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ON LOCAL BLOW-UP OF SOLUTIONS OF QUASILINEAR ELLIPTIC AND PARABOLIC INEQUALITIES

Quasilinear elliptic and parabolic inequalities with nonlinearities of the Burgers-Kardar-Parisi-Zhang type are investigated. Sufficient conditions of nonexistence of local solutions are established.

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1. Introduction.

The phenomenon of complete (local) blow-up of solutions to elliptic *semilinear* equations and inequalities was investigated in many papers of various authors (see [1] and references therein). A relatively less number of papers is devoted to the *quasilinear* case (see [1, § 6] as well as [2] and [3]). The pioneering results about the local (complete and instant) blow-up of solutions of nonlinear parabolic inequalities were obtained in [4] by the comparison method. In [1], that result was substantially strengthened (the nonlinear capacity method was applied). However, the above results refer to *semilinear* inequalities, while the *quasilinear* case is still almost open (see [1] and references therein).
In this paper, local blow-up theorems are established for solutions of inequalities of the kinds

$$\Delta u + g(u)|\nabla u|^2 + \omega(x,u) \leq 0 \tag{1}$$

and

$$\frac{\partial u}{\partial t} \geq \Delta u + g(u)|\nabla u|^2 + \omega(x,t,u),$$

which were not considered in [1, 2, 3, 4]. Nonlinearities of the specified kind arise in various applications (see, e.g., [5] and [6]); they are interesting from the purely theoretical point of view as well because the methods elaborated in [1] are used for a much more comprehensive area of investigation.
In the sequel, we denote the domain $\left\{x \in \mathbb{R}^n \middle| 0 < |x| < r\right\}$ by Ω_r (for positive values of r), assuming that $n > 1$.

2. Regular elliptic case.

In this section, we consider inequality (1), assuming that the function g is continuous on $[0, +\infty)$ and the function ω is defined on $\mathbb{R}^n \times [0, +\infty)$.
The following assertion is valid:

Theorem 1. *Let g belong to $C[0, +\infty)$. Let there exist $r > 0$, $q > 1$, and $\sigma \leq -2$ such that*

$$\omega(x,s) \geq |x|^\sigma \left(\int\limits_0^s e^{\int\limits_0^\tau g(t)dt} d\tau\right)^q e^{-\int\limits_0^s g(\tau)d\tau} \quad \textit{for all} \quad (x,s) \in \Omega_r \times [0,+\infty).$$

Then inequality (1) *has no classical nonnegative nontrivial solutions in* Ω_r.

Proof. Suppose, to the contrary, that there exists a nonnegative function $u(x)$ different from the identical zero and satisfying (in the classical sense) inequality (1) in Ω_r. Following, e.g., [7,

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Ch. V, § 1] (see also [8] and [9] and cf. [10] for the *global* blow-up), we define the following function f on $[0,+\infty)$:

$$f(s)=\int_0^s e^{\int_0^\tau g(t)dt}\,d\tau. \tag{2}$$

Then $f'(s)=e^{\int_0^s g(\tau)d\tau}$ and $f''(s)=g(s)e^{\int_0^s g(\tau)d\tau}$. Since f' is strictly positive on $[0,+\infty)$, it follows that $g(s)=\dfrac{f''(s)}{f'(s)}$ on $[0,+\infty)$. Define the following function $v(x)$ on Ω_r: $v(x)\overset{\text{def}}{=}f[u(x)]$. Then

$$\frac{\partial v}{\partial x_j}=f'(u)\frac{\partial u}{\partial x_j}\quad\text{and}\quad\frac{\partial^2 v}{\partial x_j^2}=f''(u)\left(\frac{\partial u}{\partial x_j}\right)^2+f'(u)\frac{\partial^2 u}{\partial x_j^2},\ j=\overline{1,n}.$$

Therefore,

$$\Delta v=f'(u)\Delta u+f''(u)|\nabla u|^2=f'(u)\Big[\Delta u+g(u)|\nabla u|^2\Big].$$

This implies that

$$\Delta u+g(u)|\nabla u|^2=\frac{\Delta v}{f'(u)}$$

because f' vanishes nowhere.

