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DIFFERENTIAL-OPERATOR INCLUSIONS IN BANACH SPACES WITH W_λ-PSEUDOMONOTONE MAPS

Differential-operator inclusions are studied rather strongly. By analogy with differential-operator equations are known, at the least, four methods of attack: Galerkin method, elliptic normalization, theory of semigroups, difference approximations. In present work we introduce some constructions to prove the resolvability for class of differential-operator inclusions with set-valued maps of w_λ–pseudomonotone type by Faedo–Galerkin (FG) method.

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1. Introduction.

At study of nonlinear evolutionary equations in infinite-dimensional (functional) spaces there is a series of methods, one of which is FG method [1–3]. In paper [4] the given method is disseminated to a wide class of the nonlinear differential-operational equations that contains, in particular, a multidimensional system of Navier-Stokes equations.

In present paper the substantiation of Galerkin method for differential-operator is given. Remark that evolutionary inclusions in Banach spaces, generated by the strong solutions of variational inequalities are investigated by FG method in [5–7]. The given results are generalization of [5–7].

2. Problem definition.

Let $(V_1, ||\cdot||_{V_1})$ and $(V_2, ||\cdot||_{V_2})$ be a reflexive separable Banach spaces, continuously embedded in a Hilbert space $(H, (\cdot,\cdot))$ such that

$$V_1 \cap V_2 \text{ is dense in spaces } V_1,\ V_2 \text{ and } H. \tag{1}$$

After identification $H \cong H^*$ we get

$$V_1 \subset H \subset V_1^*, \qquad V_2 \subset H \subset V_2^*, \tag{2}$$

with continuous and dense embedding [2], where $(V_i^*, ||\cdot||_{V_1})$ is topologically conjugate to V_i space with respect to the canonical bilinear form $\langle \cdot,\cdot\rangle_{V_i} : V_i^* \times V_i \to \mathbb{R}$ $(i = 1, 2)$ which coincides on H with inner product $(\cdot,\cdot)$ on H. Let us consider the functional spaces $X_i = L_{r_i}(S; H) \cap L_{p_i}(S; V_i)$, where $S = [0, T]$, $0 < T < +\infty$; $1 < p_i \le r_i$, $p_i < +\infty$ $(i = 1, 2)$. Spaces X_i are Banach spaces with norms $||y||_{X_i} = ||y||_{L_{p_i}(S;V_i)} + ||y||_{L_{r_i}(S;H)}$. Moreover if $r_i < +\infty$, then X_i is reflexive space $(i = 1, 2)$. Let us also consider the Banach space $X = X_1 \cap X_2$ with norm $||y||_X = ||y||_{X_1} + ||y||_{X_2}$. In virtue of spaces $L_{q_i}(S; V_i^*) + L_{r_i'}(S; H)$ and X_i^* $(i = 1, 2)$ are isometrically isomorphic, we identify them. Analogously, $X^* = X_1^* + X_2^* = L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r_1'}(S; H) + L_{r_2'}(S; H)$, where $r_i^{-1} + r_i'^{-1} = p_i^{-1} + q_i^{-1} = 1$ $(i = 1, 2)$. Let us define duality form on $X^* \times X$

$$\langle f, y\rangle = \int_S (f_{11}(\tau), y(\tau))_H\, d\tau + \int_S (f_{12}(\tau), y(\tau))_H\, d\tau + \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} d\tau +$$

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$$+\int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2}\,d\tau = \int_S (f(\tau), y(\tau))\,d\tau,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S;H)$, $f_{2i} \in L_{q_i}(S;V_i^*)$ $(i = 1,2)$.

Let $A : X_1 \rightrightarrows X_1^*$ and $B : X_2 \rightrightarrows X_2^*$ are multi-valued maps. We consider the next problem:

$$\begin{cases} y' + A(y) + B(y) \ni f, & (3) \\ y(0) = y_0 & (4) \end{cases}$$

where $f \in X^*$, $y_0 \in H$ arbitrary elements, y' is derivative of $y \in X$ in sense of scalar distribution space $D^*(S;V^*) = L(D(S);V_w^*)$, with $V = V_1 \cap V_2$, V_w^* equals to V^* with topology $\sigma(V^*,V)$ [8].

