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SMALL-TIME LIMIT BEHAVIOR OF THE PROBABILITY THAT A LÉVY PROCESS STAYS POSITIVE

Abstract. In the paper, we find analytically the upper and lower limits (as the time parameter tends to zero) of the probability that the Lévy process staring at 0 stays positive. We confine ourselves to the situation where the real and imaginary parts of the characteristic function are regularly varying at infinity. In this case, we can calculate the bound, and sometimes the exact values of the respective upper and lower limits.

Keywords: Lévy process, probability measure, limit behaviour in small time.

Let X be a Lévy process with the characteristic exponent

$$\psi(s) = ibs + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{-isu} - isu\,\mathbb{I}_{|u|\le 1})\mu(du). \tag{1}$$

We are interested in exact values

$$\rho_{\inf} := \liminf_{T\to 0} \mathbb{P}\{X_T > 0\} \text{ and } \rho_{\sup} := \limsup_{T\to 0} \mathbb{P}\{X_T > 0\}. \tag{2}$$

If the values in (2) coincide, i.e.,

$$\mathbb{P}\{X_T > 0\} \to \rho \in [0,1],\ T \to 0, \tag{3}$$

it is known (cf. [1, Theorem 2]) that (3) is equivalent to

$$\frac{1}{T}\int_0^T \mathbb{P}\{X_s > 0\}\,ds \to \rho,\ T \to 0. \tag{4}$$

The latter is called the Spitzer condition for X as $T \to 0$.

In the case of infinite Lévy measure and $M(0+) > 0$ the necessary and sufficient condition for $\rho = 1$ are given in [2] and [3].

The value ρ appearing in (3) is related to the ladder time process, namely, to the inverse τ of the local time L of the process, reflected at its supremum on time interval $(0, t)$. More precisely, τ is the subordinator, and the value $t^{-1}\rho(t)$, where

$$\rho(t) := \int_0^t \mathbb{P}\{X_s > 0\}\,ds,$$

is the Lévy measure of the appearing in the representation of its characteristic exponent, namely,

$$\mathbb{E}e^{-\lambda\tau} = \exp\left\{-\int_0^\infty (1 - e^{-\lambda s}) s^{-1}\rho(s)\,ds\right\}.$$

The limit (3) exists iff τ belongs to the domain of attraction (as $t \to 0$) of the ρ-stable distribution (see [4, p. 31]). Nevertheless, although the result is sharp, it does not give a clue how to calculate the value of ρ.

In this note we show that in some cases the value of $\rho \in (0,1)$ can be calculated explicitly. Our approach was inspired the paper [5], in which some particular case of this problem was considered. The method used in [5] used complex integration and analyticity of the Lévy exponent in some strip. The aim of this note is to show that this method can be extended to all Lévy processes. Since $\mathbb{P}\{M_t > c\} \to 0$ as $t \to 0$ for any $c>0$, where $M_t := \sup_{s\le qt} |X_s|$, it is enough to consider the part of the process, which has bounded jumps. Thus, in what follows we assume that $\operatorname{supp} \mu = [-1,1]$.

Denote

$$f(v) := \operatorname{Re} \psi(v), \; g(v) := bv + \operatorname{Im} \psi(v). \tag{5}$$

$$c_{f,-} v^{\alpha_-} \le \liminf_{\lambda\to\infty} \frac{f(v\lambda)}{f(\lambda)} \le \limsup_{\lambda\to\infty} \frac{f(v\lambda)}{f(\lambda)} \le c_{f,+} v^{\alpha_+}, \; v \ge 1, \tag{6}$$

$$c_{g,-} v^{\beta_-} \le \liminf_{\lambda\to\infty} \frac{g(v\lambda)}{g(\lambda)} \le \limsup_{\lambda\to\infty} \frac{g(v\lambda)}{g(\lambda)} \le c_{g,+} v^{\beta_+}, \; v \ge 1. \tag{7}$$