Thus, the inequality $\dfrac{\Delta v}{f'(u)}+\omega(x,u)\le 0$ holds in Ω_r. This means that the inequality

$$\frac{\Delta v}{f'(u)}+|x|^\sigma\frac{\left(\int_0^u e^{\int_0^\tau g(t)dt}\,d\tau\right)^q}{e^{\int_0^u g(\tau)d\tau}}\le 0$$

holds in the same domain. Then the inequality $\dfrac{\Delta v}{f'(u)}+|x|^\sigma\dfrac{f^q(u)}{f'(u)}\le 0$ holds in Ω_r as well (see (2)). By virtue of the (strict) positivity of the function f', the latter inequality is equivalent to the inequality

$$\Delta v+|x|^\sigma v^q\le 0. \tag{3}$$

However, there is no positive r such that inequality (3) has a classical nonnegative nontrivial solution in Ω_r under the restrictions imposed on q and σ (see [1, Th 6.1]).
The obtained contradiction completes the proof.

3. Singular elliptic case.

The above approach is also applicable in the case where the coefficient at the principal nonlinear term of inequality (1) has a singularity (cf. [9, Sec. 3]). Assume that $\alpha\ne 0$ and $0<\beta\ne 1$ and consider the inequality

$$\Delta u+\frac{\alpha}{u^\beta}|\nabla u|^2+\omega(x,u)\le 0. \tag{4}$$

The following assertion is valid:

Theorem 2. *Let $r>0, q>1$, and $\sigma\le -2$ be such that*

$$\omega(x,s)\ge |x|^\sigma\left(\int_0^s e^{\frac{\alpha}{1-\beta}\tau^{1-\beta}}\,d\tau\right)^q e^{\frac{\alpha}{\beta-1}s^{1-\beta}}\quad \textit{for all}\quad (x,s)\in\Omega_r\times(0,+\infty).$$

Then inequality (4) *has no classical positive solutions in* Ω_r.

Proof. Suppose, to the contrary, that there exists a nonnegative function $u(x)$ satisfying inequality (4) in Ω_r. Define the function f on $(0,+\infty)$ as follows:

$$f(s)=\int\limits_0^s e^{\frac{\alpha}{1-\beta}\tau^{1-\beta}}d\tau. \tag{5}$$

Then $f'(s)=e^{\frac{\alpha}{1-\beta}s^{1-\beta}}>0$ and $f''(s)=\frac{\alpha}{s^\beta}e^{\frac{\alpha}{1-\beta}s^{1-\beta}}$ on $(0,+\infty)$. Therefore, $\frac{\alpha}{s^\beta}=\frac{f''(s)}{f'(s)}$ on $(0,+\infty)$.
Define the following function $v(x)$ on Ω_r: $v(x)\stackrel{\text{def}}{=}f[u(x)]$. Then

$$\Delta v=f'(u)\Delta u+f''(u)|\nabla u|^2=f'(u)\Big[\Delta u+\frac{\alpha}{u^\beta}|\nabla u|^2\Big]$$

(see the proof of Th. 1).
This means that $\Delta u+\frac{\alpha}{u^\beta}|\nabla u|^2=\frac{\Delta v}{f'(u)}$ because f' vanishes nowhere.
Thus, the inequality $\frac{\Delta v}{f'(u)}+\omega(x,u)\leq 0$ holds in Ω_r. Hence, the inequality

$$\frac{\Delta v}{f'(u)}+|x|^\sigma\frac{\left(\int\limits_0^u e^{\frac{\alpha}{1-\beta}\tau^{1-\beta}}d\tau\right)^q}{e^{\frac{\alpha}{1-\beta}u^{1-\beta}}}\leq 0$$

holds in Ω_r as well.
Taking into account relation (5), we obtain (in Ω_r) the inequality $\frac{\Delta v}{f'(u)}+|x|^\sigma\frac{f^q(u)}{f'(u)}\leq 0$.
By virtue of the (strict) positivity of the function f', inequality (3) holds in Ω_r. As above (see Th. 1), this contradicts [1, Th 6.1]. This completes the proof.

4. Critical elliptic case.

In this section, inequality (4) is considered for the case where $\beta=1$. Substitution (5) is not applicable in this case, but the local blow-up of solutions still occurs.
Assuming that $\alpha>-1$, we consider the inequality

$$\Delta u+\frac{\alpha}{u}|\nabla u|^2+\omega(x,u)\leq 0. \tag{6}$$

The following assertion is valid:

THEOREM 3. *Let* $r>0,\gamma>1$, *and* $\sigma\leq-2$ *be such that*

$$\omega(x,s)\geq\frac{|x|^\sigma s^\gamma}{\alpha+1}\quad \textit{for all}\quad (x,s)\in\Omega_r\times(0,+\infty).$$

Then inequality (6) *has no classical positive solutions in* Ω_r.