Let us enter Banach space $W = \{y \in X \mid y' \in X^*\}$ with norm $||y||_W = ||y||_X + ||y'||_{X^*}$,

$$\|f\|_{X^*} = \\ = \inf_{\substack{f = f_{11}+f_{12}+f_{21}+f_{22}:\\ f_{1i} \in L_{r_i'}(S;H),\\ f_{2i} \in L_{q_i}(S;V_i^*)\,(i=1,2)}} \max\left\{ \|f_{11}\|_{L_{r_1'}(S;H)}; \|f_{12}\|_{L_{r_2'}(S;H)}; \|f_{21}\|_{L_{q_1}(S;V_1^*)}; \|f_{22}\|_{L_{q_2}(S;V_2^*)} \right\}.$$

Remark that

$$X = L_{\max\{r_1,r_2\}}(S;H) \cap L_{\max\{p_1,p_2\}}(S;V)$$

and

$$X^* = L_{\min\{r_1',r_2'\}}(S;H) + L_{\min\{q_1,q_2\}}(S;V^*).$$

So the space W is continuously and densely enclosed in $C(S;H)$ (hence condition (4) have a sense, since the solutions of the problem (3)-(4) we will search in class W). Moreover,

$$\langle u', v\rangle + \langle v', u\rangle = \Big(u(T), v(T)\Big) - \Big(u(0), v(0)\Big) \qquad \forall u, v \in W. \tag{5}$$

Under $u = v$ we have:

$$\langle u', u\rangle = \frac{1}{2}\Big(\|u(T)\|_H^2 - \|u(0)\|_H^2\Big) \qquad \forall u \in W. \tag{6}$$

3. Classes of maps.

Let $C_v(X^*)$ be a family of all nonempty closed convex bounded subsets in X^*. Let us consider classes of multi-valued maps $A : X \to C_v(X^*)$. For this map let us define upper $a_+(y,\omega) = [A(y),\omega]_+ = \sup_{d\in A(y)} \langle d, w\rangle$ and lower $a_-(y,\omega) = [A(y),\omega]_- = \inf_{d\in A(y)} \langle d, w\rangle$ forms, where $y,\omega \in X$, and also, upper $||A(y)||_+ = \sup_{d\in A(y)} ||d||_{X^*}$ and lower $||A(y)||_- = \inf_{d\in A(y)} ||d||_{X^*}$ norms.

The next properties immediately follows from mentioned above constructions and Hahn-Banach theorem.

Proposition 1. *Let A, $B : X \to C_v(X^*)$. Then $\forall y, v, v_1, v_2 \in X$ the next proportions take place:*

1) *functional* $X \ni v \to [A(y), v]_+$ *is convex opositively homogeneous and lower semicontinuous;*

2) $[A(y), v_1 + v_2]_+ \leq [A(y), v_1]_+ + [A(y), v_2]_+$, $[A(y), v_1 + v_2]_- \geq [A(y), v_1]_- + [A(y), v_2]_-$, $[A(y), v_1 + v_2]_+ \geq [A(y), v_1]_+ + [A(y), v_2]_-$, $[A(y), v_1 + v_2]_- \leq [A(y), v_1]_+ + [A(y), v_2]_-$;

3) $[A(y) + B(y), v]_+ = [A(y), v]_+ + [B(y), v]_+$, $[A(y) + B(y), v]_- = [A(y), v]_- + [B(y), v]_-$;

4) $[A(y), v]_+ \leq ||A(y)||_+ ||v||_X$, $[A(y), v]_- \leq ||A(y)||_- ||v||_X$;

5) *functional* $||\cdot||_+ : C_v(X^*) \to \mathbb{R}_+$ *defines norm on* $C_v(X^*)$;

