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AN APPROXIMATE METHOD FOR SOLVING ASSIGNMENT PROBLEM

Abstract *This paper presents an approximate method for solving assignment problem that makes it possible to assign tasks to agents in order to gain utmost overall efficiency in completing all the tasks. An example is provided for algorithm illustration.*
Key words: *approximate method, approximate solution, assignment problem*

Introduction

Evolution of market relations and management development in all kinds of purposeful human activities in a wide variety of branches (industries, agriculture, commerce, public amenities, health care, environment protection, etc.) set problems that require rendering grounded complex decisions. Particularly, it relates to modern development in Ukraine that requires expanding existing industries and founding new ones. That said, introducing new enterprises and creating new jobs is often connected with so called assignment problem, that is the problem of assignment vacant places in such the way that task performance efficiency is maximized. This problem also arises when founding new or expanding existing organizations.

The assignment problem is one of the fundamental problems of combinatorial optimization in the branch of optimization or operations research in applied mathematics. It consists in finding minimum (or maximum) weight among the elements of two finite sets. It can be presented as finding a control in weighed bipartite graph. On the other hand, the problem is among linear programming problems. It is a special case of transportation problem, that can be in turn represented as a minimum cost flow problem.

Assignment problem can be described via different application instances. For example, there are a number of agents and a number of tasks. Any agent can be assigned to perform any task. Performing tasks by an agent incur some cost that may vary depending on the agent assigned to perform a task. It is required to perform all tasks by assigning exactly one agent to each task and exactly one task to each agent in such a way that the total cost of the assignment is minimized.

If the numbers of agents and tasks are equal and the total cost of the assignment for all tasks is equal to the sum of the costs for performing each task, then the problem is called the linear assignment problem. This version of problem is basic and simplest one. Commonly, when speaking of the assignment problem without any additional qualification, then the linear assignment problem is meant.

There can be other versions of problem that include additional qualification, other methods for total cost calculation, or some changes in base conditions. For example, number of agents can be unequal to number of tasks; total cost definition can be non-linear, etc. In such cases, generalized assignment problem is suggested.

Different methods can be used for solving linear assignment problem, from common linear programming problem solving down to special methods for solving graph problems. Generally, special methods developed for solving this problem are much faster because they take advantage of its special structure. For instance, Hungarian algorithm has been one of the first algorithms, developed for solving linear assignment problem. Problem solving time is proportional to number of agents. Other algorithms used for solving the problem are adapted simplex algorithm and auction algorithm.

The objective of the paper is to demonstrate approximate method for solving assignment problem. In this case, we consider problem of assigning jobs to workers in order to gain utmost overall efficiency in completing the jobs. We also consider mathematical models for the most efficient candidate assignments to vacant positions subject to certain restrictions.

There are accurate methods for solving this kind of problem, but their program implementation is very difficult. That is why we consider method for approximate solving the problem.

An assignment problem has been investigated first in geometric shape by Gaspard Monge in 1784. Although, non-correctness of Monge's solution has been established in the early 20-th century. Further steps to solving assignment problem were made by Kőnig and Egerváry in the first third of 20-th century. Kőnig and Egerváry dealed this problem as finding perfect matching of maximum weight in weighed bipartite graph [4]. Their works settled the base for Hungarian method developed by Kuhn in 1950-s. In 1947, simplex method has been suggested by Dantzig for solving general linear programming problem to which assignment problem can be easily reduced. The assignment problem set by Dantzig and Fulkerson can be also considered as problem of maximum flow of minimum cost. In 1961, an algorithm for its solution has been published by Busacker and Gowan. As well as simplex algorithm, this algorithm for common problem has exponential complexity, and for assignment problem, polynomial. Theoretical analysis of the algorithms complexity shows that Kuhn and Busacker & Gowan algorithms feature similar theoretical complexity, which is less than that of Goldberg & Tarjan algorithm. Besides, O.F. Voloshyn and M. Y. Kvyk researched models, methods and algorithms that define decision making processes [1-2]. Although, defining the best algorithm requires empirical research.

1. Assignment problem and an approximate method for its solution

Consider assignment problem statement and an approximate method for its solution [3].

Assume that n workers has been allocated to perform n jobs. c_{ij} i-th worker's efficiency in performing j-th job is known. It is required to assign jobs to workers in order to gain utmost overall efficiency in completing all the jobs.