Proof. Suppose, to the contrary, that there exists a positive function $u(x)$ satisfying inequality (6) in Ω_r. Define a function $v(x)$ on Ω_r as follows:

$$v(x)=u^{\alpha+1}(x). \tag{7}$$

Then

$$\frac{\partial v}{\partial x_j}=(\alpha+1)u^\alpha(x)\frac{\partial u}{\partial x_j}$$

and

$$\frac{\partial^2 v}{\partial x_j^2} = \alpha(\alpha+1)u^{\alpha-1}(x)\left(\frac{\partial u}{\partial x_j}\right)^2 + (\alpha+1)u^{\alpha}(x)\frac{\partial^2 u}{\partial x_j^2} = (\alpha+1)u^{\alpha}(x)\left[\frac{\partial^2 u}{\partial x_j^2} + \frac{\alpha}{u(x)}\left(\frac{\partial u}{\partial x_j}\right)^2\right]$$

for $j = \overline{1,n}$.
Hence,

$$\Delta v = (\alpha+1)u^{\alpha}\left(\Delta u + \frac{\alpha}{u}|\nabla u|^2\right).$$

Since $(\alpha+1)u^{\alpha}$ vanishes nowhere, it follows that

$$\Delta u + \frac{\alpha}{u}|\nabla u|^2 = \frac{\Delta v}{(\alpha+1)u^{\alpha}} = \frac{\Delta v}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}}$$

(note that, due to (7), the function $v(x)$ is positive in the domain Ω_r because the function $u(x)$ is positive in the domain Ω_r).
Thus, the inequality

$$\frac{\Delta v}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}} + \omega\big(x, v^{\frac{1}{\alpha+1}}\big) \le 0$$

holds in the domain Ω_r. Therefore, the inequality

$$\frac{\Delta v}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}} + \frac{|x|^{\sigma}v^{\frac{\gamma}{\alpha+1}}}{\alpha+1} \le 0 \tag{8}$$

holds in the same domain.
Since the function $(\alpha+1)v^{\frac{\alpha}{\alpha+1}}(x)$ is positive in Ω_r, it follows from inequality (8) that the inequality

$$\Delta v + |x|^{\sigma}v^{\frac{\alpha+\gamma}{\alpha+1}} \le 0$$

is valid in Ω_r.
Now, denoting $\dfrac{\alpha+\gamma}{\alpha+1}$ by q, we see that the function $v(x)$ is positive in Ω_r and there exist $q > 1$ and $\sigma \le -2$ such that $v(x)$ satisfies inequality (3). This contradicts [1, Th. 6.1] and, therefore, completes the proof.

If $\alpha = -1$, then substitution (7) is, obviously, not applicable. However, a weakened result on local blow-up of solutions can be obtained in this critical case as well. More exactly, the following assertion is valid:

Theorem 4. *Let $r > 0, q > 1, \beta > 0$, and $\sigma \le -2$ be such that*

$$\omega(x,s) \ge |x|^{\sigma}s\left(\ln\frac{s}{\beta}\right)^q \quad \textit{for all} \quad (x,s) \in \Omega_r \times [\beta, +\infty).$$

Then the inequality

$$\Delta u - \frac{|\nabla u|^2}{u} + \omega(x,u) \le 0 \tag{9}$$

has no classical solutions in Ω_r, satisfying the condition $\beta \le u(x) \not\equiv \beta$.