6) *functional* $||\cdot||_- : C_v(X^*) \to \mathbb{R}_+$ *satisfies such conditions:*

   a) $\overline{0} \in A(y) \quad \Leftrightarrow \quad ||A(y)||_- = 0$,
   b) $||\alpha A(y)||_- = |\alpha| ||A(y)||_- \quad \forall \alpha \in \mathbb{R}, y \in X$,
   c) $||A(y) + B(y)||_- \leq ||A(y)||_- + ||B(y)||_-$;

7) $||A(y) + B(y)||_+ \geq \Big| ||A(y)||_+ - ||B(y)||_- \Big|$, $||A(y) - B(y)||_- \geq ||A(y)||_- - ||B(y)||_+$;

8) $d \in A(y) \quad \Leftrightarrow \quad \forall \omega \in X \;\; [A(y), \omega]_+ \geq \langle d, w\rangle$.

Remark 1. Together with forms a_+, a_- we consider affirmative forms

$$\overline{a}_+(y, \omega) = [[A(y), \omega]]_+ = \sup_{d \in A(y)} |\langle d, w\rangle|$$

and

$$\overline{a}_-(y, \omega) = [[A(y), \omega]]_- = \inf_{d \in A(y)} |\langle d, w\rangle| \;\forall y, \omega \in X.$$

Thus it is obvious

$$[A(y), \omega]_+ \leq |[A(y), \omega]_+| \leq [[A(y), \omega]]_+ \leq ||A(y)||_+ ||\omega||_X,$$

$$[A(y), \omega]_- \leq |[A(y), \omega]_-| \leq [[A(y), \omega]]_- \leq ||A(y)||_- ||\omega||_X.$$

Remark that $y_n \rightharpoonup y$ in Y means y_n is weakly converges to y in space Y. If Y is not reflexive, then $y_n \rightharpoonup y$ in Y^* means y_n is *-weakly converges to y in space Y^*.

Definition 1. Multi-valued map $A : X \to C_v(X^*)$ refers to:

a) *λ-pseudomonotone on W (w_λ-pseudomonotone)*, if for every such sequence $\{y_n\}_{n \geq 0} \subset W$ that $y_n \rightharpoonup y_0$ in W (i.e. $y_n \rightharpoonup y_0$ in X and $y_n' \rightharpoonup y_0'$ in X^*) from inequality

$$\varlimsup_{n \to \infty} \langle d_n, y_n - y_0\rangle \leq 0, \tag{7}$$

where $d_n \in A(y_n)$ $\forall n \geq 1$ the existence such $\{y_{n_k}\}_{k \geq 1}$ from $\{y_n\}_{n \geq 1}$ and $\{d_{n_k}\}_{k \geq 1}$ from $\{d_n\}_{n \geq 1}$ that

$$\varliminf_{k \to \infty} \langle d_{n_k}, y_{n_k} - w\rangle \geq [A(y), y_0 - w]_- \quad \forall w \in X \tag{8}$$

is follows;

b) *λ_0-pseudomonotone on W (w_{λ_0}-pseudomonotone)*, if for every such sequence $\{y_n\}_{n\geq 0}$ from W that $y_n \rightharpoonup y_0$ in W and $d_n \rightharpoonup d_0$ in X^*, where $d_n \in A(y_n)$ $\forall n \geq 1$ from inequality (7) the existence of such subsequences $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ and $\{d_{n_k}\}_{k\geq 1} \subset \{d_n\}_{n\geq 1}$ that (8) is true is follows;

c) *$+$-coercive*, if $||y||_X^{-1}\,[A(y), y]_+ \to +\infty$ at $||y||_X \to +\infty$;

d) *quasi-bounded*, if $\forall y_0 \in X$ $\forall k_1, k_2 > 0$ $\exists N = N(k_1, k_2, y_0) > 0$:

$$\forall y \in X: \ ||y||_X \leq k_1 \ \forall d \in A(y) \ \langle d, y - y_0\rangle \leq k_2 \ \Rightarrow \ ||d||_{X^*} \leq N < +\infty;$$

e) *bounded*, if A converts every bounded set in X to bounded in X^*;

satisfies:

f) *property (κ)*, if for every bounded set D in X there exists such $c \in \mathbb{R}$ that

$$[A(v), v]_+ \geq c||v||_X \quad \forall v \in D.$$

Remark 2. Bounded multi-valued map $A : X \to C_v(X^*)$ is quasi-bounded and satisfies property (κ); λ-pseudomonotone on W map is λ_0-pseudomonotone on W. The converse statement is correct for bounded multi-valued maps.

