NON-COMMUTATIVE GEOMETRY & PHYSICS[1]

M. WOHLGENANNT

Institute for Theoretical Physics, Vienna University of Technology
(8-10, Wiedner Hauptstrasse, Vienna A-1040, Austria; e-mail:
michael.wohlgenannt@univie.ac.at)

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This article is an introduction to the ideas of non-commutative geometry and star products. We will discuss consequences for physics in two different settings: quantum field theories and astrophysics. In case of quantum field theory, we will discuss two recently introduced models in detail. Astrophysical aspects will be discussed, by considering modified dispersion relations.

1. Why Non-Commutativity?

Non-commutative spaces have a long history. Even in the early days of quantum mechanics and quantum field theory, the continuous space-time and the Lorentz symmetry were considered inappropriate to describe the small-scale structure of the Universe [1]. It was also argued that one should introduce a fundamental length scale limiting the precision of position measurements. In [2, 3], the introduction of a fundamental length was suggested to cure the ultraviolet divergences occurring in quantum field theory. H. Snyder was the first who formulated these ideas mathematically [4]. He introduced non-commutative coordinates. Therefore, a position uncertainty arises naturally. The success of the renormalization program made people forget about these ideas for some time. But when the quantization of gravity was considered thoroughly, it became clear that the usual concepts of space-time are inadequate, and the space-time has to be quantized or non-commutative, in some way.

There is a deep conceptual difference between quantum field theory and gravity: The space and the time are considered as parameters in the former and as dynamical entities in the latter. In order to combine quantum theory and gravitation (geometry), one has to describe both in the same language, this is the language of algebras [5]. Geometry can be formulated algebraically in terms of Abelian $\mathbf{C}^*$ algebras and can be generalized to non-Abelian $\mathbf{C}^*$ algebras (non-commutative geometry). The quantized gravity can even act as a regulator of quantum field theories. This is encouraged by the fact that a non-commutative geometry introduces a lower limit for the precision of position measurements. There is also a very nice argument showing that, on the classical level, the self-energy of a point particle is regularized by the presence of gravity [6]. Let us consider an electron and a shell of radius ϵ around the electron. The self-energy of the electron is the self-energy of the shell $m(\epsilon)$, in the limit $\epsilon \to 0$. The quantity $m(\epsilon)$ is given by

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon},$$

where m_0 and e are, respectively, the rest mass and the charge of an electron. In the limit $\epsilon \to 0$, $m(\epsilon)$ will diverge. Including the Newtonian gravity, we have to modify this equation,

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{Gm_0^2}{\epsilon},$$

where G is Newton's gravitational constant. The self-energy $m(\epsilon)$ will still diverge for $\epsilon \to 0$, unless the mass and the charge are finely tuned. Considering general relativity, we know that the energy, particularly the energy of electron's electric field, is a source of the gravitational field. Again, we have to modify the above equation,

$$m(\epsilon) = m_0 + \frac{e^2}{\epsilon} - \frac{Gm(\epsilon)^2}{\epsilon}.$$

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The solution of this quadratic equation is straightforward:

$$m(\epsilon) = -\frac{\epsilon}{2G} \pm \frac{\epsilon}{2G}\sqrt{1 + \frac{4G}{\epsilon}(m_0 + \frac{e^2}{\epsilon})}.$$

We are interested in the positive root. Miraculously, the limit $\epsilon \to 0$ is finite,

$$m(\epsilon \to 0) = \frac{e}{\sqrt{G}}.$$

This is a non-perturbative result, since $m(\epsilon \to 0)$ cannot be expanded around $G = 0$. The quantity $m(\epsilon \to 0)$ does not depend on m_0; therefore, there is no fine tuning. Classical gravity regularizes the self-energy of an electron on a classical level. However, this does not make the quantization of space-time unnecessary, since quantum corrections to the above picture will again introduce divergences. But it provides an example for the regularization of physical quantities by introducing gravity. So the hope is raised that the introduction of gravity formulated in terms of a non-commutative geometry will regularize physical quantities even on the quantum level.

On the other hand, there is the old simple argument that a smooth space-time manifold contradicts quantum physics. If one localizes an event within a region of extension l, an energy of the order of hc/l is transferred. This energy generates a gravitational field. A strong gravity field prevents, on the other hand, signals to reach an observer. Inserting the energy density into Einstein's equations gives a corresponding Schwarzschild radius $r(l)$. This provides a limit on the smallest possible l, since it is certainly operationally impossible to localize an event beyond this resulting Planck length. To the best of our knowledge, the first time this argument was cast into precise mathematics was in the work by Doplicher, Fredenhagen, and Roberts [7]. They obtained what is now called the canonical deformation but averaged over 2-spheres. At which energies this transition to discrete structures might take place, or at which energies the non-commutative effects occur is a point much debated on.

