trajectory vary with the parameter variation. Such sensitivity is related with reorganizations of the system phase portrait at small changes of the parameter. Other trajectories pass effectively through the point of phase space fixed by initial conditions.

The structure of parameter values at which the particle with fixed initial data reaches the pre-specified energy looks like bar code. The black stripe correspond to such value of δ at which the particle reaches the pre-specified energy, the white, to that where it appears bounded. Characteristic line structure is preserved at change of scales. Each stripe at scale enlargement becomes split into more small-scale structure of stripes.

A minor change in the system parameter results in an essential change of the trajectory. Thus, a very weak, perhaps even infinitesimal, change of the system parameter or of initial data is sufficient for the closed particle becomes free and vice versa.

The obtained set of stripes, has fractal structure, as well as the set of initial data. To be convinced of that, let the set of all possible values of parameter δ be subdivides into segments of equal length, and plot the dependence of number of segments in which at least one strip has got on the length of single segment, in logarithmic scale (Fig.3). The obtained dependence is linear, thus allowing to estimate the fractal dimension of the given set as $D_F = 0.745$.

For other initial conditions, set of the system parameters which allow the infinite gain of particles energy also has fractal structure. Thus, a specific fractal dimension of the system parameter set corresponds to every initial data defining the trajectory.

Let us prove that in the trajectories considered above, the particle speed increases really indefinitely, i.e. that the speed gain will never stop. As all the trajectories obtained by means of the numerical calculations are finite, this could be proved only analytically.

Let us introduce a certain transformation F. This transformation will influence the part of the trajectory contained in one period of the wall movement. The transformation will result in the set of collisions which is also contained in one period of the wall movement and is a part of some, maybe another, trajectory. Both trajectories correspond to the same value of the system parameter δ. The transformations will be different for different δ. Thus, the transformation translates a part of one trajectory in an equal in time part of another trajectory. The average speed of particle, number of collisions within the period of the wall movement at the initial part of the trajectory and the transformed part of the trajectory will be different. The transformation F will also keep difference of number of speed-increasing and speed-reducing collisions of the particle with the wall. Thus, if in the initial part of the trajectory the particle has increased speed by some value per period, in the transformed part of the trajectory the particle will increase its speed by the same value. Applying sequentially the transformation F, it is possible to transform the part of a trajectory lying on successive periods of wall movement, including the whole trajectory.

It is possible that the initial and transformed parts of the trajectory are different parts of the same trajectory. This case is realized if coordinates of the collision following the last collision in the initial trajectory part coincide with those of the first collision in the transformed one. This condition can be named a «sewing» condition. If this condition is met, the part that is the transformation result follows immediately after the initial trajectory part. The transformed part of the trajectory, in its turn, also can be transformed. It appears that if the sewing condition for the initial and transformed parts of the trajectory is satisfied, it is sufficient for the same condition to be satisfied for the transformed part of the trajectory and the part transformed twice. Thus, applying repeatedly the transformation F to some initial part of the trajectory, it is possible to obtain infinite number of different parts of the trajectory. Those all will be sewed consecutively into a unique trajectory, if first two parts can be sewed. It is easy to see that the trajectory obtained in this way will be infinitely growing.

Thus, beyond an unrestrictedly growing trajectory consists of parts, each of which under action of transformation F gives the following one and each of which increases speed of the particle by the same value.

For each system parameter δ, we will search for transformation F in the form $F = f(m) \circ g(n)$, where f and g are two elementary transformations, m and n are natural numbers. We will define the transformation f as follows. Let us consider a part of the trajectory laying on one period of the wall movement. Let us distinguish a time interval at the the period center including two collisions of the particle with the wall. Let the interval duration be selected so that the particle got into it and left it at the same distance from the center of system. It is always possible to make. Let us note that one of these allocated collisions reduces speed of the particle by doubled wall speed, while the second increases by the same value. Thus, both the particle speed and distance to the system center at exit from the allocated zone will be same as on entrance. This means that, having thrown out the selected part from the center of the wall movement period being considered and having connected together two remnants, we will obtain again the integral part of the trajectory laying on one wall movement period. This part of trajectory will correspond to another parameter of system and will include fewer collisions than the initial one. But the difference of collisions increasing and reducing speed at the part of the trajectory obtained by action of transformation f will be the same as at the initial part of the trajectory.