Proof. Suppose, to the contrary, that a function $u(x)$ satisfies inequality (9) in Ω_r and $u(x) \ge \beta$ in Ω_r. Define a function $v(x)$ on Ω_r as follows:

$$v(x) = \ln\frac{u(x)}{\beta}. \tag{10}$$

Then

$$\frac{\partial v}{\partial x_j} = \frac{1}{u(x)}\frac{\partial u}{\partial x_j} \quad \text{and} \quad \frac{\partial^2 v}{\partial x_j^2} = \frac{\frac{\partial^2 u}{\partial x_j^2}u - \left(\frac{\partial u}{\partial x_j}\right)^2}{u^2(x)} = \frac{1}{u(x)}\left[\frac{\partial^2 u}{\partial x_j^2} - \frac{1}{u(x)}\left(\frac{\partial u}{\partial x_j}\right)^2\right],\ j=\overline{1,n}.$$

Therefore, $\Delta u - \frac{|\nabla u|^2}{u} = u\Delta v = \beta e^v \Delta v$ (note that $u(x)$ vanishes nowhere). Hence, the inequality $\beta e^v \Delta v + \omega\big(x, \beta e^v\big) \leq 0$ holds in Ω_r, i.e., the inequality $\Delta v + \frac{e^{-v}}{\beta}\omega\big(x,\beta e^v\big) \leq 0$ holds in Ω_r. This means that the inequality $\Delta v + e^{-v}|x|^\sigma\Big(\ln e^v\Big)^q e^v \leq 0$ holds in the same domain. Thus, we obtain the inequality $\Delta v + |x|^\sigma v^q \leq 0$. The latter inequality is inequality (3). By virtue of [1, Th. 6.1], it has no classical nonnegative nontrivial solutions in Ω_r (under the assumptions of Th. 4). On the other hand, it follows from (10) that $0 \leq v(x) \not\equiv 0$. The obtained contradiction concludes the proof.

Remark 1. All the assertions proved above are valid if inequalities (1), (4), (6), and (9) are replaced by the following inequality of a more general kind:

$$\Delta u + \sum_{j=1}^{n} a_j(x,u)\left(\frac{\partial u}{\partial x_j}\right)^2 + \omega(x,u) \leq 0, \tag{11}$$

where the coefficients a_j satisfy the following conditions (for $j=\overline{1,n}$):
there exists a function g from $C[0,+\infty)$ such that $a_j(x,s) \geq g(s)$ on $\Omega_r \times [0,+\infty)$ (for Th. 1);
the inequality $a_j(x,s) \geq \frac{\alpha}{s^\beta}$ holds on $\Omega_r \times (0,+\infty)$ (for Th. 2);
the inequality $a_j(x,s) \geq \frac{\alpha}{s}$ holds on $\Omega_r \times (0,+\infty)$ (for Th. 3);
the inequality $a_j(x,s) \geq -\frac{1}{s}$ holds on $\Omega_r \times (0,+\infty)$ (for Th. 4).
To prove this, it suffices to note that any function $u(x)$, satisfying inequality (11), satisfies the corresponding inequality (i.e., (1), (4), (6), or (9)) a fortiori (under the corresponding assumption above).

5. Regular parabolic case.

Consider the inequality

$$\frac{\partial u}{\partial t} \geq \Delta u + a(x,t,u)|\nabla u|^2 + \omega(x,t,u) \tag{12}$$

and the initial-value condition

$$u\Big|_{t=0} = u_0(x). \tag{13}$$

The following assertion is valid:

THEOREM 5. *Let* $r>0, t_0>0, \sigma<-2,\ q>1,$ *and* $g \in C[0,+\infty)$ *be such that* $u_0 \in C(\Omega_r)$ *and the following inequalities hold in* $\Omega_r \times (0,t_0) \times \overline{\mathbb{R}^1_+}$*:*

$$a(x,t,s) \geq g(s) \quad \textit{and} \quad \omega(x,t,s) \geq |x|^\sigma \left(\int_0^s e^{\int_0^\tau g(\theta)d\theta} d\tau\right)^q e^{-\int_0^s g(\tau)d\tau}. \tag{14}$$

Then problem (12),(13) *has no classical nonnegative nontrivial solutions in* $\Omega_r \times (0,t_0)$.