From various theories generalized to non-commutative coordinates, limits on the non-commutative scale have been derived. These generalizations have mainly considered the so-called canonical non-commutativity,

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij},$$

$\theta^{ij} = -\theta^{ji} \in \mathbb{C}$. Let us name a few estimates of the non-commutativity scale. A very weak limit on the non-commutative scale Λ_{NC} is obtained from an additional energy loss in stars due to the coupling of neutral neutrinos to photons, $\gamma \to \nu\bar{\nu}$ [8]. They get

$$\Lambda_{NC} > 81 \text{ GeV}.$$

The estimate is based on the argument that, within any new mechanism, the energy losses must not exceed the standard neutrino losses from the Standard Model by much. A similar limit is obtained in [9] from the calculation of the energy levels of a hydrogen atom and the Lamb shift within non-commutative quantum electrodynamics,

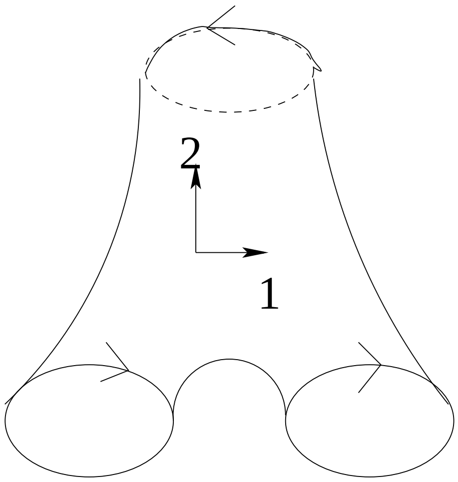
$$\Lambda_{NC} \gtrsim 10^4 \text{ GeV}.$$

If $\Lambda_{NC} = \mathcal{O}(\text{TeV})$, measurable effects may occur for the anomalous magnetic moment of a muon which may account for the reported discrepancy between the Standard Model prediction and the measured value [10]. In cosmology and astrophysics, non-commutative effects might be observable. One suggestion is that the modification of a dispersion relation due to the ($\kappa-$)non-commutativity can explain the time delay of high-energy γ rays, e.g., from the active galaxy Makarian 142 [11, 12]. We will discuss this point in Section 4. A brief introduction to non-commutative geometry will be provided in Section 2. In Section 3, we consider the quantum field theory on non-commutative spaces. We will put emphasis on scalar field theories and will only briefly discuss the case of gauge theories.

2. Some Basic Notions of a Non-Commutative Geometry

At the present time, there are three major approaches tackling the problem of quantizing gravity: String Theory, Quantum Loop Gravity, and Non-Commutative Geometry. Before we discuss some basic concepts of non-commutative geometry, let us state some advantages and disadvantages of the other theories, cf. [13]. Background independence will be a major issue. General Relativity can be described in a coordinate-free way. In some cases, theories for gravity are expanded around the Minkowski metric. They explicitly depend on the background Minkowski metric, i.e., the background independence is violated.

In String Theory, the basic constituents are 1-dimensional objects, strings. The interaction between strings can be symbolized by two-dimensional Riemann manifolds with boundary, e.g., a vertex:



The interaction region is not a point anymore. Hence, there is also the hope for that the divergences in the perturbation theory of quantum field theory are not present.

Advantages	Disadvantages
graviton contained in the particle spectrum	higher dimensions needed: superstrings $D = 10, 11$ bosonic string $D = 26$
black hole entropy	dependence on background space-time geometry
mathematical beauty (dualities, ...)	many free parameters and string-vacua
	almost no predictions

Quantum Loop Theory studies the canonical quantization of General Relativity in 3+1 dimensions.

Advantages	Disadvantages
background independence	very few predictions
quantized area operator	no matter included
$3+1$ dimensional space-time	technical difficulties

We are going to discuss the third approach in more details in the next subsection. The three approaches are connected to one another. In [14], the connection between κ-deformation and quantum loop gravity is studied. The authors conclude that the low-energy limit of quantum loop gravity is a κ-invariant field theory. This is a far reaching result which deserves a lot of attention. Also String Theory is related to certain non-commutative field theories in the limit of the vanishing string coupling [15]. A better understanding of the interrelations will provide clues how a proper theory of quantum gravity should look like.

2.1. Non-commutative geometry

In our approach, we consider a non-commutative geometry as a generalization of quantum mechanics. Thereby, we generalize the canonical commutation relations of the phase space operators $\hat{x}^i$ and $\hat{p}_j$. Most commonly, the commutation relations are chosen to be either constant or linear, or quadratic in the generators. In the canonical case, the relations are constant,

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{1}$$

where $\theta^{ij} \in \mathbb{C}$ is an antisymmetric matrix, $\theta^{ij} = -\theta^{ji}$. The linear or Lie algebra case

$$[\hat{x}^i, \hat{x}^j] = i\lambda_k^{ij}\hat{x}^k, \tag{2}$$

where $\lambda_k^{ij} \in \mathbb{C}$ are the structure constants, basically has been discussed in two different approaches, fuzzy spheres [16] and the κ-deformation [17–19]. Last but not least, we have the quadratic commutation relations

$$[\hat{x}^i, \hat{x}^j] = (\frac{1}{q}\widehat{R}^{ij}_{kl} - \delta^i_l\delta^j_k)\hat{x}^k\hat{x}^l, \tag{3}$$

where $\widehat{R}^{ij}_{kl} \in \mathbb{C}$ is the so-called $\widehat{R}$-matrix. For a reference, see, e.g., [20, 21]. The relations between coordinates and momenta (derivatives) can be constructed from the above relations in a consistent way [19, 22]. Most importantly, the usual commutative coordinates are recovered in a certain limit, $\theta^{ij} \to 0$, $\lambda^{ij}_k \to 0$ or $R^{ij}_{kl} \to 0$, respectively. In quantum mechanics, the commutation relations lead to the Heisenberg uncertainty,

