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HÖLDER REGULARITY FOR THE GRADIENTS OF SOLUTIONS OF DEGENERATE PARABOLIC SYSTEMS

We study a class of parabolic systems of the form $v_t = \operatorname{div}(F(|Dv|Dv)$. The function F satisfies a few technical hypotheses which are satisfied, for example, by $F(s) = s^{p-2}$ with $p > 1$. Hence our results extend the standard results for the parabolic p-Laplacian operator. The method of proof is similar to the usual one but uses some new ideas about Poincaré-type inequalities and a special Gehring inequality.

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Dedicated to the memory of Academician Olga Aleksandrovna Ladyzhenskaya

Introduction.

In this work, we examine the regularity of solutions to the parabolic system

$$v_t = \operatorname{div}(F(|Dv|)Dv) \tag{0.1}$$

under appropriate hypotheses on the function F. If $F(\tau) = \tau^{p-2}$ for some $p > 1$, then solutions with bounded gradient are known to have Hölder continuous gradient; see, for example [8], [9], [3], [4]. The arguments there are readily adapted to somewhat more general F's (in C^2) as described in [5]*Chapter 7. Here, we consider a more general class of functions. We assume that $F \in C^1(0,\infty)$ is positive and there are constants $\delta \in (0,1]$ and $g_0 \geq 1$ such that

$$\delta - 1 \leq \frac{\tau F'(\tau)}{F(\tau)} \leq g_0 - 1 \tag{0.2}$$

for all $\tau > 0$. We also assume a technical restriction on the modulus of continuity of F' (see (1.2b) below) which includes the results already cited, but they include other equations as well. For example (see pages 313 and 314 from [14]), the function F can map any interval of the form $(0,\varepsilon)$ onto $(0,\infty)$, so our equation need not be singular or degenerate in the usual sense. In addition, all previous proofs distinguish between $p > 2$ and $p < 2$ for $F(\tau) = \tau^{p-2}$. There are certain qualitative differences in the behavior of solutions in these two cases (a theme in [7]), but the differences are not relevant to the Hölder gradient estimate.

In Section 1, we give a basic Hölder continuity result for the gradient of a solution of (0.1) based on an alternative which we present in Propositions 1.3 and 1.4. We provide some preliminary results in Section 2: an algebraic lemma and the observations that we can replace the ordinary mean value of a function in Poincaré's inequality by a more general mean value (see [1, Lemma 2] and [17, Lemma 6.13]). Our regularity theorem is derived from these propositions in Section 3. For the convenience of the reader, we provide a brief

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proof of these propositions in Sections 4 and 5. In addition to the ideas about mean values, we use a parabolic version [11, Proposition 1.3] of Gehring's lemma [10, Lemma 3]. Thus, our approach is closer to that for elliptic systems (see [19]) than the one in [8], [9].

Of course, most of this paper could have been written fifteen years ago, and, in fact, most of it was. Recent work of Misawa [18] on regularity for solutions of inhomogeneous equations indicates a renewed interest in this problem, and we hope in future to extend his ideas to the full range of functions indicated above. In particular, the proofs of regularity (due to the present author) for a single inhomogeneous equation in [13, Theorem 1] and [16, Theorem 1.6] have flaws. In [13], the oscillation of Dv is not properly controlled if $[v]_1^* < 1$, and the second to last equation on page 558 of [16] is missing a factor of $(M_0 r)^{-\lambda\sigma/2}$.)

1. Assumptions and main results.

Let F be a $C^1(0,\infty)$ function satisfying (0.2) for constants $\delta \in (0,1]$ and $g_0 \geq 1$. We consider the function A_i^α defined by

$$A_i^\alpha(p) = F(|p|)p_\alpha^i, \tag{1.1}$$

and we define

$$A_{ij}^{\alpha\beta} = \partial A_i^\alpha / \partial p_\beta^j.$$

Observing the usual summation convention that repeated Latin indices are summed from 1 to N, and repeated Greek indices are summed from 1 to n, we see that these conditions guarantee that there are positive constants λ_0 and Λ_0 such that

$$A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq \lambda_0 F(|p|)|\xi|^2 \quad \text{and} \quad |A_{ij}^{\alpha\beta}| \leq \Lambda_0 F(|p|) \tag{1.2a}$$

for all $\xi \in \mathbb{R}^{nN}$. (In fact, $\lambda_0 = \delta$ and $\Lambda_0 = g_0$, but we shall use these conditions directly.) We also assume that there is a continuous increasing function ω defined on (0,1/2] with $\omega(0) = 0$ such that

$$|A_{ij}^{\alpha\beta}(p) - A_{ij}^{\alpha\beta}(p')| \leq \omega(\frac{|p-p'|}{|p|})F(|p|) \tag{1.2b}$$

for $|p - p'| \leq (1/2)|p|$.

Our main result is the following oscillation estimate, which implies the Hölder continuity of Dv in the scaled cylinders

$$Q(R, M) = \{X = (x,t) : |x - x_0| < R, -R^2/F(M) < t - t_0 < 0\}.$$

THEOREM 1.1. *Let R_0 and M_0 be positive constants and let v be a bounded weak solution of*

$$v_t = D_\alpha(A^\alpha(Dv)) \text{ and } |Dv| \leq M_0 \quad \text{in } Q(R_0, M_0). \tag{1.3}$$

If (0.2), (1.1), *and* (1.2) *hold, then there are constants C and μ depending only on δ, g_0, Λ, λ, N, n, and ω such that*

$$osc_{Q(r,M_0)} Dv \leq CM_0 \left(\frac{r}{R_0}\right)^\mu \tag{1.4}$$

for all $r \in (0, R)$.

We can also give an oscillation estimate in terms of the usual cylinders

$$Q(r) = \{X : |x - x_0| < r,\ -r^2 < t - t_0 < 0\}$$

in a form only slightly weaker than the ones in Theorems 1.1′ and 1.1″ of [7, Chapter IX].