Theorem 1. Let X be a Lévy process with Lévy exponent (1). Assume that $\mu(\mathbb{R}_+) > 0$. Then

a) if $\alpha_- > \beta_+$, then $\rho = \rho_{\inf} = \rho_{\sup} = 1/2$. This case clearly illustrates the fact that “symmetry wins”;

b) suppose

$$\alpha_\pm = \beta_\pm = \alpha, \; c_{f,\pm} = c_f, \; c_{g,\pm} = c_g, \tag{8}$$

then

$$\rho_{\sup} = \rho_{\inf} = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan \frac{c_g}{c_f}; \tag{9}$$

b′) suppose that

$$\alpha_\pm = \beta_\pm = \alpha, \text{ but } c_{f,+} \ne c_{f,-}, \; c_{g,+} \ne c_{g,-},$$

then

$$\rho_{\inf} \ge \frac{1}{2} - \frac{1}{\pi\alpha} \frac{c_{g,-}}{c_{f,+}}, \; \rho_{\sup} \le \frac{1}{2} + \frac{1}{\pi\alpha} \frac{c_{g,+}}{c_{f,-}};$$

c) if $\alpha_+ < 1$, then $\rho = \rho_{\inf} = \rho_{\sup}$, and is equal 1 or 0.

Remark 1. 1. In fact, case b) corresponds to the situation when the distribution of X_1 is in the zone of attraction of the α-stable law Y (cf. [6]) with the sqewness parameter $\beta := \frac{1}{\tan(\pi\alpha/2)} \frac{c_g}{c_f}$, i.e. $\mathbb{E} e^{i\xi Y} = e^{-c_f |\xi|^\alpha (1 - i\beta \operatorname{sign} \xi)}$. In this case

$$\rho = 1 - G(0) = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right),$$

where $G(x)$ is the distribution function of Y; this result is known (see [7, Chapter 8.9.2], also [6].

2. Of course, case b′) is informative only of $\rho_{\inf} > 0$, $\rho_{\sup} < 1$.

Remark 2. Intuitively, statement c) is obvious: when the non-symmetric part dominates, the value ρ should be equal 0 or 1. Indeed, when $\alpha_+ < 1$, then $\int_{-1}^{1} |u|^{\alpha_+} \mu(du) < \infty$ (see [8]). This implies that $\int_{-1}^{1} |u| \mu(du) < \infty$, and thus c) is the particular case of the Doney and Maller’s result. Our proof is analytic, and completely different.

Proof of Theorem 1. Let

$$M(s,T):=e^{-T\psi(is)}=\exp\left\{Tbs+T\int_{-1}^{1}(e^{su}-1-su)\mu(du)\right\},\ T>0, \tag{10}$$

and

$$\phi(s):=bs+\int_{-1}^{1}(e^{su}-1-su)\mu(du). \tag{11}$$

Observe that $\operatorname{Im} M(\bar{z},T)=-\operatorname{Im} M(z,T)$, implying that $\operatorname{Im}\frac{M(\bar{z},T)}{\bar{z}}=\operatorname{Im}\frac{M(z,T)}{z}$. Then for any $a>0$ we have

$$\mathbb{P}[X_T\ge 0]=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{s}M(s,T)\,ds=\operatorname{Im}\frac{1}{\pi}\int_{a}^{a+i\infty}\frac{1}{s}M(s,T)ds=$$

$$=\frac{1}{\pi}\operatorname{Im}\left(i\int_0^\infty\frac{1}{a+iv}M(a+iv,T)\,dv\right)=\frac{1}{\pi}\operatorname{Re}\left(\int_0^\infty\frac{1}{a+iv}M(a+iv,T)\,dv\right)=$$

$$=\frac{1}{\pi}\operatorname{Re}\left(\int_0^\infty\frac{a-iv}{a^2+v^2}M(a+iv,T)\,dv\right)=$$

$$=\frac{1}{\pi}\int_0^\infty\frac{a\operatorname{Re} M(a+iv,T)-v\operatorname{Im} M(a+iv,T)}{a^2+v^2}dv=$$

$$=\frac{1}{\pi}\int_0^\infty\frac{ae^{T\operatorname{Re}\phi(a+iv)}\cos(T\operatorname{Im}\phi(a+iv))}{a^2+v^2}dv+$$

$$+\frac{1}{\pi}\int_0^\infty\frac{ve^{T\operatorname{Re}\phi(a+iv)}\sin(T\operatorname{Im}\phi(a+iv))}{a^2+v^2}dv=:I_1(a,T)+I_2(a,T).$$