Proof. Suppose, to the contrary, that a nonnegative function $u(x,t)$ is different from the identical zero and satisfies (in the classical sense) problem (12),(13) in $\Omega_r \times (0,t_0)$. Then the following inequality holds in $\Omega_r \times (0,t_0)$:

$$\frac{\partial u}{\partial t} \geq \Delta u + g(u)|\nabla u|^2 + |x|^\sigma \left(\int\limits_0^u e^{\int\limits_0^\tau g(\theta)d\theta} d\tau \right)^q e^{-\int\limits_0^u g(\tau)d\tau}.$$

Similarly to Th. 1, we define the function f on $[0,+\infty)$ by relation (2). Defining the function $v(x,t) \stackrel{\text{def}}{=} f[u(x,t)]$ on $\Omega_r \times (0,t_0)$, we see that

$$\frac{\partial v}{\partial t} = f'(u)\frac{\partial u}{\partial t},\ \frac{\partial v}{\partial x_j} = f'(u)\frac{\partial u}{\partial x_j},\ \text{and}\ \frac{\partial^2 v}{\partial x_j^2} = f''(u)\Big(\frac{\partial u}{\partial x_j}\Big)^2 + f'(u)\frac{\partial^2 u}{\partial x_j^2},\ j = \overline{1,n}.$$

Therefore,

$$\Delta v = f'(u)\Delta u + f''(u)|\nabla u|^2 = f'(u)\Big[\Delta u + g(u)|\nabla u|^2\Big].$$

Hence, $\Delta u + g(u)|\nabla u|^2 = \dfrac{\Delta v}{f'(u)}$ and $\dfrac{\partial u}{\partial t} = \dfrac{1}{f'(u)}\dfrac{\partial v}{\partial t}$ (because f' vanishes nowhere).
Thus, the following inequality holds in $\Omega_r \times (0,t_0)$:

$$\frac{1}{f'(u)}\frac{\partial v}{\partial t} \geq \frac{\Delta v}{f'(u)} + |x|^\sigma \frac{f^q(u)}{f'(u)}.$$

By virtue of the (strict) positivity of the function f', it is equivalent to the inequality

$$\frac{\partial v}{\partial t} \geq \Delta v + |x|^\sigma v^q. \tag{15}$$

It follows from (2) that the function $v(x,t)$ satisfies the following initial-value condition in Ω_r:

$$v\Big|_{t=0} = v_0(x), \tag{16}$$

where $v_0(x) = \int\limits_0^{u_0(x)} e^{\int\limits_0^\tau g(t)dt} d\tau$.
The latter function is continuous in Ω_r, and its sign coincides with the sign of $u_0(x)$ at each point of Ω_r. However, $u_0(x)$ is nonnegative in Ω_r because $u(x,t)$ is a classical nonnegative solution of problem (12),(13). Finally, $v(x,t)$ is different from the identical zero because $u(x,t)$ would be equal to the identical zero otherwise (due to (2)), while it is different from the identical zero by assumption. However, by virtue of [1, Ch. 3, §28], problem (15),(16) has no classical nonnegative nontrivial solutions in $\Omega_r \times (0,t_0)$ under the restrictions imposed on q and σ.
The obtained contradiction completes the proof.

Remark 2. Th. 5 is valid if the term $a(x,t,u)|\nabla u|^2$ in (12) is replaced by the following term of a more general kind:

$$\sum_{j=1}^n a_j(x,t,u)\left(\frac{\partial u}{\partial x_j}\right)^2 \tag{17}$$

provided that its coefficients a_j, $j \in \overline{1,n}$, satisfy condition (14) for the function a.
This is proved in the same way as Remark 1.

6. Singular parabolic case.

The above approach can be applied for the case were the coefficient at the principal nonlinear term of inequality (12) is singular (cf. Sec. 3). More exactly, the following assertion is valid:

THEOREM 6. *Let* $r > 0, t_0 > 0, \sigma < -2$, $q > 1$, $\alpha \neq 0$, *and* $\beta \in (0,1) \cup (1,+\infty)$ *be such that* $u_0 \in C(\Omega_r)$ *and the following inequalities are valid in* $\Omega_r \times (0,t_0) \times \mathbb{R}^1_+$:

$$a(x,t,s) \geq \frac{\alpha}{s^\beta} \quad \textit{and} \quad \omega(x,t,s) \geq |x|^\sigma \left(\int_0^s e^{\frac{\alpha}{1-\beta}\tau^{1-\beta}} d\tau \right)^q e^{\frac{\alpha}{\beta-1}s^{1-\beta}}. \tag{18}$$

Then problem (12),(13) *has no classical positive solutions in* $\Omega_r \times (0,t_0)$.