$$\Delta x^i \, \Delta p_j \gtrsim \delta^i_j \frac{\hbar}{2}.$$

Similarily, we obtain an uncertainty relation for the coordinates in the non-commutative case, e.g.,

$$\Delta x^i \, \Delta x^j \gtrsim \frac{|\theta^{ij}|}{2}. \tag{4}$$

In a next step, we need to know which functions of the non-commutative coordinates are. Classically, the smooth functions can be approximated by power series. So, a function $f(x)$ can be written as

$$f(x) = \sum_I a_I \, (x^1)^{i_1}(x^2)^{i_2}(x^3)^{i_3}(x^4)^{i_4},$$

where $I = (i_1, .., i_4)$ is a multiindex, in a four-dimensional space. The commutative algebra of functions generated by the coordinates x^1, x^2, x^3, and x^4 is denoted by

$$\mathcal{A} = \frac{\mathbb{C}\langle\langle x^1, ..., x^4\rangle\rangle}{[x^i, x^j]} \equiv \mathbb{C}[[x^1, ..., x^4]], \tag{5}$$

i.e., $[x^i, x^j] = 0$. The generalization to the non-commutative algebra of functions $\widehat{\mathcal{A}}$ on a non-commutative space

$$\hat{\mathcal{A}} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ..., \hat{x}^n\rangle\rangle}{\mathcal{I}}, \tag{6}$$

where $\mathcal{I}$ is the ideal generated by the commutation relations of coordinate functions, can be found in (1-3). Again, an element $\hat{f}$ of $\hat{\mathcal{A}}$ is defined by a power series in the non-commutative coordinates. There is one complication: Since the coordinates do not commute, the monomials $\hat{x}^i\hat{x}^j$ and $\hat{x}^j\hat{x}^i$, e.g., are different operators. Therefore, we have to specify basis monomials with some care. This means that we have to give an ordering prescription. Let us discuss two different orderings briefly which will be denoted by : :. The normal ordering means the following:

$$\begin{aligned} :\hat{x}^i: &= \hat{x}^i,\ i = 1,2,3,4 \\ :\hat{x}^2\hat{x}^4\hat{x}^2\hat{x}^1: &= \hat{x}^1(\hat{x}^2)^2\hat{x}^4. \end{aligned} \tag{7}$$

Powers of $\hat{x}^1$ come first, then powers of $\hat{x}^2$, and so on. A non-commutative function is given by the formal expansion

$$\hat{f}(\hat{x}) = \sum_I b_I\ :(\hat{x}^1)^{i_1}(\hat{x}^2)^{i_2}(\hat{x}^3)^{i_3}(\hat{x}^4)^{i_4} :=$$

$$= \sum_I b_I\,(\hat{x}^1)^{i_1}(\hat{x}^2)^{i_2}(\hat{x}^3)^{i_3}(\hat{x}^4)^{i_4}. \tag{8}$$

A second choice is the symmetric ordering. There, we define

$$\begin{aligned} :\hat{x}^i: &= \hat{x}^i, \\ :\hat{x}^i\hat{x}^j: &= \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i), \\ :\hat{x}^i\hat{x}^j\hat{x}^k: &= \frac{1}{6}\Big(\hat{x}^i\hat{x}^j\hat{x}^k + \hat{x}^i\hat{x}^k\hat{x}^j + \hat{x}^j\hat{x}^i\hat{x}^k + \\ &\quad +\hat{x}^j\hat{x}^k\hat{x}^i + \hat{x}^k\hat{x}^i\hat{x}^j + \hat{x}^k\hat{x}^j\hat{x}^i\Big), \\ &\ \vdots \end{aligned} \tag{9}$$

A non-commutative function is given by the formal expansion

$$\hat{f}(\hat{x}) = \sum_I c_I\ :(\hat{x}^1)^{i_1}(\hat{x}^2)^{i_2}(\hat{x}^3)^{i_3}(\hat{x}^4)^{i_4}: .$$

symmetric ordering can also be achieved by exponentials,

$$e^{ik_i\hat{x}^i} = 1 + ik_i\hat{x}^i - \frac{1}{2}k_i\hat{x}^i k_j\hat{x}^j + \dots = 1 + ik_i\hat{x}^i -$$

$$-\frac{1}{2}(k_1\hat{x}^1 + \dots + k_4\hat{x}^4)(k_1\hat{x}^1 + \dots + k_4\hat{x}^4) + \dots, \tag{10}$$

and, therefore,

$$\hat{f}(\hat{x}) = \int d^4k\, c(k) e^{ik_i\hat{x}^i}, \tag{11}$$

with a coefficient function $c(k)$. This formula will be of vital importance in the next subsection.

The normal and symmetric orderings define different choices of a basis in the same non-commutative algebra $\hat{\mathcal{A}}$. Most importantly, many concepts of differential geometry can be formulated using the non-commutative function algebra $\hat{\mathcal{A}}$ such as differential structures.

In the following sections, we will concentrate on the first two cases of non-commutative coordinates, namely canonical (1) and κ-deformed (2) space-time structures.

2.2. Star product

Star products are a way to return to the familiar concept of commutative functions $f(x)$ within the non-commutative realm. In addition, we have to include a non-commutative product denoted by $*$. Earlier, we have introduced the algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ and have discussed the choice of a basis or an ordering in the latter. We need to establish an isomorphism between the non-commutative algebra $\hat{\mathcal{A}}$ and the commutative function algebra $\mathcal{A}$.