Here

$$\phi(a+iv)=b(a+iv)+\int_{-1}^{1}(e^{(a+iv)u}-1-(a+iv)u)\mu(du)=$$

$$=ba+\int_{-1}^{1}(e^{au}\cos vu-1-au)\mu(du)+i\left(bv+\int_{-1}^{1}(e^{au}\sin vu-vu)\mu(du)\right)=:$$

$$=:\operatorname{Re}\phi(a+iv)+i\operatorname{Im}\phi(a+iv).$$

Note that since $\operatorname{supp}\mu=[-1,1]$, then for any $a>0$ the function $\phi(a+iv)$ is well defined, and

$$\operatorname{Re}\phi(a+iv)=\phi(a)-\int_{-1}^{1}(1-\cos vu)\mu_a(du),$$

where

$$\mu_a(du):=e^{au}\mu(du). \tag{12}$$

By the dominated convergence theorem, we can pass to the limit as $T\to 0$ in the first integral:

$$I_1(a,T)\to\frac{1}{\pi}\int_0^\infty\frac{a}{a^2+v^2}dv=\frac{1}{2},\ T\to 0. \tag{13}$$

Consider now $I_2(a,T)$. Put

$$f_a(v) := \int_{-1}^{1} (1-\cos uv)\mu_a(du),$$

$$g_a(v) := \operatorname{Im}\phi(a+iv) = bv + \int_{-1}^{1} (e^{au}\sin vu - vu)\mu(du). \qquad (14)$$

Clearly, $f_0(v) = f(v)$, $g_a(v) = g(v)$. Since

$$g_a(v) = \left(b + \int_{-1}^{1} (e^{au}-1)u\mu(du) \right) v + o(v^2), \ v \to 0,$$

by the dominated convergence theorem we can pass to the limit as $a \to 0$:

$$\lim_{a\to 0} I_2(a,T) = J(T) := \frac{1}{\pi}\int_0^\infty e^{-Tf(v)} \frac{\sin(Tg(v))}{v} dv. \qquad (15)$$

Thus, if the limits $\lim_{a\to 0}$ and $\liminf_{T\to 0}$ or $\limsup_{T\to 0}$ are interchangeable, it is enough to calculate the limit $\lim_{T\to 0} J(T)$.

From now we consider the cases a), b), and c) separately.

In case a) we have by (7) the estimate $g(v) \le Cv^{\beta_+}$ for $v \ge 1$. Then, by the dominated convergence theorem we get

$$\limsup_{T\to 0} J_T = \limsup_{T\to 0} \pi^{-1} \int_0^\infty e^{-Tf(v)} \frac{\sin Tg(v)}{v} dv =$$

$$= \pi^{-1} \limsup_{T\to 0} \left[\int_0^{T^{1/\alpha_-}} + \int_{T^{1/\alpha_-}}^\infty \right] e^{-Tf(vT^{-1/\alpha_-})} \frac{\sin Tg(vT^{-1/\alpha_-})}{v} dv =$$

$$= \pi^{-1} \limsup_{T\to 0} \int_{T^{1/\alpha_-}}^\infty e^{-Tf(vT^{-1/\alpha_-})} \frac{\sin Tg(vT^{-1/\alpha_-})}{v} dv \le$$

$$\le C\pi^{-1} \limsup_{T\to 0} \int_{T^{1/\alpha_-}}^\infty e^{-Tf(vT^{-1/\alpha_-})} T^{1-\beta^+/\alpha_-} v^{\beta_+ - 1} dv = 0. \qquad (16)$$

In case b) denote by (T_k) the sequence on which the supremum in $\limsup_{T\to 0} J_T$ is achieved:

$$\limsup_{T\to 0} J_T = \lim_{k\to\infty} \pi^{-1} \int_0^\infty e^{-T_k f(v)} \frac{\sin T_k g(v)}{v} dv. \qquad (17)$$

Then by (8) and the dominated convergence theorem we get

$$\limsup_{T\to 0} J_T = \pi^{-1} \int_0^\infty \lim_{k\to\infty} e^{-T_k f(T_k^{-1/\alpha} v)} \frac{\sin T_k g(T_k^{-1/\alpha} v)}{v} dv =$$

$$= \pi^{-1} \int_0^\infty e^{-c_f v^\alpha} \frac{\sin(c_g v^\alpha)}{v} dv = \frac{1}{\alpha\pi} \arctan \frac{c_g}{c_f}.$$

In case b′) proceeding as in case b), we get by the dominated convergence theorem

$$\limsup_{T\to 0} J_T = \pi^{-1}\int_0^\infty \lim_{k\to\infty} e^{-T_k f(T_k^{-1/\alpha} v)} \frac{\sin T_k g(T_k^{-1/\alpha} v)}{v} dv \le$$

$$\le \pi^{-1}\int_0^\infty e^{-c_{f,-} v^\alpha} c_{g,+} v^{\alpha-1} dv = \frac{1}{\alpha\pi}\frac{c_{g,+}}{c_{f,-}}.$$

The argument for the $\liminf_{T\to 0} J_T$ is similar.