Let us define $W_i = \{y \in X_i \mid y' \in X^*\}$ $(i = 1, 2)$.

Proposition 2. *Let $A : X_1 \to C_v(X_1^*)$ and $B : X_2 \to C_v(X_2^*)$* be *λ-pseudomonotone on W_1 and correspondingly on W_2 multi-valued maps. Then multi-valued map $C := A + B : X \to C_v(X^*)$ is λ-pseudomonotone on W.*

Definition 2. *Multi-valued maps $A : X_1 \to C_v(X_1^*)$ and $B : X_2 \to C_v(X_2^*)$ is called s-mutually bounded, if for every $M > 0$ there exists such $K(M) > 0$ that from $||y||_X \leq M$ and $\langle d_1(y) + d_2(y), y\rangle \leq M$ we have or $||d_1(y)||_{X_1^*} \leq K(M)$ or $||d_2(y)||_{X_2^*} \leq K(M)$. Here $d_1 \in A$ and $d_2 \in B$ are some selectors.*

Remark 3. *If the pair $(A; B)$ is s-mutually bounded, then proposition 2 takes place for λ_0-pseudomonotone (correspondingly on W_1 and on W_2) maps.*

Remark 4. *Obviously, if one of operators from the pair $(A; B)$ is bounded, then the pair $(A; B)$ is s-mutually bounded. Moreover, if the pair $(A; B)$ is s-mutually bounded, then operator $C = A + B : X \to X^*$ has property (π)* [3].

Proof. Let us prove this statement 2 for λ_0-pseudomonotone maps. It is obvious that $C(y) \in C_v(X^*)$ $\forall y \in X$.

Let it be $y_n \rightharpoonup y_0$ in X, $y_n' \rightharpoonup y_0'$ in $X_1^* + X_2^* = X^*$ and $C(y_n) \ni d_n \rightharpoonup d_0$ in X^*, moreover the inequality (7) takes place. Hence, $d_n = d_n' + d_n''$, where $d_n' \in A(y_n)$, $d_n'' \in B(y_n)$. Because of the pair $(A; B)$ is s-mutually bounded from estimation $\langle d_n(y), y\rangle = \langle d_n'(y) + d_n''(y), y\rangle \leq M$ we have or $||d_n'(y)||_{X_1^*} \leq K(M)$ or $||d_n''(y)||_{X_2^*} \leq K(M)$. Then passing, if it is necessary to a subsequence, we claim

$$d_n' \rightharpoonup d_0' \text{ in } X_1^* \quad \text{i} \quad d_n'' \rightharpoonup d_0'' \text{ in } X_2^*.$$

From inequality (7) we have

$$\underline{\lim_{n\to\infty}}\langle d''_n, y_n - y_0\rangle + \overline{\lim_{n\to\infty}}\langle d'_n, y_n - y_0\rangle \le \overline{\lim_{n\to\infty}}\langle d_n, y_n - y_0\rangle \le 0,$$

or symmetrically

$$\underline{\lim_{n\to\infty}}\langle d'_n, y_n - y_0\rangle + \overline{\lim_{n\to\infty}}\langle d''_n, y_n - y_0\rangle \le \overline{\lim_{n\to\infty}}\langle d_n, y_n - y_0\rangle \le 0.$$