Having introduced variables x_{ij} defined according to formula

$$x_{ij} = \begin{cases} 1, \text{if } \ i\text{-th job is assigned to } \ j\text{-th worker} \\ 0 \text{ in opposite case,} \end{cases}$$

then the problem's mathematical model is

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \to \max \tag{1}$$

subject to the conditions

$$\sum_{i=1}^{n} x_{ij} = 1,\ j = 1,2,...,n; \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,...,n; \tag{3}$$

$$x_{ij} \in \{0,1\},\ i = 1,2,...,n,\ j = 1,2,...,n. \tag{4}$$

It follows from formulas (2) to (4) that one agent is allocated to one task. Agents distribution depends on their work performance c_{ij}.

Method algorithm. Let's form a matrix of elements c_{ij}

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ & & \dots & \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix}.$$

The algorithm includes n steps. In the first step, find maximum among c_{ij} elements. If such element is the only one c_{ik}, then $\mathrm{x}_{ik} = 1$, and the first step is finished here. Assume that the max element is not the only in the *i*-th row. Assume that *i*-th row contains elements c_{ik_1}, c_{ik_2}, …, c_{ik_m} equal to each other. Let us call

$$\max_{j \neq i} c_{jk_1} = c_{j_1k_1}, \max_{j \neq i} c_{jk_2} = c_{j_2k_2}, ..., \max_{j \neq i} c_{jk_m} = c_{j_mk_m}. \tag{5}$$

Then

$$c_{ik} = \min\{c_{j_1k_1}, c_{j_2k_2}, ..., c_{j_mk_m}\} \tag{6}$$

and $x_{ik} = 1$. The first step is finished here. The second step is carried out similar to the first one, but the *i*-th row and *k*-column of C matrix is not engaged. The elements of that row and column $x_{ik} = 1)$ are assigned zero. Approximate problem solution will be found in n steps.

2. Implementation example

Let us take a certain matrix C, which elements c_{ij} are given performance efficiencies of i-th worker in j-th job.

$$C = \begin{pmatrix} 5 & 1 & 2 & 3 & 4 \\ 4 & 7 & 5 & 7 & 3 \\ 3 & 4 & 4 & 6 & 6 \\ 5 & 3 & 2 & 4 & 5 \\ 4 & 5 & 6 & 5 & 4 \end{pmatrix}$$

Step 1. $\max c_{ij} = c_{22} = c_{24}$. It arises from (5) and (6) that $\max\{c_{12}, c_{32}, c_{42}, c_{52}\} < \max\{c_{14}, c_{34}, c_{44}, c_{54}\}$, then $c_{ik} = c_{22}$ and $x_{22} = 1$. As a result, we obtain matrix

$$C = \begin{pmatrix} 5 & 0 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 4 & 6 & 6 \\ 5 & 0 & 2 & 4 & 5 \\ 4 & 0 & 6 & 5 & 4 \end{pmatrix}.$$

Step 2. $\max c_{ij} = c_{34} = c_{35} = c_{53}$. By virtue of the fact of $\max\{c_{13}, c_{33}, c_{34}\} < \max\{c_{14},\ c_{44}, c_{45}\}$ and $\max\{c_{13}, c_{33}, c_{34}\} < \max\{c_{15},\ c_{45}, c_{55}\}$, then $c_{ik} = c_{53}$ and $x_{53} = 1$. As a result, we obtain matrix

$$C = \begin{pmatrix} 5 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 6 & 6 \\ 5 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Step 3. $\max c_{ij} = c_{34} = c_{35}$. By virtue of the fact of $\max\{c_{14}, c_{44}\} < \max\{c_{15}, c_{45}\}$, then $c_{ik} = c_{34}$ and $x_{34} = 1$. As a result, we obtain matrix

$$C=\begin{pmatrix}5&0&0&0&4\\0&1&0&0&0\\0&0&0&1&0\\5&0&0&0&5\\0&0&1&0&0\end{pmatrix}.$$

Step 4. $\max c_{ij}=c_{41}=c_{45}$. By virtue of the fact of $c_{15}<c_{11}$, then $c_{ik}=c_{45}$, and $x_{45}=1$. A matrix is obtained

$$C=\begin{pmatrix}5&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}.$$

Step 5. $x_{11}=1$ The final matrix is obtained

$$C=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\end{pmatrix}.$$

Objective function value for this distribution $L=5+7+6+5+6=29$.