Considering the particle speed to be much higher than the wall one, it is possible to estimate the selected interval duration to be equal to the time of two particle collisions with a wall, i.e. $\frac{4L_{MAX}}{v_{MIN}}$, where L_{MAX} is the maximal distance from the wall to the system center; v_{MIN}, the particle minimal speed in the wall movement period being considered. After transformation f, L_{MAX} will decrease by $\Delta L_f = \frac{2L_{MAX}}{v_{MIN}}u$, where u is the wall movement speed.

It is possible to carry out the actions described above in the opposite sequence, i.e. to take the transformed part of the trajectory and to add two additional collisions into its center, in order to obtain the initial part of the trajectory. Such a transformation $f{-}1$ will be opposite to transformation f. The transformation g is similar to $f{-}1$, except for that it adds two collisions not to the center, but to the onset of the wall movement period. Since one collision increasing the particle speed and one reducing it are added, transformation g as well as f keeps the difference between collisions increasing and reducing the particle speed. The particle speed before the first speed-reducing collision at the transformed trajectory part will be double wall speed higher than the similar speed in the initial trajectory part.

Applying repeatedly the transformations f and g on an initial part of the trajectory, we will obtain parts of trajectories corresponding to various system parameters. For any system parameter δ, it is possible to find such m and n values that the trajectory part obtained by transformation $f(m)og(n)$ will correspond to the same system parameter δ as the initial one. Let us consider the part of the system trajectory laying on one wall movement period. Each action of transformation f reduces the maximum distance from the wall to the system center L_{MAX} by ΔL_f. Each action of transformation g reduces the minimum distance from the wall to the system center by $\Delta L_g = \frac{2L_{MIN}}{v_{MAX}}u$ (Fig.4).

If the transformed part of the trajectory corresponds to the same system parameter δ, as the initial one, then:

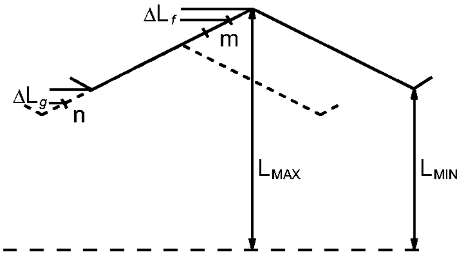

Fig. 4. One period of wall movement is shown. before (solid line) and after (dashed line) F transformation.

$$\delta=\frac{L_{MAX}-L_{MIN}}{L_{MAX}+L_{MIN}}=\frac{(L_{MAX}-m\Delta L_f)-(L_{MIN}-n\Delta L_g)}{(L_{MAX}-m\Delta L_f)+(L_{MIN}-n\Delta L_g)}$$

from where it follows:

$$\frac{L_{MAX}}{L_{MIN}}=\frac{n\Delta L_g}{m\Delta L_f}$$

Substituting ΔL_f and ΔL_g, we get:

$$\frac{m}{n}=\frac{v_{MIN}}{v_{MAX}}$$

Before, we have obtained that $v=\frac{const}{L}$, wherefrom $L_{MIN}v_{MAX}=L_{MAX}v_{MIN}$, and since $L_{MIN}=\frac{1}{\delta}-1$ and $L_{MAX}=\frac{1}{\delta}+1$, then

$$\frac{m}{n}=\frac{1-\delta}{1+\delta}$$

Thus, having taken the minimal natural numbers m and n, satisfying the obtained formula, we obtain the transformation F to be sought. The trajectory part resulting from transformation F on the initial part will have the same difference between the number of speed-increasing and speed-reducing collisions as the initial one; it will correspond to the same system parameter δ as the initial trajectory part; it will have the first collision speed exceeding by $2n$ wall speed than that at the initial part of the trajectory. The number of collisions will be larger by $2(n-m)$ than at the initial trajectory part.