Further let us consider the last inequality. It is obvious there exists such subsequence $\{y_m\} \subset \{y_n\}_{n\ge 1}$ that

$$\underline{\lim_{n\to\infty}}\langle d''_n, y_n - y_0\rangle + \overline{\lim_{n\to\infty}}\langle d'_n, y_n - y_0\rangle \ge \underline{\lim_{m\to\infty}}\langle d''_m, y_m - y_0\rangle + \lim_{m\to\infty}\langle d'_m, y_m - y_0\rangle \tag{9}$$

Hence, it follows:

$$\overline{\lim_{m\to\infty}}\langle d'_m, y_m - y_0\rangle \le 0, \quad \text{or} \overline{\lim_{m\to\infty}}\langle d''_m, y_m - y_0\rangle \le 0. \tag{10}$$

Without loss of generality we consider that $\overline{\lim\limits_{m\to\infty}}\langle d'_m, y_m - y_0\rangle \le 0$. Then, in virtue of λ_0-pseudomonotony A on W_1, there exists such subsequence $\exists\{y_{m_k}\}_{k\ge 1}$ from $\{y_m\}_m$ that

$$\underline{\lim_{k\to\infty}}\langle d'_{m_k}, y_{m_k} - v\rangle \ \ge\ [A(y), y_0 - v]_- \quad \forall v \in X_1. \tag{11}$$

Let us take in last inequality $v = y_0$. We find $\langle d'_{m_k}, y_{m_k} - y_0\rangle \to 0$. Then, due to (9), $\overline{\lim\limits_{m\to\infty}}\langle d''_m, y_m - y_0\rangle \le 0$. In virtue of λ_0-pseudomonotony B on W_2, passing to a subsequence $\{y_{m'_k}\} \subset \{y_{m_k}\}_{k\ge 1}$ we find

$$\underline{\lim_{m'_k\to\infty}}\langle d''_{m'_k}, y_{m'_k} - w\rangle \ge [B(y), y - w]_- \quad \forall w \in X_2. \tag{12}$$

Then from proportions (11), (12) finally obtain

$$\underline{\lim_{m'_k\to\infty}}\langle d_{m'_k}, y_{m'_k} - x\rangle \ge \underline{\lim_{m'_k\to\infty}}\langle d'_{m'_k}, y_{m'_k} - x\rangle + \underline{\lim_{m'_k\to\infty}}\langle d''_{m'_k}, y_{m'_k} - x\rangle \ge$$

$$\ge [A(y), y - x]_- + [B(y), y - x]_- = [C(y), y - x]_- \quad \forall x \in X = X_1 \cap X_2.$$

The proposition is proved.

PROPOSITION 3. *Let* $A : X_1 \overset{\rightarrow}{\rightarrow} X_1^*$ *and* $B : X_2 \overset{\rightarrow}{\rightarrow} X_2^*$ *be multi-valued coercive maps, that satisfies condition* (κ). *Then multi-valued operator* $C := A + B : X \overset{\rightarrow}{\rightarrow} X^*$ *is coercive.*

Proof. We obtain this statement arguing by contradiction. Let $\exists\{x_n\}_{n\ge 1} \subset X : ||x_n||_X = ||x_n||_{X_1} + ||x_n||_{X_2} \to +\infty$ as $n \to \infty$, but

$$\sup_{n\ge 1} \frac{[C(x_n), x_n]_+}{||x_n||_X} < +\infty. \tag{13}$$

Case 1. $||x_n||_{X_1} \to +\infty$ as $n \to \infty$, $||x_n||_{X_2} \le c\ \forall n \ge 1$.

$$\gamma_A(r) := \inf_{||v||_{X_1}=r} \frac{[A(v), v]_+}{||v||_{X_1}}, \quad \gamma_B(r) := \inf_{||w||_{X_2}=r} \frac{[B(w), w]_+}{||w||_{X_2}}, \quad r > 0.$$

Note that $\gamma_A(r) \to +\infty$, $\gamma_B(r) \to +\infty$ as $r \to +\infty$. Then $\forall n \geq 1$ $||x_n||^{-1}_{X_1}[A(x_n), x_n]_+ \geq \gamma_A(||x_n||_{X_1})||x_n||_{X_1}$ and $\frac{[A(x_n),x_n]_+}{||x_n||_X} \geq \gamma_A(||x_n||_{X_1})\frac{||x_n||_{X_1}}{||x_n||_X} \to +\infty$ as $||x_n||_{X_1} \to +\infty$ and $||x_n||_{X_2} \leq c$.