COROLLARY 1.2. *Let R_0 and M_0 be positive constants and let v be a bounded weak solution of*

$$v_t = D_\alpha(A^\alpha(Dv)) \text{ and } |Dv| \le M_0 \text{ in } Q(R_0). \tag{1.5}$$

If (0.2), (1.1), *and* (1.2) *hold, then there are positive constants C and μ determined only by δ, g_0, λ, Λ, M, n, and ω such that*

$$osc_{Q(r)} Dv \le C M_0 \left(\frac{rF^*(M_0)}{R_0}\right)^\mu, \tag{1.6}$$

where $F^*(\tau) = \max\{F(\tau)^{1/2}, 1/F(\tau)^{1/2}\}$.

Proof. If $F(M_0) \ge 1$, (1.6) is clear when $r \ge R_0 F(M_0)^{1/2}$, so we may assume that $r < R_0 F(M_0)^{1/2}$ and then we set $\rho = rF(M_0)^{1/2}$. Then $Q(r) \subset Q(\rho, M_0)$ and $Q(R_0, M_0) \subset Q(R_0)$ so Theorem 1.1 implies that

$$osc_{Q(r)} Dv \le osc_{Q(\rho,M_0)} Dv \le C M_0 \left(\frac{\rho}{R_0}\right)^\mu = C M_0 \left(\frac{rF^*(M_0)}{R_0}\right)^\mu.$$

If $F(M_0) < 1$, we set $\rho = R_0 F(M_0)^{1/2}$. Then $Q(r) \subset Q(r, M_0)$ and $Q(\rho, M_0) \subset Q(R_0)$, so

$$osc_{Q(r)} Dv \le osc_{Q(r,M_0)} Dv \le C M_0 \left(\frac{r}{\rho}\right)^\mu = C M_0 \left(\frac{rF^*(M_0)}{R_0}\right)^\mu.$$

□

The proof of Theorem 1.1 is based on two simple propositions which are proved in Sections 4 and 5. To state these propositions, we consider solutions of the parabolic system

$$v_t = D_\alpha(A_i^\alpha(Dv)) \quad \text{and} \quad |Dv| \le M \quad \text{in} \quad Q(R, M). \tag{1.7}$$

The first proposition gives an estimate when $|Dv|$ is large on most of $Q(R, M)$.

PROPOSITION 1.3. *Let v satisfy* (1.7) *for some positive constants M and R. If* (0.2), (1.1), *and* (1.2) *hold, then there are positive constants $\sigma < 1$ and C_1 determined only by δ, g_0, n, N, and ω such that*

$$|\{|Dv| > (1 - \sigma)M\} \cap Q(R, M)| \ge (1 - \sigma)|Q(R, M)| \tag{1.8}$$

implies

$$osc_{Q(r,M)} Dv \le C_1 \left(\frac{r}{R}\right)^{1/2} osc_{Q(R,M)} Dv \tag{1.9}$$

for all $r \in (0, R)$.

Usually, a related estimate on mean oscillations is proved, and we shall do so as part of the proof of this proposition. This estimate implies (1.4) by virtue of da Prato's result [6]*Theorem 3.Ib; a little more care is needed to infer (1.9).

The second proposition is an estimate on how fast $|Dv|$ shrinks if (1.8) fails.

PROPOSITION 1.4*Let v satisfy* (1.7) *for some positive constants M and R. If* (0.2) *and* (1.1) *hold, then for any $\sigma \in (0,1)$, there is a constant $\eta \in (0,1)$, determined only by δ, g_0, n, N, and σ, such that*

$$|\{|Dv| > (1-\sigma)M\} \cap Q(R,M)| < (1-\sigma)|Q(R,M)| \tag{1.8$'$}$$

implies

$$\sup_{Q(\sigma R/2,M)} |Dv| \leq \eta M. \tag{1.9$'$}$$

2. Preliminaries.

We now give some results which are used to prove Theorem 1.1 as well as Propositions 1.3 and 1.4. A key step is the following algebraic lemma, which is similar to Lemmata 5.1 and 5.2 of [7, Chapter X].

LEMMA 2.1. *Let U and V be tensors in $\mathbb{R}^{nN}$ with $|U| \leq |V|$ and suppose that A and F satisfy* (0.2) *and* (1.2a). *Then there is a constant $c_1(\delta, \Lambda_0)$ such that, for any $\kappa \in [0,1]$,*

$$|A(U) - A(V)| \leq c_1 F(|V|) V^{1-\kappa} |U-V|^{\kappa}. \tag{2.1a}$$

In addition, there is a positive constant $c_2(g_0, \lambda_0)$ such that

$$[A(U) - A(V)] \cdot [U - V] \geq c_2 F(|V|)|U-V|^2. \tag{2.1b}$$

Finally, if (1.2b) *also holds, then for any $\varepsilon \in (0,1)$ and $\eta > 0$, there is a constant $c_3(\delta, \varepsilon, \eta, \Lambda_0, \omega)$ such that*

$$|A^{\alpha\beta}_{ij}(V)(U^i_\beta - V^i_\beta) - A^\alpha_i(U) + A^\alpha_i(V)| \leq F(|V|)|U-V|(\varepsilon + c_3|V|^{-\eta}|U-V|^{\eta}). \tag{2.1c}$$

Proof. To prove (2.1a), we suppose first that $|U-V| \leq |V|/2$. In this case,

$$\frac{1}{2}|V| \leq |V + \tau(U-V)| \leq |V|, \tag{2.2}$$

so, after using the integral form of the mean value theorem for $A(U) - A(V)$, we see that

$$|A(U) - A(V)| \leq 2\Lambda_0 F(|V|)|U-V| \leq 2\Lambda_0 F(|V|)|V|^{1-\kappa}|U-V|^{\kappa}.$$

On the other hand, if $|U-V| > |V|/2$, then

$$\begin{aligned} |A(U) - A(V)| &\leq \frac{2\Lambda_0}{\delta}(|U|F(|U|) + F(|V|)|V|) \leq \frac{4\Lambda_0}{\delta}F(|V|)|V| \\ &= \frac{4\Lambda_0}{\delta}F(|V|)|V|^{1-\kappa}|V|^{\kappa} \leq \frac{8\Lambda_0}{\delta}F(|V|)|V|^{1-\kappa}|U-V|^{\kappa}. \end{aligned}$$

For (2.1b), we use the integral form of the mean value theorem for $A(U) - A(V)$ to infer that

$$[A(U) - A(V)] \cdot [U - V] \geq \lambda_0 |U - V|^2 \int\limits_0^{1/4} F(|V + \tau(U - V)|)\, d\tau$$

For $\tau \in [0, 1/4]$, we have (2.2), so $F(|V + \tau(U - V)|) \geq 2^{-g_0} F(|V|)$. We immediately infer (2.1b) with $c_2 = \lambda_0 / 2^{2+g_0}$.