Let us choose symmetrically ordered monomials as a basis in $\hat{\mathcal{A}}$. We now map the basis monomials in $\mathcal{A}$ onto the according symmetrically ordered basis elements of $\hat{\mathcal{A}}$

$$\begin{aligned} W : \mathcal{A} &\to \hat{\mathcal{A}}, \\ x^i &\mapsto \hat{x}^i, \\ x^i x^j &\mapsto \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i) \equiv :\hat{x}^i\hat{x}^j: . \end{aligned} \tag{12}$$

The ordering is indicated by : :; W is an isomorphism of vector spaces. In order to extend W to an algebra isomorphism, we have to introduce a new non-commutative multiplication $*$ in $\mathcal{A}$. This $*$-product is defined by

$$W(f * g) := W(f) \cdot W(g) = \hat{f} \cdot \hat{g}, \tag{13}$$

where $f, g \in \mathcal{A}$, $\hat{f}, \hat{g} \in \hat{\mathcal{A}}$. Explicitly, we have

$$\begin{aligned} f(x) &= \sum_I a_I (x^1)^{i_1}(x^2)^{i_2}(x^3)^{i_3}(x^4)^{i_4} \\ &\| \\ &\| \quad \text{Quantization map W} \\ &\Downarrow \\ \hat{f}(\hat{x}) &= \sum_I a_I : (\hat{x}^1)^{i_1}(\hat{x}^2)^{i_2}(\hat{x}^3)^{i_3}(\hat{x}^4)^{i_4} : . \end{aligned}$$

The star product is constructed in the following way:

$$\begin{aligned} \hat{f}\cdot\hat{g} &= \sum_{I,J} a_I b_J : (\hat{x}^1)^{i_1}(\hat{x}^2)^{i_2}(\hat{x}^3)^{i_3}(\hat{x}^4)^{i_4} : \\ &: (\hat{x}^1)^{j_1}(\hat{x}^2)^{j_2}(\hat{x}^3)^{j_3}(\hat{x}^4)^{j_4} := \qquad (14) \\ &= \sum_K c_K : (\hat{x}^1)^{k_1}(\hat{x}^2)^{k_2}(\hat{x}^3)^{k_3}(\hat{x}^4)^{k_4} : , \qquad (15) \end{aligned}$$

where $\hat{g} = \sum_J b_J : (\hat{x}^1)^{j_1}(\hat{x}^2)^{j_2}(\hat{x}^3)^{j_3}(\hat{x}^4)^{j_4} :$. Consequently, we obtain

$$f * g\,(x) = \sum_J b_J (\hat{x}^1)^{j_1}(\hat{x}^2)^{j_2}(\hat{x}^3)^{j_3}(\hat{x}^4)^{j_4}\,. \qquad (16)$$

The information on the non-commutativity of $\hat{\mathcal{A}}$ is encoded in the $*$-product. The Weyl quantization procedure uses the exponential representation of the symmetrically ordered basis. The above procedure yields

$$\hat{f} = W(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k\, e^{ik_j \hat{x}^j}\, \tilde{f}(k), \qquad (17)$$

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x\, e^{-ik_j x^j}\, f(x), \qquad (18)$$

where we have replaced the commutative coordinates by non-commutative ones ($\hat{x}^i$) in the inverse Fourier transformation (17). Hence, we obtain

$$(\mathcal{A}, *) \cong (\hat{\mathcal{A}}, \cdot), \qquad (19)$$

i.e., W is an algebra isomorphism. Using Eq. (13), we are able to compute the star product explicitly,

$$W(f*g) = \frac{1}{(2\pi)^n} \int d^n k\, d^n p\, e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j}\, \tilde{f}(k)\tilde{g}(p). \qquad (20)$$

Because of the non-commutativity of the coordinates $\hat{x}^i$, we need the Campbell–Baker–Hausdorff (CBH) formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]-\frac{1}{12}[[A,B],A]+\cdots}. \qquad (21)$$

Clearly, we need to specify the commutation relations of the coordinates in order to evaluate the CBH formula. We will consider the canonical and linear cases as examples.

2.2.1. Canonical case

Due to the constant commutation relations

$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$,

the CBH formula will terminate, terms with more than one commutator will vanish,

$$\exp(ik_i\hat{x}^i)\exp(ip_j\hat{x}^j) =$$

$$= \exp\left(i(k_i+p_i)\hat{x}^i - \frac{i}{2}k_i\theta^{ij}p_j\right). \qquad (22)$$

Relation (20) now reads

$$f * g\,(x) = \frac{1}{(2\pi)^n} \int d^n k d^n p\, e^{i(k_i+p_i)x^i - \frac{i}{2}k_i\theta^{ij}p_j}\, \tilde{f}(k)\tilde{g}(p), \qquad (23)$$

and we get, for the $*$-product the Moyal–Weyl product [23],

$$f * g\,(x) = \exp\left(\frac{i}{2}\frac{\partial}{\partial x^i}\,\theta^{ij}\,\frac{\partial}{\partial y^j}\right) f(x)g(y)\Big|_{y\to x}. \qquad (24)$$