Problems with following mathematical models can be solved in a similar way:

1. $$L=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\to \max$$

subject to the conditions

$$0\le\sum_{j=1}^{n}x_{ij}\le 1,\ i=1,2,...,m;$$

$$\sum_{i=1}^{m}x_{ij}=1,\ j=1,2,...,n;$$

$$x_{ij}\in\{0,1\},\ i=1,2,\ldots,n,\ j=1,2,\ldots,n$$

in case that candidates number m is greater then vacant positions number n. In other words, only the most qualified workers receive jobs.

$$2.\quad L=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\rightarrow \max$$

subject to the conditions

$$\sum_{j=1}^{n}x_{ij}=1,\ i=1,2,...,m;$$

$$0\leq\sum_{i=1}^{m}x_{ij}\leq 1,\ j=1,2,...,n;$$

$$x_{ij}\in\{0,1\},\ i=1,2,...,n,\ j=1,2,...,n$$

in case that candidates number m is less then vacant positions number n. In other words, all workers receive jobs, since vacant positions number in greater then candidates number.

It's worth to notice that in the first model implementation, the final matrix C will have all zeros in m-n rows, while in the second model implementation, the final matrix C will have all zeros in n-m columns.

We also consider mathematical models for the most efficient candidate assignments to vacant positions in groups.

Here

n – number of vacant position groups;

m – number of candidates for positions;

k_j – number of vacant positions in j-th group;

c_{ij} – expert estimate of i-th candidate's for positions in j-th group.

$$x_{ij}=\begin{cases}1, if\ i\text{ - th candidate is qualified for a postion in } j\text{ - th group;}\\ 0\text{ in opposite case.}\end{cases}$$

Let us also introduce restrictions on candidates for vacancies and vacant position numbers in groups:

1. Assume that position candidates number equals to number of vacant positions in groups, where each group has k_j vacant positions in j-th group

$$m=\sum_{j=1}^{n}k_j\ .$$

Then, the objective function

$$L=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\rightarrow \max$$

subject to the conditions

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,...,m;$$

$$\sum_{i=1}^{m} x_{ij} = k_j,\ j = 1,2,...,n;$$

$$x_{ij} \in \{0,1\},\ i = 1,2,...,n,\ j = 1,2,...,n\,.$$

That is, each candidate will receive positions according to expert estimates.

2. Assume that position candidates number is greater than number of vacant positions in groups, where each group has k_j vacant positions in j-th group

$$m > \sum_{j=1}^{n} k_j\ .$$

Then, the objective function

$$L = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \to \max$$

subject to the conditions

$$0 \le \sum_{j=1}^{n} x_{ij} \le 1,\ i = 1,2,...,m;$$

$$\sum_{i=1}^{m} x_{ij} = k_j,\ j = 1,2,...,n;$$

$$x_{ij} \in \{0,1\},\ i = 1,2,...,n,\ j = 1,2,...,n\,.$$

Consequently, not all the candidates will receive positions.

3. Assume that position candidates number is less than number of vacant positions in groups, where each group has k_j vacant positions in j-th group

$$m < \sum_{j=1}^{n} k_j\ .$$

Then, the objective function

$$L = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \to \max$$

subject to the conditions

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,...,m;$$

$$\sum_{i=1}^{m} x_{ij} \le k_j ,\ j = 1,2,...,n;$$

$$x_{ij} \in \{0,1\},\ i = 1,2,...,n,\ j = 1,2,...,n .$$

That is, each candidate will receive a position according to expert estimate.

Conclusions

Hence, this problem solving algorithm allows us to allocate workers to jobs in order to gain utmost overall efficiency in completing all the tasks, if efficiency c_{ij} of i-th worker in performing j-th job is known. This algorithm is applicable in industries, agriculture, commerce, etc. Some additional mathematical models shown in this paper are applicable in cases when candidates number m is greater than vacant positions number n and when candidates number m is less than vacant positions number n. We have also presented mathematical models for the most efficient candidate assignments to vacant positions in groups.

REFERENCES

1. Voloshyn O.F. Decision making models and methods: textbook / O.F. Voloshyn, S.O. Mashscenko. – Kyiv: Kyiv University printing and publishing centre, 2010. — 336 p.
2. Kvyk M.Y. Mathematical methods and models for decision making support in small enterprises management: author’s abstract of Candidate of Economic Sciences, specialty: 08.00.11 – Mathemantical methods, models and information technologies in economics / M.Y. Kvyk. – Cherkasy: East European University of Economics and Management, 2015. – 20 p.
3. Marko M.Y., Tsehelyk H.H. An approximate method for solving assignment problem: talk abstracts/M.Y. Marko, H.H. Tsehelyk. – Pereiaslav-Khmelnytskyi: Hryhoriy Skovoroda State Pedagogical University of Pereyaslav-Khmelnytsky. 2017. – 24 p.
4. Kőnig, Dénes. Theory of finite and infinite graphs. — Birkhäuser, 1990.

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