Proof. Suppose, to the contrary, that a positive function $u(x,t)$ satisfies problem (12),(13) in $\Omega_r \times (0,t_0)$. Then the following inequality holds in $\Omega_r \times (0,t_0)$:

$$\frac{\partial u}{\partial t} \geq \Delta u + \frac{\alpha}{u^\beta}|\nabla u|^2 + |x|^\sigma \frac{\left(\int_0^u e^{\frac{\alpha}{1-\beta}\tau^{1-\beta}} d\tau \right)^q}{e^{\frac{\alpha}{1-\beta}u^{1-\beta}}}.$$

Define the function f on $(0,+\infty)$ by relation (5) and define the function $v(x,t) \stackrel{\text{def}}{=} f[u(x,t)]$ on $\Omega_r \times (0,t_0)$. This yields the relation

$$\Delta v = f'(u)\Delta u + f''(u)|\nabla u|^2 = f'(u)\left[\Delta u + \frac{\alpha}{u^\beta}|\nabla u|^2\right]$$

(see the proofs of Th. 1 and Th. 5).
This implies that $\Delta u + \frac{\alpha}{u^\beta}|\nabla u|^2 = \frac{\Delta v}{f'(u)}$ because f' vanishes nowhere. In the same way, we have $\frac{\partial u}{\partial t} = \frac{1}{f'(u)}\frac{\partial v}{\partial t}$.
Thus, the inequality $\frac{1}{f'(u)}\frac{\partial v}{\partial t} \geq \frac{\Delta v}{f'(u)} + |x|^\sigma \frac{f^q(u)}{f'(u)}$ holds in $\Omega_r \times (0,t_0)$. Due to the (strict) positivity of the function f', this inequality is equivalent to inequality (15). The function $v(x,t)$ satisfies condition (16) in Ω_r and the sign of the initial-value function $v_0(x)$ coincides with the sign of the function $u_0(x)$ at each point of Ω_r (due to (5)). The latter function is nonnegative because the function $u(x,t)$ is a classical positive solution of problem (12),(13). Finally, the function $v(x,t)$ is different from the identical zero because the function $u(x,t)$ would be equal to the identical zero otherwise (by virtue of (5)), while it is positive by assumption.
Similarly to Th. 5, we obtain a contradiction with [1]. This concludes the proof.

Remark 3. Th. 6 remains valid if the term $a(x,t,u)|\nabla u|^2$ in (12) is replaced by a term of kind (17) provided that its coefficients $a_j, j \in \overline{1,n}$, satisfy condition (18) for the function a.
The proof coincides with the proof of Remark 2.

7. Critical parabolic case.

In this section, condition (18) for the function a is considered in the case where $\beta = 1$. Substitution (5) is not applicable in this case, but the local blow-up still occurs.
The following assertion is valid:

THEOREM 7. *Let* $r > 0, t_0 > 0, \sigma < -2$, $\gamma > 1$, *and* $\alpha > -1$ *be such that* $u_0 \in C(\Omega_r)$ *and the inequalities*

$$a(x,t,s) \geq \frac{\alpha}{s} \quad \textit{and} \quad \omega(x,t,s) \geq \frac{|x|^\sigma s^\gamma}{\alpha+1} \tag{19}$$

hold in $\Omega_r \times (0, t_0) \times \mathbb{R}^1_+$. *Then problem* (12),(13) *has no classical positive solutions in* $\Omega_r \times (0, t_0)$.

Proof. Suppose, to the contrary, that a positive function $u(x,t)$ satisfies problem (12),(13) in $\Omega_r \times (0, t_0)$. Then the following inequality holds in $\Omega_r \times (0, t_0)$:

$$\frac{\partial u}{\partial t} \geq \Delta u + \frac{\alpha}{u}|\nabla u|^2 + \frac{|x|^\sigma u^\gamma}{\alpha + 1}.$$

Define the following function $v(x,t)$ on $\Omega_r \times (0, t_0)$:

$$v(x,t) = u^{\alpha+1}(x,t). \tag{20}$$

Then

$$\frac{\partial v}{\partial t} = (\alpha+1)u^\alpha(x,t)\frac{\partial u}{\partial t},\ \frac{\partial v}{\partial x_j} = (\alpha+1)u^\alpha(x,t)\frac{\partial u}{\partial x_j},$$