It is easy to see that if we worked with the values having a dimensiality, the wall movement period would change. The new period would be equal $T_1=T-\Delta T$. However, as we have taken the wall movement period as a unit time, the wall movement period after transformation F, as well as prior to it, is equal to one. The collisions phases at the transformed trajectory part thus should be measured in the new wall movement periods. So, if the first collision phase before transformation F was ξ_0, after transformation it will be equal to $\xi_0'=\xi_0\frac{T}{T_1}=\xi_0\frac{T}{T-\Delta T}$. Now let the initial trajectory part be supposed to possess the following property: the first collision has coordinates (ξ_0,v_0), and one of following collisions has coordinates $(\xi_N=\xi_0\frac{T}{T-\Delta T},v_n=v_0)$. In other words, coordinates of the first collision of the initial trajectory part after transformation coincide with coordinates of another collision of the initial trajectory part. If coordinates of a collision from one trajectory part coincide completely with those from other part of the trajectory, then coordinates of all previous and subsequent collisions coincide also, i.e. the considered parts of trajectories are different parts of the same trajectory. Let the transformed trajectory part be subjected once again by transformation F. The collision with coordinates (ξ_0',v_0) will pass to $(\xi_o'\frac{T-\Delta T}{T-2\Delta T},v_0)=(\xi_n\frac{T-\Delta T}{T-2\Delta T},v_0)$. It is easy to see that coordinates of this collision coincide to within infinitesimal values of the order of ΔT^2 with coordinates $(\xi_n\frac{T}{T-\Delta T},v_0)$ of unitary transformed collision (ξ_n,v_0). Thus, if coordinates of unitary transformed collision (ξ_0,v_0) coincide with coordinates of collision (ξ_n,v_0), then coordinates of twice transformed collision (ξ_0,v_0) coincide with coordinates of one-fold transformed collision (ξ_n,v_0), etc. It means, that all parts of the trajectory obtained by transformation F on the initial part of the trajectory belong to the same trajectory as the initial part, forming a unique trajectory.

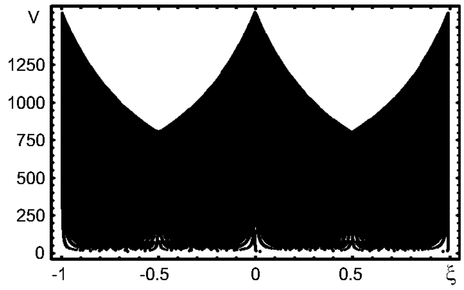

Fig. 5. Phase portrait of infinitely growing trajectory with the greatest possible acceleration. ξ_0=0.0002, v_0=134.0, $\delta = \frac{1}{3}$, N=200000

This proof cannot be considered to be mathematically strict, however, it provides understanding how infinitely growing trajectories are constructed. Besides, the laws obtained in this way are proved to be true by numerical calculations. So, for example, the calculated size the particle speed gain for one period of trajectory increase coincides completely with observed one.

Let us exemplicify the use of such transformation to search for an accelerating trajectory. Let the parameter of system δ be equal to $\frac{1}{3}$. It is easy to see that the system parameter will not be changed by the combination of one transformation f_{-2} and two transformations f_{+2}. Thus, at the specified parameter, an infinitely growing trajectory increases its speed by 4for one wall period. Let us consider a trajectory with the initial phase close to one. This trajectory will be an infinitely growing one if after a number of collisions again will hit the wall at a collision phase close to one, but at speed exceeding the initial one by 4. It is easy to see that two speed-increasing collisions of the particle are enough for this purpose. For the initial particle speed of V_0, we have:

$$\xi_1 = \frac{\frac{\delta+1}{2} - \xi_1}{V_0 + 2}$$

$$1 - \xi_1 = \frac{\frac{\delta+1}{2} - \xi_1}{V_0 + 4}$$

Having solved this system of equations, we get $V_0 = 0$. Thus, if the system parameter $\delta = \frac{1}{3}$, the trajectory with the initial phase $\xi_0 = 0.9999$ and initial speed $V_0 = 0, 4, 8...$ will be infinitely growing, and for each wall movement period the particle will increase its speed by 4. The phase portrait of this trajectory is shown in Fig. 5. This is most fast-growing trajectory of all infinitely Growing ones.

It is similarly possible to consider the case at other parameters of the system. For example, if $\delta = \frac{1}{4}$, the system parameter is not changed by the combination of three transformations of the first type and five transformations of the second type, and an infinitely growing trajectory increases its speed by 10 per period. For this purpose, the trajectory period should be not less than three the wall movement periods; accordingly, the greatest acceleration of the trajectory possible at the specified parameter is equal $\frac{10}{3}$. Having made and solved the system of 17 equations corresponding to such trajectory, it is possible to obtain the phase and speed of the initial collision for this trajectory.

Similarly, it is possible to obtain the maximum accelerations values for the particle at various values of δ.parameter

Let us consider the limiting case when the parameter δ is close to 1. Physically, it means that walls are closely approach to one another during the period. This limiting case is opposite to often used limiting case when the wall movement amplitude is neglected in comparison with distance between walls.