In proving (2.1c), first we set

$$H_i^\alpha = A_{ij}^{\alpha\beta}(V)(U_\beta^j - V_\beta^j) - A_i^\alpha(U) + A_i^\alpha(V) \tag{2.3}$$

and we fix $\theta \in (0, 1/2)$ so small that $\omega(\theta) \leq \varepsilon$. If $|U - V| \leq \theta |V|$, then the integral form of the mean value theorem for H gives

$$|H| \leq \int_0^1 \omega\left(\tfrac{\tau|U-V|}{|V|}\right) d\tau F(|V|)|U - V| \leq \omega(\theta) F(|V|)|U - V| \leq \varepsilon F(|V|)|U - V|,$$

which implies (2.1c) in this case.

On the other hand, if $|U - V| > \theta |V|$, then

$$|H| \leq \Lambda_0 F(|V|)|U - V| + \frac{2\Lambda_0}{\delta} F(|V|)|V| \leq \Lambda_0 \left(1 + \frac{2}{\delta\theta}\right) F(|V|)|U - V|.$$

Using the inequality $|U - V| > \theta|V|$ in the forms $|V| < |U - V|/\theta$ and $|U - V| < \theta^{-\eta}|V|^{-\eta}|U - V|^{-\eta}$ gives (2.1c) in this case with $c_3 = \Lambda_0(1 + 2/(\delta\theta))\theta^{-\eta}$.

□

We also consider an alternative mean value. For $r > 0$, we say that $\zeta \in L^\infty(B(r))$ is a *weight on* $B(r)$ if ζ is nonnegative and $\int_{B(r)} \zeta\, dx = 1$. For a function $u \in L^1(B(r))$, we call the number $\int_{B(r)} \zeta u\, dx$ the ζ*-mean value of* u. If ζ is a weight on $B(r)$, then [17, (6.18)] says that, for any $p > 1$, there is a constant $c_0(n, p, r^n \sup \zeta)$ such that

$$\int_{B(r)} |u|^p\, dx \leq c_0 r^{-p} \int_{B(r)} |Du|^p\, dx \tag{2.4}$$

for any $u \in W^{1,p}$ with ζ-mean value equal to zero. In addition (by arguing as in [15, Lemma 1.1]), there is a constant $s_0(n, r^n \sup \zeta)$ such that

$$\int_{B(r)} u^2\, dx \leq s_0 \left(\int_{B(r)} |Du|^{2n/(n+2)}\, dx\right)^{(n+2)/n} \tag{2.5}$$

for any $u \in W^{1,2n/(n+2)}$.

Next, we define the integral average

$$\fint_S w\, dX = \frac{1}{|S|} \int\limits_S w\, dX$$

for any measurable subset S of $\mathbb{R}^{n+1}$ with Lebesgue measure $|S|$ and any $w \in L^1(S)$. We also recall that, for any measurable set S with positive measure,

$$\int_S |u - U^*|^2 \, dx = \inf_{L \in \mathbb{R}} \int_S |u - L|^2 \, dx, \tag{2.6}$$

where $U^* = \fint_S u \, dX$ is the usual mean value of u.

Finally, we note that there is (at least) one weight ζ for $B(r)$ such that $\zeta \in C^2(\overline{B(r)})$, ζ and $D\zeta$ vanish on $\partial B(r)$, and $|\zeta| + r|D\zeta| + r^2|D^2\zeta| \le C(n)r^{-n}$. We call any such function a *cut-off weight function.*

3. Proof of Theorem 1.1.

We follow the argument of [9] to prove spatial continuity. We first choose σ from Proposition 1.3 and then η from Proposition 1.4, and finally γ so that $C_1\gamma^{1/2} \le \eta$ and $\gamma \le \sigma\eta^{g_0/2}/2$ for the constant C_1 from Proposition 1.3. We now define $R_j = \gamma^j R_0$ and $M_j = \eta^j M_0$ for any positive integer j. If $(1.8)'$ holds, then

$$\sup_{Q(\sigma R_0/2, M_0)} |Dv| \le \eta \sup_{Q(R_0, M_0)} |Dv| \le \eta M_0,$$

and

$$\frac{\sigma^2 R_0^2}{4F(M_0)} = \frac{\sigma^2 R_1^2}{4\gamma^2 F(M_0)} \ge \frac{\sigma^2 \eta^{g_0} R_1^2}{4\gamma^2 F(M_1)} \ge \frac{R_1^2}{F(M_1)}.$$

It follows that $Q(R_1, M_1) \subset Q(\sigma R_0/2, M_0)$ and hence $|Dv| \le M_1$ in $Q(R_1, M_1)$. Similar reasoning shows that as long as $(1.8)'$ holds with R and M replaced by R_{j-1} and M_{j-1}, respectively, we have $|Dv| \le M_j$ in $Q(R_j, M_j)$, and hence

$$osc_{Q(R_j, M_j)} Dv \le 2\eta^j M_0. \tag{3.1a}$$

On the other hand, if J is the first integer j for which (1.8) holds with R and M replaced by R_{j-1} and M_{j-1}, respectively, then we have

$$osc_{Q(R_j, M_J)} Dv \le \eta^{j-J} osc_{Q(R_J, M_J)} Dv \le 2\eta^j M_0 \tag{3.1b}$$

for any integer $j \ge J$.

To proceed, we fix a point $X_1 = (x_0, t_1)$ with $t_0 \ge t_1 \ge t_0 - \frac{1}{2}R_0^2/F(M_0)$ and let $r \le R_0/2$. We use (3.1a) and (3.1b) with $R_0/2$ in place of R_0 to define R_j and $Q(r, M_0, X_1)$ in place of $Q(r, M_0)$. By choosing j so that $R_j \le r < R_{j-1}$, and setting $\theta = \log_{1/\gamma}(1/\eta)$, we see that

$$|Dv(y, t_1) - Dv(x_0, t_1)| \le CM_0 \left(\frac{r}{R_0}\right)^\theta \tag{3.2}$$

as long as $|y - x_0| \le r$.