The same reasoning can be applied to the case of normal ordering. In this basis, a non-commutative function f is given by

$$\hat{f}(\hat{x}) = \frac{1}{(2\pi)^{n/2}} \int d^n k\, \tilde{f}(k) e^{ik_1\hat{x}^1} e^{ik_2\hat{x}^2} e^{ik_3\hat{x}^3} e^{ik_4\hat{x}^4}. \qquad (25)$$

Relation (20) has to be replaced by

$$\hat{f}\cdot\hat{g} = \frac{1}{(2\pi)^n} \int d^n k d^n p \tilde{f}(k)\tilde{g}(p) e^{ik_1\hat{x}^1} \ldots$$

$$\ldots e^{ik_4\hat{x}^4} e^{ip_1\hat{x}^1} \ldots e^{ip_4\hat{x}^4}. \qquad (26)$$

Using the CBH formula, $e^{ia\hat{x}^i}e^{ib\hat{x}^j} = e^{ib\hat{x}^j}e^{ia\hat{x}^i}e^{-iab\theta^{ij}}$, we obtain, for the star product for normal ordering,

$$f *_N g\,(x) = \exp\left(\sum_{i>j} i\frac{\partial}{\partial x^i}\,\theta^{ij}\,\frac{\partial}{\partial y^j}\right) f(x)g(y)\Big|_{y\to x}. \qquad (27)$$

In both cases, we can now explicitly show that Eq. (1) is satisfied. The star product enjoys a very important property,

$$\int d^4x\, f \star g \star h = \int d^4x\, h \star f \star g,$$

$$\int d^4x f \star g = \int d^4x\, f \cdot g. \qquad (28)$$

This is called the trace property.

2.2.2. κ-Deformed case

The following choice of a linear commutation relation is called κ-defomation:

$$[\hat{x}^1, \hat{x}^p] = ia\hat{x}^p, \qquad [\hat{x}^q, \hat{x}^p] = 0, \tag{29}$$

where $p, q = 2, 3, 4$. Because the structure is more involved, the computation of a star product is not as easy as in the canonical case. Therefore, we will just state the result. The symmetrically ordered star product is given by [22]

$$f * g\,(x) = \int d^4k\, d^4p\, \tilde{f}(k)\tilde{g}(p)\, e^{i(\omega_k+\omega_p)x^1} \times$$

$$\times e^{i\mathbf{x}(\mathbf{k}e^{a\omega_p}A(\omega_k,\omega_p)+\mathbf{p}A(\omega_p,\omega_k))}, \tag{30}$$

where $k = (\omega_k, \mathbf{k})$, and $\mathbf{x} = (x^2, x^3, x^4)$. We have used the definition

$$A(\omega_k, \omega_p) \equiv \frac{a(\omega_k+\omega_p)}{e^{a(\omega_k+\omega_p)}-1}\frac{e^{a\omega_k}-1}{a\omega_k}. \tag{31}$$

The normal ordered star product has the form [22]

$$f *_N g\,(x) = \lim_{\substack{y\to x\\ z\to x}} e^{x^j \frac{\partial}{\partial y^j}\left(e^{-ia\frac{\partial}{\partial z^1}}-1\right)} f(y)g(z) =$$

$$= \int \frac{d^4p\, d^4k}{(2\pi)^4} e^{ix^1(\omega_k+\omega_p)} e^{i\mathbf{x}(\mathbf{k}e^{a\omega_p}+\mathbf{p})} \tilde{f}(k)\tilde{g}(p). \tag{32}$$

In the κ-deformed case, the trace property is modified. We have to introduce an integration measure $\mu(x)$:

$$\int d^4x\mu(x)\,(f\star g)\,(x) = \int d^4x\mu(x)\,(g\star f)\,(x). \tag{33}$$

The above relation also determines the function $\mu(x)$, see, e.g., [19].

3. Non-Commutative Quantum Field Theory

Many models of non-commutative quantum field theory have been studied in recent years, and a coherent picture is beginning to emerge. One of the surprising features is the so-called ultraviolet (UV)/infrared (IR) mixing, where the usual divergences of field theory in the UV are reflected by new singularities in the IR. This is essentially a reflection of the uncertainty relation: determining some coordinates to a very high precision (UV) implies a large uncertainty (IR) for others. This leads to a serious problem for the usual renormalization procedure of quantum field theories which has only recently been overcome for a scalar-field theoretical model on the canonical deformed Euclidean space [24, 25]. This model will be discussed in subsection 3.1. Most models constructed so far use the canonical space-time, the simplest deformation. Therefore, we will also describe a quantum field theory on a more complicated structure, namely a κ-deformed space, here. Nevertheless, the problem of UV/IR mixing could not be solved by this deformation.

3.1. Scalar field theory

In this subsection, we want to sketch two different models of scalar fields on a non-commutative space-time. The first model is formulated on a κ-deformed Euclidean space [26], the second model given in [24, 25] on a canonically deformed Euclidean space.