Neglecting the difference between the wall oscillation amplitude and distance between the walls, i.e. considering $\delta = 1$ and assuming the particle speed to be high enough, we obtain a simplified mapping:

$$\begin{aligned} u_{n+1} &= u_n + 2 \\ \xi_{n+1} &= \xi_n + 2\frac{1-\xi_n}{u_{n+1}} \end{aligned} \tag{3}$$

It is easy to see that the mapping (3) has an invariant

$$I_n = u_n(1 - \xi_n) = const \tag{4}$$

Let us consider the value of this invariant on $n+1$-th step and transform it using the mapping (3):

$$u_{n+1}(1-\xi_{n+1}) = (u_n+2)(1-\xi_n - 2\frac{1-\xi_n}{u_n+2}) = (1-\xi_n)(u_n+2)(1-\frac{2}{u_n+2}) =$$

$$= (1-\xi_n)(u_n+2)(\frac{u_n}{u_n+2}) \equiv u_n(1-\xi_n)$$

Thus,

$$I_{n+1} = I_n$$

and this value is preserved at iterations. Hence, each orbit of the mapping is evolved remaining in the curve.

$$u_n = \frac{const}{(1-\xi_n)} \tag{5}$$

Here, the value $const = u_0(1-\xi_0)$ is defined by initial conditions. It means that in this limiting case, the mode of explosive acceleration of the particle is observed. Therefore, in this limiting case, the system phase space is stratified by invariant curves along which all trajectories move. At $\delta \neq 1$, I_n is not the exact invariant of the mapping, but gets the sense of adiabatic invariant.

The considered system can be described not only exactly, but also using high-energy approximation. In this approach, the particle speed is considered high enough to neglect the effects observed at low speeds, such as double collisions. The distance between walls is considered to be large enough, so it is possible to neglect displacement of the wall in comparison with it. In this approximation, the wall speed is considered to be the same as in an exact case. Under this approach, the mapping describing the system is essentially simpler as compared to the exact one, and has the form

$$v_{n+1} = \left| v_n + 2sign\left(|\xi_n| - \frac{1}{2}\right)\right|$$

$$\xi_{n+1} = -\left(\xi_n + sign(\xi_n)\frac{2}{\delta v_{n+1}}\right) \pmod 1$$

This mapping, despite its simplicity, has an essential Shortcoming that hinders its use. During evolution, the particle can lower its speed, so that the system can fall outside the limits of the mapping applicability. In advance, it is not known, whether the particle will descend to inadmissibly low speeds or not. Therefore, to neglect the displacement of the wall is not trivial and demands an additional comparative analysis.

Cosidering trajectories under high-energy approach, it is possible to establish that the main difference from behavior of trajectories in exact mapping is reduced to disappearance of trajectories with infinitely increasing speed. So, if to take the initial data corresponding in exact mapping to an infinitely growing trajectory, that data in high-energy approach will correspond to a limited trajectory.

This conclusion remains its validity for all trajectories under high-energy approximation. In other words, in high-energy approximation, all trajectories are limited. This circumstance demands a cautious approach to use of high-energy approximation. The physical reason of presence of such fundamental distinction is connected with preservation of adiabatic invariant (similar to (4)) in high-energy approach and its infringement in the exact one.

Thus, the particle speed gain is considered at the walls movement being continuous and periodic, but not smooth. The one-parametrical family of exact mappings for description of particle behavior in such systems has been obtained. The main characteristics of particles in such systems have been studied. The existence of fractal set of initial conditions at which of the particle speed increases infinitely has been demonstrated. The absence of of particle speed limit in such trajectories has been shown analytically.

Dependence of the particle average acceleration factor on the system parameter is analyzed. The mode of «superacceleration» of particles in some area of system parameters is revealed. In this mode, an explosive particle speed increase is observed. Comparison with high-energy approximation is carried out. Qualitative difference in the particle behavior in comparison with exact mapping is shown. So, in high-energy approach, there are no trajectories with unlimited growth of speed.

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Мінімальна модель Фермі

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У роботі розглянута проста модель Фермі прискорення частинки між двома стінками, які періодично коливаються. Закон руху стінок є неперевным, але не гладким. Отримано точне відображення для цієї системи. Доведено існування фрактальної множини траекторій зі швидкістю, яка необмежено зростає. Розглянуто головні характеристики таких траєкторій. Проведено порівняння з високоенергетичним наближенням. Знайдено якісні відмінності у поведінці точного та наближенного описів. Так у високоенергетичному наближенні відсутні траекторії з необмеженим зростанням швидкості.