We prove the continuity in time via a different approach. Setting $B = B(r)$, we let ζ be a cut-off weight function in B and we define $W(t)$ to be the ζ-mean value of $Dv(\cdot, X)$. The triangle inequality implies that

$$|Dv(X) - Dv(x_0, t_0)| \le CM \left(\frac{r}{R_0}\right)^\theta + |W(t) - W(t_0)|. \tag{3.3}$$

for any $X \in Q(r, M_0)$. In addition, an integration by parts along with the weak form of the differential equation in (1.7) gives

$$W^i_\alpha(t) - W^i_\alpha(t_0) = \int_B D_\alpha\zeta(y)v^i(y,t_0)\,dy - \int_B D_\alpha\zeta(y)v^i(y,t)\,dy$$
$$= -\int_t^{t_0}\int_B D_{\alpha\beta}\zeta(y)[A^\beta_i(Dv(Y)) - A^\beta_i(W(s))]\,dY$$

because $D\zeta$ vanishes on ∂B and $A(W(s))$ is independent of y. It follows that

$$|W(t) - W(t_0)| \le C(n)r^{-n-2}\int_{Q(r,M_0)} |A(Dv(Y)) - A(W(s))|\,dY.$$

Next, we use (2.1a) with $\kappa = \delta$ to infer that

$$|W(t) - W(t_0)| \le Cr^{-n-2}\int_{Q(r,M_0)} F(h)h^{1-\delta}|Dv(Y) - W(s)|^\delta\,dY, \tag{3.4}$$

where $h = h(Y) = \max\{|Dv(Y)|, |W(s)|\}$. Noting that $0 \le h \le M_0$ and that $W(s) = Dv(x_1, s)$ for some $x_1 \in B(r)$, we infer that

$$|W(t) - W(t_0)| \le Cr^{-n-2}|Q(r,M_0)|F(M_0)M_0^{1-\delta}\left[M_0\left(\frac{r}{R_0}\right)^\theta\right]^\delta = CM_0\left(\frac{r}{R_0}\right)^{\delta\theta}.$$

Combining this estimate with (3.3) and (3.2) yields (1.4) with $\mu = \delta\theta$ because the inequality is obvious for $r \ge R_0/2$.

4. Proof of Proposition 1.3.

We always assume in this section that v is a solution of (1.7) and that A and F satisfy (0.2) and (1.2a). Further assumptions will be made as needed.

Our first step is an mean oscillation estimate for a related constant coefficient problem. To state this result, we write $\{w\}_R = \int_{Q(R,\overline{M})} w\,dX$.

LEMMA 4.1. *Let V be a tensor such that $M/2 \le |V| \le M$. If $\bar v$ solves*

$$-\bar v_t = D_\alpha(A^{\alpha\beta}_{ij}(V)D_\beta\bar v^j) \text{ in } Q(R/2,M),\ \bar v = v \text{ on } \mathcal{P}Q(R/2,M), \tag{4.1}$$

then there is a constant C_1, determined only by n, N, λ, Λ, g_0, and δ such that

$$\int_{Q(\rho,M)} |D\bar v - \{D\bar v\}_\rho|^2\,dX \le C_1\left(\frac{\rho}{R}\right)^{n+4}\int_{Q(R/2,M)} |D\bar v - V|^2\,dX \tag{4.2}$$

for all $\rho \in (0, R/2)$.

Proof. Since $\bar v$ satisfies a system of constant coefficient differential equations, a straightforward modification of the Campanato technique ([2]; see [9, Lemma 3.2] or [7, Theorem IX.6.1] for details of the modification) implies (4.2).

□

Our next step is a reverse Hölder inequality. The statement is the same as [9, Lemma 3.3] but the proof is essentially that of [19, Lemma 6.1].

LEMMA 4.2 *Let V be a tensor such that $M/2 \le |V| \le M$. Then there are positive constants η and C_2, determined only by δ, g_0, λ, Λ, n, and N such that*

$$\fint_{Q(R/2,M)} |Dv - V|^{2+2\eta}\,dX \le C_2 \left(\fint_{Q(R,M)} |Dv - V|^2\,dX \right)^{1+\eta}. \tag{4.3}$$

Proof. Fix $X_1 \in Q(R,M)$. For brevity, we write $K(r)$ for $Q(r,M,X_1)$. We now choose $r > 0$ so that $K(4r) \subset Q(R,M)$ and we set $t_2 = t_1 - 4r^2/F(M)$. We define w by $w(X) = v(X) - V\cdot(x - x_1)$, we take ζ to be a cut-off weight function in $B(X_1,4r)$, and we set

$$w_0 = r^{-2}F(M)\int_{K(4r)} \zeta w\,dX, \quad \bar w = w - w_0, \quad W(t) = \int_{B(x_1,4r)} \zeta(x)w(x,t)\,dx.$$

We use ψ to denote a $C^2(\overline{K(4r)})$ function which vanishes on the parabolic boundary of $K(4r)$ with $0 \le \psi \le 1$ in $K(4r)$ and $\psi \equiv 1$ on $K(r)$. In addition $|\psi_t| \le C(n)r^{-2}F(M)$ and $|D\psi| \le C(n)/r$. With $q \ge 2$ to be further specified, we then use $\psi^q \bar w$ as test function in the weak form of the equation for v. Some simple rearrangement, along with Lemma 2.1, yields

$$S_q + F(M)\int_{K(4r)} |Dw|^2\psi^q\,dX \le Cq^2r^{-2}F(M)\int_{K(4r)} |\bar w|^2\psi^{q-2}\,dX. \tag{4.4}$$

for

$$S_q = \sup_{t_2<t<t_1} \int_{B(X_1,4r)\times\{t\}} |\bar w|^2\psi^q\,dx.$$

We now choose $q = 4$ and conclude from (4.4) and Hölder's inequality that

$$\int_{K(r)} |Dw|^2\,dX \le Cr^{-2}S_2^{2/(n+2)} \int_{t_2}^{t_1} \left(\int_{B(X_1,4r)\times\{t\}} |\bar w|^2\,dx \right)^{n/(n+2)} dt. \tag{4.5}$$