The commutation relations of coordinates for the κ-deformed case are given by Eq. (29). For simplicity, we concentrate on the Euclidean version and use the symmetrically ordered star product given in Eq. (30). The κ-deformed spaces allow for a generalized coordinate symmetry, the so-called κ-Poincaré symmetry [17, 19]. Therefore, also the action should be invariant under this symmetry. In [19], the κ-Poincaré algebra and the action of its generators on commutative functions are explicitly calculated starting from the commutation relations (29). In order to describe scalar fields on a κ-deformed space, we need to write down an action. Therefore, we have to know the κ-deformed version of the Klein–Gordon operator and an integral invariant under κ-Poincaré transformations. The Klein–Gordon operator is a Casimir one in the translation generators (momenta) [19]. Acting on commutative functions, it is given by the expression

$$\Box^* = \sum_{i=1}^{4} \partial_i\partial_i \frac{2(1-\cos a\partial_1)}{a^2\partial_1^2}. \tag{34}$$

A κ-Poincaré invariant integral is given in [27] and has the form

$$(\phi, \psi) = \int d^4x\phi(K\bar{\psi}), \tag{35}$$

where K is a suitable differential operator,

$$K = \left(\frac{-ia\partial_1}{e^{-ia\partial_1}-1}\right)^3. \tag{36}$$

In the momentum space, this amounts to

$$(\phi, \psi) = \int d^4q \left(\frac{-a\omega_q}{e^{-a\omega_q}-1}\right)^3 \tilde{\phi}(q)\bar{\tilde{\psi}}(q). \tag{37}$$

Therefore, the action for a scalar field with the ϕ^4 interaction is given by

$$S[\phi] = -(\phi, (\Box^* - m^2)\psi)+$$

$$+\frac{g}{4!}\left(b(\phi * \phi,\, \phi * \phi) + d(\phi * \phi * \phi * \phi, 1)\right). \qquad (38)$$

In the above action, we have not included all possible interaction terms. A term proportional to $(\phi \star \phi \star \phi,\, \phi)$ is missing. This term would lead, however, to a peculiar behavior; therefore, it is ignored. For more details, see [26].

In the momentum space, the action has the form

$$S[\phi] = \int d^4q \left(\frac{-a\omega_q}{e^{-a\omega_q}-1}\right)^3 \tilde{\phi}(q) \left(q^2 \frac{2(\cosh a\omega_q - 1)}{a^2\omega_q^2} +\right.$$

$$\left. + m^2\right) \bar{\tilde{\phi}}(q) + b\,\frac{g}{4!}\int d^4z \prod_{i=1}^{4} d^4k_i \times$$

$$\times \left(\frac{a(\omega_{k_3}+\omega_{k_4})}{e^{a(\omega_{k_3}+\omega_{k_4})}-1}\right)^3 \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\times$$

$$\times e^{iz^1\sum\omega_{k_i}}\, exp\Big(i\mathbf{z}\,[\mathbf{k}_1 e^{a\omega_{k_2}} A(\omega_{k_1},\omega_{k_2})+$$

$$+\mathbf{k}_2 A(\omega_{k_2},\omega_{k_1}) + \mathbf{k}_3 e^{-a\omega_{k_4}} A(-\omega_{k_3},-\omega_{k_4})+$$

$$+\mathbf{k}_4 A(-\omega_{k_4},-\omega_{k_3})]\Big) + d\,\frac{g}{4!}\int d^4z \prod_{i=1}^{4} d^4k_i \times$$

$$\times e^{iz^1\sum\omega_{k_i}}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\times$$

$$\times \exp\left[i\mathbf{z}(\mathbf{k}_1 e^{a\omega_{k_2}} A(\omega_{k_1},\omega_{k_2}) + \mathbf{k}_2 A(\omega_{k_2},\omega_{k_1}))\,\times\right.$$

$$\left.\times\, e^{a(\omega_{k_3}+\omega_{k_4})} A(\omega_{k_1}+\omega_{k_2},\omega_{k_3}+\omega_{k_4})\right]\times$$

$$\times \exp\left[i\mathbf{z}(\mathbf{k}_3 e^{a\omega_{k_4}} A(\omega_{k_3},\omega_{k_4}) + \mathbf{k}_4 A(\omega_{k_4},\omega_{k_3}))\,\times\right.$$

$$\left.\times\, A(\omega_{k_3}+\omega_{k_4},\omega_{k_1}+\omega_{k_2})\right]. \qquad (39)$$

Note that $\bar{\tilde{\phi}}(k) = \tilde{\phi}(-k)$ for real fields $\phi(x)$. The x-dependent phase factors are a direct result of the star product (30), and b and d are real parameters. In the case of a canonical deformation, the phase factor is independent of x. Like the commutative case, we want to extract the amplitudes for Feynman diagrams from a generating functional by differentiation. The generating functional can be defined as

$$Z_\kappa[J] = \int \mathcal{D}\phi e^{-S[\phi]+\frac{1}{2}(J,\phi)+\frac{1}{2}(\phi,J)}. \qquad (40)$$

The n-point functions $\tilde{G}_n(p_1,\ldots,p_n)$ are given by functional differentiation:

$$\tilde{G}_n(p_1,\ldots,p_n) = \frac{\delta^n}{\delta\tilde{J}(-p_1)\ldots\delta\tilde{J}(-p_n)} Z_\kappa[J]\Big|_{J=0}. \qquad (41)$$