Due to condition (κ) for every $n \geq 1$

$$\frac{[B(x_n), x_n]_+}{||x_n||_X} \geq \gamma_B(||x_n||_{X_2})\frac{||x_n||_{X_2}}{||x_n||_X} \geq c_1\frac{||x_n||_{X_2}}{||x_n||_X} \to 0 \quad \text{as } n \to \infty,$$

where $c_1 \in \mathbb{R}$ is a constant in condition (κ) with $D = \{y \in X_2 \mid ||y||_{X_2} \leq c\}$. It is obvious that

$$\frac{[C(x_n), x_n]_+}{||x_n||_X} = \frac{[A(x_n), x_n]_+}{||x_n||_X} + \frac{[B(x_n), x_n]_+}{||x_n||_X} \to +\infty \quad \text{as} \quad n \to \infty.$$

This is a contradiction with (13).

Case 2. Case when $||x_n||_{X_1} \leq c$ $\forall n \geq 1$ and $||x_n||_{X_2} \to \infty$ as $n \to \infty$ examines analogously.

Case 3. Let us consider situation, when $||x_n||_{X_1} \to \infty$ and $||x_n||_{X_2} \to \infty$ as $n \to \infty$. Then

$$+\infty > \sup_{n\geq 1}\frac{[C(x_n), x_n]_+}{||x_n||_X} \geq \gamma_A(||x_n||_{X_1})\frac{||x_n||_{X_1}}{||x_n||_{X_1} + ||x_n||_{X_2}} + +\gamma_B(||x_n||_{X_2})\frac{||x_n||_{X_2}}{||x_n||_{X_1} + ||x_n||_{X_2}}. \tag{14}$$

It is obvious, that $\forall n \geq 1$ $\frac{||x_n||_{X_1}}{||x_n||_X} > 0$ and $\frac{||x_n||_{X_2}}{||x_n||_X} > 0$ and moreover, if even one of the boundaries, for example, $\frac{||x_n||_{X_1}}{||x_n||_X} \to 0$, then $\frac{||x_n||_{X_2}}{||x_n||_X} = 1 - \frac{||x_n||_{X_1}}{||x_n||_X} \to 1$. We have a contradiction in (14).

The proposition is proved.

Definition 3. Operator $L : D(L) \subset X \to X^*$ refers to:

1) *monotone*, if for every $y_1, y_2 \in X$ $\langle L(y_1) - L(y_2), y_1 - y_2\rangle \geq 0$;

2) *maximal monotone*, if it is monotone and from $\langle w - L(u), v - u\rangle \geq 0$ for all $u \in D(L)$ follows that $v \in D(L)$ and $L(v) = w$.

4. Auxiliary statements.

In virtue of (1) and (2) $V = V_1 \cap V_2 \subset H$ with continuous and dense embedding. As V is separable Banach space, then there exists complete in V and consequently in H countable vector system $\{h_i\}_{i\geq 1} \subset V$. Let for every $n \geq 1$ H_n be a linear capsule stretched on $\{h_i\}_{i=1}^n$. On H_n we consider the inner product induced from H that we also denote as $(\cdot,\cdot)$. Let $P_n : H \to H_n \subset H$ be an operator of orthogonal projection from H to H_n, i.e. $\forall h \in H$ $\quad P_n h = \arg\min_{h_n\in H_n} ||h - h_n||_H$.