We then invoke (4.4) with $q = 2$ to infer that

$$S_2 \le CF(M)r^{-2}\int_{K(4r)} |\bar w|^2\,dX.$$

Now, (2.4) tells us that

$$\int_{B(X_1,4r)} |w - W(t)|^2\,dx \le Cr^2\int_{B(X_1,4r)} |Dw|^2\,dx$$

and it is clear that

$$|w_0 - W(t)| \le \sup_{t_2<\tau'<\tau<t_1} |W(\tau) - W(\tau')|.$$

The differential equation for v shows that

$$W^i(\tau) - W^i(\tau') = \int_{\tau'}^{\tau}\int_{B(X_1,4r)} D_\alpha\zeta(x)(A_i^\alpha(Dv) - A_i^\alpha(V))\,dX,$$

and then (2.1a) with $\kappa = 0$ and Hölder's inequality imply that

$$|W(\tau) - W(\tau')| \le Cr \left(\fint_{K'(2r)} |Dw|^2 \, dX \right)^{1/2}.$$

It follows that

$$S_2 \le CF(M) \int_{K(4r)} |Dw|^2 \, dX. \tag{4.6}$$

If we use (2.5) in place of (2.4), the preceding argument shows that

$$\int_{t_2}^{t_1} \left(\int_{B(X_1,4r)\times\{t\}} |\bar{w}|^2 \, dx \right)^{n/(n+2)} dt \le C \int_{K(4r)} |Dw|^{2n/(n+2)} \, dX. \tag{4.7}$$

Upon combining (4.5), (4.6), and (4.7), we infer that, for any $\theta > 0$,

$$\fint_{K(r)} |Dw|^2 \, dX \le \theta \fint_{K(4r)} |Dw|^2 \, dX + C(\theta) \left(\fint_{K(4r)} |Dw|^{2n/(n+2)} \, dX \right)^{(n+2)/n},$$

and then [11]*Proposition 1.3 implies (4.3) if θ is sufficiently small (determined only by n).

□

Lemma 4.3. *In addition to the hypotheses of Lemma 4.1, suppose that A has the form* (1.1) *and that* (1.2b) *holds. Then, for any $\varepsilon_0 \in (0,1)$, there is a constant C_3 determined only by n, N, δ, ε_0, g_0, λ, Λ, and ω such that*

$$\int_{Q(R/2,M)} |D\bar{v} - Dv|^2 \, dX \le \left[\varepsilon_0 + C_3 \left(M^{-2} \fint_{Q(R,M)} |Dv - V|^2 \, dX \right)^{\eta} \right] \int_{Q(R,M)} |Dv - V|^2 \, dX \tag{4.8}$$

for η the constant from Lemma 4.2.

Proof. We define H by (2.3) with Dv in place of U and rewrite the differential equation in (1.7) as

$$-v_t + D_\alpha(A_{ij}^{\alpha\beta}(V) D_\beta v^i) = D_\alpha(H_i^\alpha).$$

By using $\bar{v} - v$ as test function in the equations for $\bar{v}$ and v and then using (2.1c), we find that

$$\begin{aligned}\int_{Q(R/2,M)} |D\bar{v} - Dv|^2 \, dX &\le \frac{C}{F(M)^2} \int_{Q(R/2,M)} |H|^2 \, dX, \\ &\le \varepsilon_0 \int_Q |Dv - V|^2 \, dX + C \int_Q M^{-\eta} |Dv - V|^{2+\eta} \, dX.\end{aligned}$$

The proof is completed by applying Lemma 4.2.

□

We now give a useful weak version of the differential equation. Set $t_1 = t_0 - R^2/F(M)$, let ψ be a nonnegative $C^1(\overline{Q(R,M)})$ function which vanishes on $\partial B(R) \times (t_1, t_0)$, let Γ be a nonnegative increasing $C^1([0,\infty))$, and define H by

$$H(s) = \int_0^s \sigma\Gamma(\sigma)\, d\sigma.$$

(The choices for Γ and ψ will vary depending on the context.) If we multiply the equation for v^i by $D_\gamma(\Gamma(|Dv|)D_\gamma v^i\psi^2)$, we find that

$$\begin{aligned}\int\limits_{Q(R,M)} & |Dv|F(|Dv|)a^{\alpha\beta}D_\alpha|Dv|D_\beta|Dv|\Gamma'\psi^2\, dX \\ &+ 2\int\limits_{Q(R,M)} |Dv|F(|Dv|)\Gamma a^{\alpha\beta}D_\alpha|Dv|D_\beta\psi\psi\, dX \\ &+ \int\limits_{Q(R,M)} A^{\alpha\beta}_{ij}D_{\beta\gamma}v^jD_{\alpha\gamma}v^i\Gamma\psi^2\, dX + \int_{B(R)\times\{t_0\}} H\psi^2\, dx \\ &= 2\int\limits_{Q(R,M)} H\psi\psi_t\, dX + \int_{B(R)\times\{t_1\}} H\psi^2\, dx.\end{aligned}$$

where the argument $|Dv|$ is omitted from Γ, Γ', and H, and we define

$$a^{\alpha\beta} = \delta^{\alpha\beta} + \frac{F'(|Dv|)}{F(|Dv|)}\frac{D_\alpha v^iD_\beta v^i}{|Dv|}$$

Now the matrix $[a^{\alpha\beta}]$ is symmetric and satisfies the matrix inequalities $\delta I \le [a^{\alpha\beta}] \le g_0 I$. It follows that

$$\begin{aligned}\int_{B(R)\times\{t_0\}} H\psi^2\, dx + \frac{\delta}{2}\int\limits_{Q(R,M)} \Gamma' F|Dv||D|Dv||^2\psi^2\, dX + \lambda_0\int\limits_{Q(R,M)} F|D^2v|^2\Gamma\psi^2\, dX \\ \le \int_{B(R)\times\{t_1\}} H\psi^2\, dx + \int\limits_{Q(R,M)} H\psi\psi_t\, dX + C\int\limits_{Q(R,M)} \frac{\Gamma^2}{\Gamma'}|Dv|F|D\psi|^2\, dX.\end{aligned} \tag{4.9}$$

Now we prove the crucial estimates which allow us to show that the mean oscillation of Dv decreases sufficiently fast. (This lemma is based on [8, Lemma 4.4.)