In case c) since α_+ is the upper index, we can write $f(v)$ as $f(v) = v^{\alpha_+}(1+\phi_f(v))$, where $\phi_f(v)$ is bounded. Since we assumed that $\int_{-1}^{1} u\mu(du)$ is finite, then $\beta_+ = 1$, and $\alpha_- < 1$. Then we can write $g(v) = \tilde{b} v(1+\phi_g(v))$, where $\phi_g(v) \to 0$ as $v \to \infty$. Making the change of variables $v = T^{-1}u$ we get

$$J(T) = \frac{1}{\pi}\int_0^\infty e^{-T^{1-\alpha_+} u^{\alpha_+}(1+\phi_f(T^{-1}u))} \frac{\sin(u+\phi_g(T^{-1}u))}{u} du.$$

Observe that $e^{-t\lambda^\alpha}$ is the Laplace transform of the transition probability density of an α-stable subordinator:

$$e^{-t\lambda^\alpha} = \int_0^\infty e^{-\lambda x} p_t(x) dx. \tag{18}$$

Then using the Fubini theorem we get

$$J(T) = \int_0^\infty\int_0^\infty e^{-T^{1-\alpha_+}\phi_f(T^{-1}u)} e^{-ux} \frac{\sin(u+\phi_g(T^{-1}u))}{u} p_{T^{1-\alpha_+}}(x)\, dudx =$$

$$= \int_0^\infty p_{T^{1-\alpha_+}}(x) V_T(x) dx,$$

where

$$V_T(x) = \frac{1}{\pi}\int_0^\infty e^{-T^{1-\alpha_+}\phi_f(T^{-1}u)} e^{-ux} \frac{\sin \tilde{b}(u+\phi_g(T^{-1}u))}{u} du.$$

Note that by the dominated convergence theorem we get

$$\lim_{T\to 0} V_T(x) \to V(x) = \frac{1}{\pi}\arctan\frac{\tilde{b}}{x},$$

uniformly in x (because $V_T(x)$ is bounded and continuous). Therefore,

$$\lim_{T\to 0}\int_0^\infty p_{T^{1-\alpha_+}}(x) V_T(x) dx = V(0) = \frac{1}{2}.$$

□

The theorem is proved.

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ПРО ГРАНИЧНУ ПОВЕДІНКУ У МАЛОМУ ЧАСІ ЙМОВІРНОСТІ ЗНАХОДЖЕННЯ ПРОЦЕСУ ЛЕВІ НА ДОДАТНІЙ ПІВОСІ

Анотація. Розглянуто аналітичний метод знаходження верхньої та нижньої границь при часовому параметрі, що прямує до нуля, ймовірності того, що процес Леві, який стартує з нуля, залишається на додатній півосі. Досліджено тільки випадок, коли уявна та дійсна частини характеристичної експоненти регулярно змінюються на нескінченності. У цьому випадку знайдено оцінки, а у деяких випадках і точні значення розглянутих границь.

Ключові слова: процес Леві, ймовірнісна міра, гранична поведінка у малому часі.

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О ПРЕДЕЛЬНОМ ПОВЕДЕНИИ В МАЛОМ ВРЕМЕНИ ВЕРОЯТНОСТИ НАХОЖДЕНИЯ ПРОЦЕССА ЛЕВИ НА ПОЛОЖИТЕЛЬНОЙ ПОЛУОСИ

Аннотация. Рассмотрен аналитический метод нахождения верхней и нижней границ при временном параметре, стремящемся к нулю, вероятности того, что процесс Леви, стартующий из нуля, остается на положительной полуоси. Рассмотрен только случай, когда действительная и мнимая части характеристической экспоненты регулярно меняются на бесконечности. В этом случае найдены оценки, а в некоторых случаях и точные значения вышеназванных верхней и нижней границ.

Ключевые слова: процесс Леви, вероятностная мера, предельное поведение в малом времени.

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