Let us first consider the free case. For the free generating functional $Z_{0,\kappa}$, Eq. (40) yields

$$Z_{0,\kappa}[J] = \int \mathcal{D}\phi \exp\left[-\frac{1}{2}\int d^4k \left(\frac{-a\omega_k}{e^{-a\omega_k}-1}\right)^3 \tilde{\phi}(k)\,\times\right.$$

$$\times(\mathcal{M}_k + m^2)\tilde{\phi}(-k)+$$

$$+\frac{1}{2}\int d^4k \left(\left(\frac{-a\omega_k}{e^{-a\omega_k}-1}\right)^3 + \left(\frac{a\omega_k}{e^{a\omega_k}-1}\right)^3\right)\times$$

$$\left.\times\ \tilde{J}(k)\tilde{\phi}(-k)\right], \qquad (42)$$

where we have defined

$$\mathcal{M}_k := \frac{2k^2(\cosh a\omega_k - 1)}{a^2\omega_k^2}. \qquad (43)$$

The same manipulations, as in the classical case, yield

$$Z_{0,\kappa}[J] = Z_{0,\kappa}[0] e^{\frac{1}{2}\int d^4k \left(\frac{-a\omega_k}{e^{-a\omega_k}-1}\right)^3 \frac{\tilde{J}(k)\tilde{J}(-k)}{\mathcal{M}_k+m^2}}. \qquad (44)$$

We will always consider the normalized functional obtained by dividing by $Z_{0,\kappa}[0]$. Now, the free propagator is given by

$$\tilde{G}(k,p) = \frac{\delta^2}{\delta\tilde{J}(-k)\delta\tilde{J}(-p)} Z_{0,\kappa}[J]\Big|_{J=0} =$$

$$= L(\omega_k)\frac{\delta^{(4)}(k+p)}{\mathcal{M}_k + m^2} \equiv \delta^{(4)}(k+p) Q_k. \qquad (45)$$

For the sake of brevity, we have introduced

$$L(\omega_k) := \frac{1}{2}\left(\left(\frac{-a\omega_k}{e^{-a\omega_k}-1}\right)^3 + \left(\frac{a\omega_k}{e^{a\omega_k}-1}\right)^3\right). \qquad (46)$$

We can rewrite the full generating functional in the form

$$Z_\kappa[J] = e^{-S_I[1/L(\omega_k)\frac{\delta}{\delta \tilde{J}(-k)}]} Z_{0,\kappa}[J]. \tag{47}$$

The full propagator to the first order in the coupling parameter is given by the connected part of the expression

$$\tilde{G}^{(2)}(p,q) = \frac{\delta^2}{\delta\tilde{J}(-p)\delta\tilde{J}(-q)} Z_\kappa[J]\Big|_{J=0}. \tag{48}$$

The aim is to compute tadpole diagram contributions. In order to do so, we expand the generating functional (47) in powers of the coupling constant g. Using Eq. (50), we obtain

$$Z_\kappa[J] = Z_{0,\kappa}[J] + Z^1_\kappa[J] + \mathcal{O}(g^2). \tag{49}$$

The details of the calculation are given in [26]. Let us just state the results. As in the canonically deformed case, we can distinguish between planar and non-planar diagrams. The planar diagrams display a linear UV divergence. The non-planar diagrams are finite for generic external momenta, p and p, respectively. However, in the exceptional case $\omega_p = \omega_k = 0$, the amplitudes also diverge linearly in the UV cut-off.

Let us switch the second model. Remarkably, the problem of UV/IR divergences is solved in this case, and the model turns out to be renormalizable. We will briefly sketch the model and its peculiarities. Again, it is a scalar field theory. It is defined on the 4-dimensional quantum plane $\mathbb{R}^4_\theta$ with the commutation relations $[x_\mu, x_\nu] = i\theta_{\mu\nu}$. The UV/IR mixing was taken into account through a modification of the free Lagrangian, by adding an oscillator term which modifies the spectrum of the free action:

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)+$$

$$+\frac{\mu^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x) \ . \tag{50}$$

Here, $\star$ is the Moyal star product (24). The harmonic oscillator term in Eq. (50) was found as a result of the renormalization proof. The model is covariant under the Langmann–Szabo duality relating the short-distance and long-distance behaviors.

The renormalization proof proceeds by using a matrix base b_{nm}. The remarkable feature of this base is that the star product is reduced to a matrix product,

$$b_{kl}\star b_{mn} = \delta_{lm} b_{kn}. \tag{51}$$

We can expand the fields in terms of this base:

$$\phi = \sum_{m,n} \phi_{nm} b_{nm}(x). \tag{52}$$

This leads to a dynamical matrix model of the type

$$S = (2\pi\theta)^2 \times$$

$$\times \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{mn}\Delta_{mn;kl}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big) \ , \tag{53}$$

where

$$\Delta_{\substack{m^1 n^1 \ k^1 l^1 \\ m^2 n^2 ; k^2 l^2}} = \big(\mu^2 + \frac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\big)\times$$

$$\times\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} - \frac{2-2\Omega^2}{\theta}\times$$

$$\times\big(\sqrt{k^1l^1}\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{m^1n^1}\times$$

$$\times\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} -$$

$$-\frac{2-2\Omega^2}{\theta}\Big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} +$$

$$+\sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\Big)\delta_{n^1k^1}\delta_{m^1l^1} \ . \tag{54}$$

The interaction part becomes the trace of a product of matrices, and no oscillations occur in this basis. In the κ-deformed case we have discussed before, $x-$dependent phases occurred. Here, the interaction terms have a very simple structure, but the propagator obtained from the free part is quite complicated. For the details, see [25].