Lemma 4.4. *Let $\theta < 1/4$, ε, and ε_0 be positive constants and suppose that*

$$|\{Dv\}_R| \ge \frac{1}{2}M, \tag{4.10a}$$

$$⨍_{Q(R,M)} |Dv - \{Dv\}_R|^2\, dX \le \varepsilon M^2. \tag{4.10b}$$

Then

$$\int_{Q(\theta R,M)} |Dv - \{Dv\}_{\theta R}|^2\, dX \le \left(\varepsilon_1 + C_4\theta^{n+4}\right)\int\limits_{Q(R,M)} |Dv - \{Dv\}_R|^2\, dX, \tag{4.11}$$

where $\varepsilon_1 = 2(\varepsilon_0 + C_3\varepsilon^\eta)$ *and* $C_4 = 4C_1[1+\varepsilon_1]$.

Proof. Let $\bar v$ solve (4.1) with $V = \{Dv\}_R$. Then we have from (2.6) and the triangle inequality that

$$\int_{Q(\theta R,M)} |Dv - \{Dv\}_{\theta R}|^2\, dX \le 2\int_{Q(\theta R,M)} |Dv - D\bar v|^2\, dX + 2\int_{Q(\theta R,M)} |D\bar v - \{D\bar v\}_{\theta R}|^2\, dX.$$

We estimate the first integral on the right-hand side of this equation by using Lemma 4.3, (4.10) and the observation that increasing the region of integration of a nonnegative function increases the integral. Estimating the second integral via Lemma 4.1 then gives

$$\int_{Q(\theta R,M)} |Dv - \{Dv\}_{\theta R}|^2\, dX \le \varepsilon_1 \int_{Q(R,M)} |Dv - V|^2\, dX + 2C_1\theta^{n+4}\int_{Q(R/2,M)} |D\bar v - V|^2\, dX.$$

Now the triangle inequality implies that

$$\int\limits_{Q(R/2,M)} |D\bar v - V|^2\, dX \le 2 \int\limits_{Q(R/2,M)} |Dv - D\bar v|^2\, dX + 2\int\limits_{Q(R/2,M)} |Dv - V|^2\, dX.$$

We estimate the first integral here via Lemma 4.3 and the second integral by increasing the integration region to conclude that

$$\int\limits_{Q(R/2,M)} |D\bar v - V|^2\, dX \le 2[\varepsilon_1 + 2]C_1\theta^{n+4} \int\limits_{Q(R,M)} |Dv - V|^2\, dX.$$

Combining all our estimates yields (4.11).

□

Our next lemma is a restatement of [8, Lemma 4.5] in our present language.

LEMMA 4.5. *There are positive constants* θ *and* ε *such that if*

$$|\{Dv\}_R| \ge \frac{3}{4}M \tag{4.12}$$

and if (4.10b) *holds, then, for every nonnegative integer* i, *we have*

$$|\{Dv\}_{R(i)}| \ge \left(\frac{1}{2} + \frac{1}{2^{i+2}}\right) M, \tag{4.13a}$$

$$\int_{Q[i]} |Dv - \{Dv\}_{R(i+1)}|^2\, dX \le \theta^{n+7/2}\int_{Q[i]} |Dv - \{Dv\}_{R(i)}|^2\, dX, \tag{4.13b}$$

where $R(i) = \theta^i R$ *and* $Q[i] = Q(R(i), M)$.

Proof. Choose θ so that $[C_1 + 1]\theta \le 1/4$, then set $\varepsilon_0 = \theta^{n+3}/4$ and $\varepsilon = \min\{\varepsilon_0/C_2, \theta^{2n+4}/64\}$.

Then (4.11) implies (4.13b) for $i = 0$. In addition,

$$\begin{aligned} |\{Dv\}_{\theta R} - \{Dv\}_R| &= \left| \fint_{Q(\theta R, M)} [Dv - \{Dv\}_R] \, dX \right| \\ &\le \fint_{Q(\theta R, M)} |Dv - \{Dv\}_R| \, dX \le \theta^{-n-2} \fint_{Q(R,M)} |Dv - \{Dv\}_R| \, dX \\ &\le \theta^{-n-2} \left(\fint_{Q(R,M)} |Dv - \{Dv\}_R|^2 \, dX \right)^{1/2} \le \theta^{-n-2} \varepsilon^{1/2} M \le \frac{1}{8} M, \end{aligned}$$

and therefore (4.13a) holds for $i = 1$.

If conditions (4.13a) hold for all i less than or equal to some positive integer k and (4.13b) holds for all $i < k$, then Lemma 4.4 with $R(k)$ in place of R along with the argument outlined above implies (4.13b) with $i = k$. An easy induction argument shows that

$$\fint_{Q(R(k),M)} |Dv - \{Dv\}_{R(k)}|^2 \, dX \le \theta^k \varepsilon,$$

so $|\{Dv\}_{R(k+1)} - \{Dv\}_{R(k)}| \le \theta^{-n-2} (\theta^{3k/2} \varepsilon)^{1/2} \le 2^{-(k+3)}$, and hence (4.13a) holds also for $i = k + 1$.

□

Next, we show that condition (1.8), with suitable σ, implies that Dv stays close to its mean on most of $Q(R/2, M)$ and that this mean is comparable to M. This result is the same as [8, Lemma 5.1] but the proof is rather different.