and

$$\frac{\partial^2 v}{\partial x_j^2} = \alpha(\alpha+1)u^{\alpha-1}(x,t)\left(\frac{\partial u}{\partial x_j}\right)^2 + (\alpha+1)u^\alpha(x)\frac{\partial^2 u}{\partial x_j^2} = (\alpha+1)u^\alpha(x)\left[\frac{\partial^2 u}{\partial x_j^2} + \frac{\alpha}{u(x)}\left(\frac{\partial u}{\partial x_j}\right)^2\right]$$

for $j = \overline{1,n}$.
Therefore, $\Delta v = (\alpha+1)u^\alpha\left(\Delta u + \frac{\alpha}{u}|\nabla u|^2\right)$. Since $(\alpha+1)u^\alpha$ vanishes nowhere, it follows that

$$\Delta u + \frac{\alpha}{u}|\nabla u|^2 = \frac{\Delta v}{(\alpha+1)u^\alpha} = \frac{\Delta v}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{1}{(\alpha+1)u^\alpha}\frac{\partial v}{\partial t} = \frac{1}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}}\frac{\partial v}{\partial t}$$

(note that, by virtue of (20), the function $v(x,t)$ is positive in the domain $\Omega_r \times (0, t_0)$ because the function $u(x,t)$ is positive in the same domain). Thus, the following inequality holds in the domain $\Omega_r \times (0, t_0)$:

$$\frac{1}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}}\frac{\partial v}{\partial t} \geq \frac{\Delta v}{(\alpha+1)v^{\frac{\alpha}{\alpha+1}}} + \frac{|x|^\sigma v^{\frac{\gamma}{\alpha+1}}}{\alpha+1}.$$

This inequality is equivalent to the inequality $\frac{\partial v}{\partial t} \geq \Delta v + |x|^\sigma v^{\frac{\alpha+\gamma}{\alpha+1}}$ by virtue of the positivity of $(\alpha+1)v^{\frac{\alpha}{\alpha+1}}$.
Denoting $\frac{\alpha+\gamma}{\alpha+1}$ by q, we see that there exist $q > 1$ and $\sigma \leq -2$ such that $v(x,t)$ is a nontrivial (because it is positive) solution of inequality (15) in $\Omega_r \times (0, t_0)$. The function $v(x,t)$ also satisfies condition (16) with the initial-value function $v_0(x) \stackrel{\text{def}}{=} u_0^{\alpha+1}(x)$, which is continuous in Ω_r due to the continuity of the function $u_0(x)$ in the domain Ω_r.
Thus, we obtain a contradiction with [1]. This completes the proof.

If $\alpha = -1$, then substitution (20) is, obviously, not applicable. However, a (weakened) result on the local blow-up of solutions can be obtained in this critical case as well. More exactly, the following assertion is valid:

THEOREM 8. *Let* $r > 0, t_0 > 0, \sigma \leq -2$, $\beta > 0$, *and* $q > 1$ *be such that* $u_0 \in C(\Omega_r)$ *and the inequalities*

$$a(x,t,s) \geq -\frac{1}{s} \quad \textit{and} \quad \omega(x,t,s) \geq |x|^\sigma s\left(\ln\frac{s}{\beta}\right)^q$$

hold in $\Omega_r \times (0, t_0) \times [\beta, +\infty)$. *Then problem* (12),(13) *has no classical solutions in* $\Omega_r \times (0, t_0)$, *satisfying the condition* $\beta \leq u(x,t) \not\equiv \beta$.

Proof. Suppose, to the contrary, that a function $u(x,t)$ satisfies problem (12),(13) in the domain $\Omega_r \times (0,t_0)$ and is such that the inequality $\beta \not\equiv u(x,t) \geq \beta$ is valid $\Omega_r \times (0,t_0)$. Then the following inequality holds in the domain $\Omega_r \times (0,t_0)$:

$$\frac{\partial u}{\partial t} \geq \Delta u - \frac{|\nabla u|^2}{u} + |x|^\sigma u \left(\ln \frac{u}{\beta}\right)^q.$$

Define the following function $v(x,t)$ on $\Omega_r \times (0,t_0)$:

$$v(x,t) = \ln \frac{u(x,t)}{\beta}. \tag{21}$$

Then

$$\frac{\partial v}{\partial x_j} = \frac{1}{u(x,t)} \frac{\partial u}{\partial x_j} \text{ and } \frac{\partial^2 v}{\partial x_j^2} = \frac{\frac{\partial^2 u}{\partial x_j^2} u - \left(\frac{\partial u}{\partial x_j}\right)^2}{u^2(x,t)} = \frac{1}{u(x,t)} \left[\frac{\partial^2 u}{\partial x_j^2} - \frac{1}{u(x,t)} \left(\frac{\partial u}{\partial x_j}\right)^2\right], j = \overline{1,n},$$

and

$$\frac{\partial v}{\partial t} = \frac{1}{u(x,t)} \frac{\partial u}{\partial t}.$$