Definition 4. We say that the triple $\Big(\{h_i\}_{i\geq 1}; V; H\Big)$ satisfies *condition (γ)* if

$$\sup_{n\geq 1} ||P_n||_{L(V,V)} < +\infty,$$

i.e. there exists such $C \geq 1$ that for every $v \in V_1 \cap V_2$ and $n \geq 1$

$$||P_n v||_{V_1} \leq C \cdot ||v||_{V_1} \quad ||P_n v||_{V_2} \leq C \cdot ||v||_{V_2}. \tag{15}$$

Remark 5. In case when vector system $\{h_i\}_{i\geq 1} \subset V$ is orthogonal in H condition (γ) means that the given system is a Schauder base in space V (in particular in V_1 and in V_2) [9].

Due to equivalence of H^* and H it follows that $H_n^* \cong H_n$. Further for every $n \geq 1$ let us consider the Banach space $X_n = L_{p_0}(S; H_n) \subset X$ (where $p_0 := \max\{r_1, r_2\}$) with norm $\|\cdot\|_{X_n}$ induced from space X. This norm is equivalent to the natural norm in $L_{p_0}(S; H_n)$ [2]. The space $L_{q_0}(S; H_n)$ ($q_0^{-1} + p_0^{-1} = 1$) with norm $\|f\|_{X_n^*} := \sup\limits_{x \in X_n \setminus \{\bar{0}\}} \frac{|\langle f, x\rangle|}{\|x\|_X} = \sup\limits_{x \in X_n \setminus \{\bar{0}\}} \frac{|\langle f, x\rangle_{X_n}|}{\|x\|_{X_n}}$ is isometrically isomorphic to conjugate to X_n space X_n^* (further the given spaces are identified).

$$\mathbb{R} \ni \langle f, x\rangle = \int_S (f(\tau), x(\tau))d\tau = \int_S (f(\tau), x(\tau))_{H_n} d\tau =$$

$$= \langle f, x\rangle_{X_n} \leftarrow \{f, x\} \in X_n^* \times X_n$$

be duality form on $X_n^* \times X_n$. This statement is correct due to $X_n^* = L_{q_0}(S; H_n) \subset L_{q_0}(S; H) \subset L_{r_1'}(S; H) + L_{r_2'}(S; H) + L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) = X^*$ (see [2]).

For every $n \geq 1$ as I_n we denote the canonical embedding X_n in X, i.e. $\forall x \in X_n \; I_n x = x$. It is necessary to notice that $I_n \in L\Big((X_n, \|\cdot\|_{X_n}); (X, \|\cdot\|_X)\Big)$ and, moreover,

$$||I_n||_{L\big((X_n, \|\cdot\|_{X_n}) \; ; \; (X, \|\cdot\|_X)\big)} = 1.$$

From [10] follows, that conjugate operator $I_n^* \in L\Big((X^*, \|\cdot\|_{X^*}) \; ; \; (X_n^*, \; \|\cdot\|_{X_n^*})\Big)$ and

$$\langle I_n^* y, x\rangle = \langle y, x\rangle_X \quad \forall x \in X_n \;\; \forall y \in X^*, \quad ||I_n^*||_{L((X^*, \|\cdot\|_{X^*}); \; (X_n^*, \; \|\cdot\|_{X_n^*}))} = 1. \tag{16}$$

Lemma 1. *Let the condition* (γ) *is satisfied. Then for the constant* $C \geq 1$ *from* (15) *and norm* $\|\cdot\|_n^*$*, induced from space* X^* *to* X_n^**, it is fulfilled*:

a) $||f_n||_{X_n^*} \leq ||f_n||_n^* \leq C \cdot ||f_n||_{X_n^*} \qquad \forall n \geq 1, \; \forall f_n \in X_n^*$;

b) $\forall f_n \in X_n^* \qquad I_n^* f_n = f_n$;
moreover, the following takes place:

c) *if for arbitrary* $u \in X$ *to put* $u_n(\cdot) := P_n u(\cdot) \in X_n$*, then*

$$\forall f \in X_n^* \quad \langle f, u_n\rangle = \langle f, u\rangle, \quad ||u_n||_X \leq C \cdot ||u||_X. \tag{17}$$