Lemma 4.6. *Given a positive number ε, there is a constant $\sigma \in (0, 1)$ such that if (1.8) holds, then*

$$\frac{7}{8} M \le |\{Dv\}_{R/2}| \le M, \tag{4.14a}$$

$$\fint_{Q(R/2,M)} |Dv - \{Dv\}_{R/2}|^2 \, dX \le \varepsilon M^2. \tag{4.14b}$$

Proof. With $\theta \in (0, 1/4)$ at our disposal, we take $\Gamma(\tau) = (\tau - (1 - 2\theta)M)_+$ and we note that there is a positive constant C determined only by g_0 such that $F(|Dv|)/C \le F(M) \le CF(|Dv|)$ if $\Gamma(|Dv|) > 0$. We now choose ψ so that $|D\psi| \le 4/R$ and $|\psi_t| \le 16F(M)/R^2$ in $Q(R, M)$, $\psi \equiv 1$ in $Q(R/2, M)$, and $\psi(\cdot, t_2) = 0$. Writing

$$A(k, r) = \{X \in Q(r, M) : |Dv(X)| > k\},$$

we conclude from (4.9) that

$$\int_{A((1-\theta)M,R/2)} |D^2 v|^2 \, dX \le C\theta M^2 R^{-2} |Q(R/2, M)|. \tag{4.15}$$

Now, let h_0 be an increasing, $C^1(\mathbb{R})$ such that $h_0(\tau) = 0$ for $\tau \le 3M/4$, $h_0(\tau) = 1$ for $\tau \ge 7M/8$, and $h_0' \le 16/M$ on $\mathbb{R}$, and set $h = h_0(|Dv|)Dv$. Then $|Dh| \le C(n, N)|D^2 v|$.

To proceed, we let ζ be a cut-off weight function in $B(R)$, we write $W(t)$ and $W_0(t)$ for the ζ-means of $Dv(\cdot,t)$ and $h(\cdot,t)$, respectively, and we set

$$w = R^{-2}F(M) \int_{Q(R,M)} \zeta(x)Dv(X)\,dX.$$

Since $|h - W_0(t)|^2 \le C(n)M^{2/(n+1)}|h - W_0(t)|^{2n/(n+1)}$, (2.4) implies that

$$\int_{B(R/2)\times\{t\}} |h_0 - W(t)|^2\,dx \le C(|B(R)|M)^{2/(n+1)} \int_{B(R/2)\times\{t\}} |Dh|^{2n/(n+1)}\,dx$$

and hence

$$\int_{Q(R/2,M)} |h - W_0(t)|^2\,dX \le C(|B(R)|M)^{2/(n+1)} \int_{Q(R/2,M)} |Dh|^{2n/(n+1)}\,dX.$$

Now, we set

$$\Sigma = A((1-\sigma)M, R/2), \quad S = A(3M/4, R/2) \setminus \Sigma.$$

Since $|Dh| = 0$ on $Q(R/2) \setminus A(3M/4, R/2)$, we have

$$\int_{Q(R/2,M)} |Dh|^{2n/(n+1)}\,dX = \int_S |Dh|^{2n/(n+1)}\,dX + \int_\Sigma |Dh|^{2n/(n+1)}\,dX$$

Taking $\theta = 1/4$ in (4.15) then gives

$$\begin{aligned}\int_S |Dh|^{2n/(n+1)}\,dX &\le C|S|^{1/(n+1)} \left(\int_S |Dh|^2\,dX\right)^{n/(n+1)} \\ &\le C\sigma^{1/(n+1)}M^{2n/(n+1)}R^{-2n/(n+1)}|Q(R,M)|.\end{aligned}$$

A similar argument with $\theta = \sigma$ shows that

$$\int_\Sigma |Dh|^{2n/(n+1)}\,dX \le C\sigma^{n/(n+1)}M^{2n/(n+1)}R^{-2n/(n+1)}|Q(R,M)|.$$

Since $\sigma \le 1$, we conclude that

$$\int_{Q(R/2,M)} |h - W(t)|^2\,dX \le C\sigma^{1/(n+1)}M^2|Q(R/2,M)|. \tag{4.16}$$

To simplify the notation, we use $\|\cdot\|$ to denote the $L^2(Q(R/2,M))$ norm. It then follows from the triangle inequality that

$$\|Dv - w\| \le \|Dv - h\| + \|h - W_0\| + \|W - W_0\| + \|W - w\|.$$

Since $Dv = h$ on Σ, we have

$$\int_{Q(R/2,M)} |Dv - h|^2\,dX = \int_{Q(R/2,M)\setminus\Sigma} |Dv - h|^2\,dX,$$

so

$$\|Dv - h\|^2 \le C\sigma M^2 |Q(R/2, M)|. \tag{4.17}$$

Next,

$$\|h - W_0\|^2 \le C\sigma^{1/(n+1)} M^2 |Q(R/2, M)|$$

by (4.16). From Jensen's inequality and the estimate $|\zeta| \le CR^{-n}$, we have

$$\begin{aligned} \|W - W_0\|^2 &= \int_{t_1}^{t_0} \left| \int_{B(R/2)} \zeta(x)[Dv - h](X)\, dx \right|^2 |B(R/2)|\, dt \\ &\le C \int_{Q(R/2,M)} |Dv - h|^2\, dx \le C\sigma M^2 |Q(R/2, M)|. \end{aligned}$$

Finally,

$$\|W - w\| \le |Q(R/2, M)| \sup_{t_1 \le \tau \le \tau' \le t_0} |W(\tau) - W(\tau')|.$$

We then estimate $W(\tau) - W(\tau')$ as in the proof of Theorem 1.1. From (3.4) and Hölder's inequality, we now infer that

$$|W(\tau) - W(\tau')| \le CM^{1-\delta} \left(\fint_{Q(R/2,M)} |Dv - W(t)|^2\, dX \right)^{\delta/2}.$$

This integral is estimated via (4.16) and (4.17); we conclude that

$$|W(\tau) - W(\tau')| \le C\sigma^{\delta/(2n+2)} M.$$

Combining all our inequalities then yields

$$\int_{Q(R/2,M)} |Dv - \{Dv\}_{R/2}|^2\, dX \le C\sigma^{\delta/(2n+2)} M^2 |Q(R/2, M)|, \tag{4.18}$$

which implies (4.14b) provided σ is sufficiently small.