Therefore,

$$\Delta u - \frac{|\nabla u|^2}{u} = u\Delta v = \beta e^v \Delta v, \quad \frac{\partial u}{\partial t} = \beta e^v \frac{\partial v}{\partial t}$$

(note that u vanishes nowhere), and $u\left(\ln \frac{u}{\beta}\right)^q = \beta e^v v^q$ (note that v is nonnegative due to (21)). Thus, the inequality

$$\beta e^v \frac{\partial v}{\partial t} \geq \beta e^v \Delta v + \beta e^v |x|^\sigma v^q$$

holds in $\Omega_r \times (0,t_0)$. Therefore, inequality (15) holds in the same domain. Further, the function $v(x,t)$ satisfies condition (16) with $v_0(x) = \ln \frac{u_0(x)}{\beta}$. The latter function is continuous in Ω_r because $u_0 \in C(\Omega_r)$ and $u_0(x) \geq \beta$ in Ω_r (the latter inequality is valid because the function $u(x,t)$ satisfies it, while this function is a classical solution of problem (12),(13)). Finally, the function $u(x,t)$ is different from the constant β. Then relation (21) implies that $v(x,t)$ is not only nonnegative, but is different from the identical zero.

Thus, the function $v(x,t)$ is a classical nonnegative nontrivial solution of problem (15),(16) in $\Omega_r \times (0,t_0)$ and the initial-value function of the latter problem is continuous. This contradicts [1]. The obtained contradiction completes the proof.

Remark 4. Th. 7 and Th. 8 remain valid if the term $a(x,t,u)|\nabla u|^2$ in (12) is replaced by a term of kind (17) provided that its coefficients $a_j, j \in \overline{1,n}$, satisfy condition (19) (in the case of Th. 8, it is assigned $\alpha = -1$ in the latter condition) for the function a.

The proof coincides with the proof of Remark 3.

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1. *Mitidieri E., Pohozaev S.I.* A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities. Proc. Steklov Inst. Math. 2001. Vol. 234.
2. *Moschini L., Pohozaev S.I., Tesei A.* Existence and nonexistence of solutions of nonlinear Dirichlet problems with first order terms // J. Funct. Anal. 2000. Vol. 177. P. 365–382.
3. *Galakhov E.* Some nonexistence results for quasilinear elliptic problems // J. Math. Anal. and Appl. 2000. Vol. 252. P. 256–277.

4. *Brezis H., Cabré X.* Some simple nonlinear PDE's without solutions // Boll. Unione Mat. Ital. VII. Ser. B 1998. Vol. 8. P. 223–262.
5. *Kardar M., Parisi G., Zhang Y.-C.* Dynamic scaling of growing interfaces // Phys. Rev. Lett. 1986. Vol. 56. P. 889–892.
6. *Medina E., Hwa T., Kardar M., Zhang Y.-C.* Burgers equation with correlated noise: Renormalization group analysis and applications to directed polymers and interface growth // Phys. Rev. A. 1989. Vol. 39. P. 3053–3075.
7. *Bitsadze A.V.* Some classes of partial differential equations. Nauka (*in Russian*), 1981.
8. *Burgan J.R., Munier A., Feix M.R., Fijalkow E.* Homology and the nonlinear heat diffusion equation // SIAM J. Appl. Math. 1984. Vol. 44. P. 11–18.
9. *Denisov V.N., Muravnik A.B.* On stabilization of the solution of the Cauchy problem for quasilinear parabolic equations // Differ. Equ. 2002. Vol. 38. No. 3. P. 369–374.
10. *Muravnik A.B.* On a quasilinear analog of Gidas-Spruck theorem // Nonlinear Bound. Value Probl. 2004. Vol. 14. P. 105–111.

Computing Centre of the 4th Clinical Polyclinic of Voronezh City,
Lizyukova 24, Voronezh 394077, Russia
amuravnik@yandex.ru

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