These propagators show asymmetric decay properties: they decay exponentially on particular directions, but have power law decay in others. These decay properties are crucial for the perturbative renormalizability (respectively, the nonrenormalizability) of models. The renormalization proof follows the ideas of Polchinski [28]. The integration of the Polchinski equation from some initial scale down to the renormalization scale leads to divergences after removing the cutoff. For relevant/marginal operators, one therefore has to fix certain initial conditions. The goal is then to find a procedure involving only a finite number of such operators. Through the invention of a mixed integration procedure and by proving a certain power counting theorem, they were able to reduce the divergences to only four relevant/marginal operators. A somewhat long sequence of estimates and arguments leads then to the proof of renormalization. Afterwards, they could also derive β-functions for the coupling constant flow, which shows that the ratio of the coupling constants λ/Ω^2 remains bounded along the renormalization group flow up to the first order. The renormalizability of this model is a very important result and, so far, the only example of a renormalizable non-commutative model.

3.2. Gauge theories

At present, particles and their interactions are described by gauge theories. The most prominent gauge theory is the Standard Model which incudes the electromagnetic force and the strong and weak nuclear forces. Therefore, it is of vital interest to extend the ideas of non-commutative geometry and the renormalization method described above to gauge field theories. Let us sketch two approaches:

1. Non-commutative gauge theories can be formulated by introducing the so-called Seiberg–Witten maps [15, 29]. There, the non-commutative gauge fields are given as a power series in non-commutativity parameters. They depend on the commutative gauge field and the gauge parameter and are solutions of gauge equivalence conditions. Therefore, no additional degrees of freedom are introduced. A major advantage of this approach is that there are no limitations to the gauge group. For an introduction, see, e.g., [30]. The Standard Model of elementary particle physics is discussed in [31–33] using this approach. However, these theories seemingly have to be considered as effective theories, since the non-renormalizability of non-commutative QED has explicitly been shown in [34].

2. The second approach starts from covariant coordinates $B_\mu = \theta^{-1}_{\mu\nu} x^\nu + A_\mu$ [29]. These objects are transformed covariantly under the gauge transformations

   $$B_\mu \to U^* \star B_\mu \star U,$$

   with $U^* \star U = U \star U^* = 1$. This is analogous to the introduction of covariant derivatives. Covariant coordinates only exist on non-commutative spaces. We can write down a gauge-invariant version of action (50):

   $$S = \int d^4 \left(\frac{1}{2} \phi \star [B_\nu \overset{\star}{,} [B^\nu \overset{\star}{,} \phi]] + \frac{\Omega^2}{2} \phi \star B_\nu \overset{\star}{,} \{\{B^\nu \overset{\star}{,} \phi\}\} \right), \tag{55}$$

   where we have used $[x^\mu \overset{\star}{,} f] = i\theta^{\mu\nu}\partial_\nu f$.

4. Astrophysical Considerations

In this section, we want to discuss a modification of the dispersion relations in a κ-deformed space-time. This modification leads to a bound on the non-commutativity parameter. We will follow the presentation given in [11].

In Section 3, we have discussed a scalar model on a κ-deformed Euclidean space. Here, we consider a κ-Minkowski space-time with the relations

$$[\hat{x}^i, \hat{t}] = i\lambda\hat{x}^i, \qquad [\hat{x}^i, \hat{x}^j] = 0, \tag{56}$$

$i, j = 1, 2, 3$. The modification of the dispersion relation by a modified d'Alembert operator has been briefly discussed in Section 3:

$$\lambda^{-2}\left(e^{\lambda\omega} + e^{-\lambda\omega} - 2\right) - \mathbf{k}^2 e^{-\lambda\omega} = m^2. \tag{57}$$

In the commutative limit, $\lambda \to 0$, we obtain, of course, the usual relation

$$\omega^2 - \mathbf{k}^2 = m^2\,.$$

The velocity for a massless particle is given by

$$\mathbf{v} = \frac{d\omega}{d\mathbf{k}} = \frac{\lambda\mathbf{k}}{\lambda^2\mathbf{k}^2 + \frac{\lambda\omega}{|\lambda\omega|}\sqrt{\lambda^2\mathbf{k}^2}}. \tag{58}$$

One obtains

$$v = e^{-\lambda\omega} \approx 1 - \lambda\omega. \tag{59}$$

This means that the velocity of a particle depends on its energy. Particles with different energies will take different amounts of time for the same distance.

Let us consider γ-ray bursts from active galaxies such as Makarian 142. The time difference δt in the arrival times for photons with different energies can be estimated as

$$|\delta t| \approx \lambda \frac{L}{c} \delta\omega, \tag{60}$$

where L is the distance of the galaxy, $\delta\omega$ is the energy range of a burst, and λ is the non-commutativity parameter. A usual γ-ray burst spreads over a range of $0.1 - 100$ MeV. Data already available seem to imply that $\lambda < 10^{-33}$ m.

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НЕКОМУТАТИВНА ГЕОМЕТРІЯ І ФІЗИКА

М. Волґенант

Р е з ю м е

Робота містить деякі основні ідеї некомутативної геометрії. Її застосування в фізиці розглянуто у двох напрямках: у квантовій теорії поля та астрофізиці. Детально описано деякі сучасні моделі в квантовій теорії поля. В контексті астрофізичних аспектів некомутативної геометрії отримано модифіковані дисперсійні співвідношення.