We now use the triangle inequality, followed by (1.8), Hölder's inequality and (4.18), to see that

$$|\{Dv\}_{R/2}| \ge \fint_{Q(R/2,M)} |Dv|\, dX - \fint_{Q(R/2,M)} |Dv - \{Dv\}_{R/2}|\, dX \ge [(1-\sigma)^2 - C\sigma^{\delta/(4n+4)}]M,$$

which yields (4.14a) upon taking σ sufficiently small.

□

To prove (1.9) from the mean oscillation estimate of Lemmata 4.5 and 4.6, we suppose that X_0 is taken so that (1.8) holds. With ε and ε' positive constants to be determined, we take σ from Lemma 4.6 corresponding to $\varepsilon/2$ in place of ε. If $X_1 \in Q(R, M)$ and $|X_0 - X_1| < \varepsilon' R$, then

$$|Q(R/2, M, X_1) \setminus Q(R/2, M, X_0)| \le C(n)\varepsilon' |Q(R, M)|$$

and therefore

$$\int_{Q(R/2,M,X_1)} |Dv - \{Dv\}_{X_0,R/2}|^2 \, dX \le |Q(R,M)| \left(C(n)\varepsilon' + \frac{\varepsilon}{2} \right).$$

It follows that

$$\int_{Q(R/2,M,X_1)} |Dv - \{Dv\}_{X_1,R/2}|^2 \, dX \le \varepsilon M^2$$

and $|\{Dv\}_{X_1,R/2}| \ge 3M/4$ if ε and ε' are sufficiently small. We then conclude that

$$\int_{Q(r,M,X_1)} |Dv - \{Dv\}_{X_1,r}|^2 \, dX \le C \left(\frac{r}{R} \right)^{1/2} \int_{Q(R,M,X_1)} |Dv - \{Dv\}_{X_1,R}|^2 \, dX$$

for any $r \le R$ and any X_1 as above. We then infer (1.9) from [6, Theorem 3.I.b] (see also [17, Lemma 4.3] for an alternative approach).

5. Proof of Proposition 1.4.

The proof of Proposition 1.4 is essentially the same as for the case of the parabolic p-Laplacian system ([8, Proposition 3.3] and [9, Proposition 2.2]). We follow the proof the corresponding result for a single equation [13, Lemmata 1.2–1.4], which is based on the proofs in [8], [9]; the major difference is that [13] uses Moser iteration rather than DeGiorgi iteration.

First, we introduce some notation. For $\theta \in (0, 1/2)$, we write

$$S(\theta) = \{X \in Q(R,M) : |Dv(X)| > (1-\theta)M\},$$
$$S(\theta, t) = \{x \in B(R) : |Dv(x,t)| > (1-\theta)M\}.$$

We also set $t_1 = t_0 - R^2/F(M)$ and $t_2 = t_0 - \sigma R^2/(2F(M))$, where $\sigma \in (0,1)$ is fixed.

Lemma 5.1. *If* (1.8)$'$ *holds, then there is* $t' \in (t_1, t_2)$ *such that* $|S(\sigma, t')| \le \left(1 - \frac{\sigma}{2}\right) |B(R)|$.

Proof. The proof of [13, Lemma 1.2] implies that

$$\left(1 - \frac{\sigma}{2}\right) \inf_{t_1 < s < t_2} |S(\sigma, s)| \le (1-\sigma)|B(R)|,$$

and the proof is completed by noting that $(1-\sigma)/(1-\sigma/2) < 1 - \sigma/2$.

□

Next, we define $\nu = (1 - (\sigma/2))^{1/(n+2)}$ and we note that $\nu \in (1/2, 1)$.

Lemma 5.2. *If* (0.2), (1.1) *and* (1.8)$'$ *hold, then there is a positive integer* r *such that*

$$\sup_{t_2 \le t \le t_0} |S'(2^{-r}\sigma, t) \cap B(\nu R)| \le \nu |B(\nu R)|. \tag{5.1}$$

Proof. Let t' be as in Lemma 5.1 and let $t'' \in (t_2, t_0)$. Now we define Ψ by

$$\Psi(s) = \ln^+ \left(\frac{\sigma}{1 - (s/M) + 2^{1-r}\sigma} \right).$$

(This is the same function Ψ as in [8] and [13] since the argument of the logarithm here is less than one if $s < (1-\sigma)M$.) With $\Gamma = 2\Psi\Psi'$ and ψ independent of t, it follows from (4.9) that

$$\int_{B(R)\times\{t''\}} \psi^2\Psi^2\,dX \le \int_{B(R)\times\{t'\}} \psi^2\Psi^2\,dX + CF(M)\int_{t'}^{t''}\int_{B(R/2)} \Psi|D\psi|^2\,dX.$$

Now take ψ so that $\psi \equiv 1$ in $B(\nu R)$ and $|D\psi| \le C(\nu)/R$ and estimate the terms in this inequality as in [13]*Lemma 1.3 (with ν in place of $\nu/2$) to infer (5.1) for r sufficiently large.

□

To prove Proposition 1.4, we set $w(\tau) = (\tau - (1-\theta)M)_+$ with θ to be further specified. Then for $q > 2$, we set $\Gamma = w^{q-2}$ and we take $\zeta \in C^1(Q(R,M))$ with support in $B(\nu R)\times(t', t_1]$ such that

$$\zeta(X) = 1 \text{ if } |x| \le \frac{\nu}{2}R \text{ and } t \ge t_0 - \frac{\sigma R^2}{4F(M)},$$
$$0 \le \zeta \le 1, |\zeta_t| \le \frac{8F(M)}{\sigma R^2}, |D\zeta| \le \frac{8}{\nu R} \text{ in } Q(R,M),$$

and we set $\bar{w} = \zeta^{n+2}w$. It then follows from (4.9) with $\psi = (\zeta^{(n+2)q-n})^{1/2}$ that

$$\sup_{t_2<t<t_0}\int_{B(R)\times\{t\}} \bar{w}^q\,dx + \int_{Q(R,M)} |D[\bar{w}^{q/2}\zeta^{-n/2}]|^2\,dX \le Cq^2\frac{F(M)}{R^2}\int_{Q(R,M)} \bar{w}^q\zeta^{-n-2}\,dX.$$

The proof is completed by arguing as on pages 507 and 508 of [13] (with w in place of $D_k v$).

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