Proof. b) This statement is correct. It is remained to prove that for every $f_n \in X_n^*$ and $x_n \in X_n$ $\langle f_n - I_n^* f_n, x_n\rangle = 0$. From definitions I_n^* and I_n follows

$$\forall x_n \in X_n \quad \langle f_n - I_n^* f_n, x_n\rangle = \langle f_n, x_n\rangle - \langle I_n^* f_n, x_n\rangle =$$

$$= \langle f_n, x_n\rangle - \langle f_n, I_n x_n\rangle = \langle f_n, x_n\rangle - \langle f_n, x_n\rangle = 0.$$

c) For every $u \in X$ let $u_n(\cdot) := P_n u(\cdot) \in X_n$, i.e. $u_n(t) = P_n u(t)$ almost everywhere (a.e.) in S. Because of P_n is linear and continuous on V_1, V_2 and H we have that $u_n \in X_n \subset X$. In virtue of (15) and definitions of $\|\cdot\|_{L_{p_i}(S;V_i)}$, $\|\cdot\|_{L_{r_i}(S;H)}$ $(i=1,2)$ it follows: $\|u_n\|_{L_{p_i}(S;V_i)} \leq C \cdot \|u\|_{L_{p_i}(S;V_i)}$, $\|u_n\|_{L_{r_i}(S;H)} \leq \|u\|_{L_{r_i}(S;H)}$ $(i=1,2)$. Thus $||u_n||_X \leq C \cdot ||u||_X$.

Now prove that for all $f \in X_n^*$ $\langle f, u_n\rangle = \langle f, u\rangle$. As $f \in L_{q_0}(S; H_n)$ then

$$\langle f, u\rangle = \int_S (f(\tau), u(\tau))d\tau = \int_S (f(\tau), P_n u(\tau))d\tau = \int_S (f(\tau), u_n(\tau))d\tau = \langle f, u_n\rangle,$$

because of for all $n \geq 1$ $h \in H$ $v \in H_n$ $(h - P_n h, v)_H = (h - P_n h, v) = 0$. So, (17) is proved.

a) For every $f \in X_n^* \subset X^*$

$$\|f\|_n^* = \|f\|_{X^*} = \sup_{x \in X \setminus \{\bar{0}\}} \frac{|\langle f, x\rangle|}{\|x\|_X} \geq \sup_{x \in X_n \setminus \{\bar{0}\}} \frac{|\langle f, x\rangle|}{\|x\|_X} = \sup_{x \in X_n \setminus \{\bar{0}\}} \frac{|\langle f, x\rangle|}{\|x\|_{X_n}} = ||f||_{X_n^*}. \tag{18}$$

In virtue of (17)

$$\|f\|_n^* = \|f\|_{X^*} = \sup_{x \in X \setminus \{\bar{0}\}} \frac{|\langle f, x\rangle|}{\|x\|_X} \leq$$

$$\leq \sup_{x \in X \setminus \{\bar{0}\}} \frac{C \cdot |\langle f, x_n(x)\rangle|}{\|x_n(x)\|_X} \leq \sup_{x_n \in X_n \setminus \{\bar{0}\}} \frac{C \cdot |\langle f, x_n\rangle|}{\|x_n\|_{X_n}} = C \cdot ||f||_{X_n^*},$$

that with (18) finishes the proof of a).

The lemma is proved.

Corollary 1. *For every $f \in X^*$ and $n \geq 1$*

$$||I_n^* f||_{X^*} \leq C \cdot ||f||_{X^*}. \tag{19}$$

The proof immediately follows from (16) and lemma 1 a).

For all $n \in \mathbb{N}$ let us define the Banach space $W_n = \left\{ y \in X_n | \ y' \in X_n^* \right\}$ with norm $\|y\|_{W_n} = \|y\|_{X_n} + \|y'\|_{X_n}^*$, where the derivative y' is considered in sense of scalar distributions space $D^*(S; H_n)$. As far as $D^*(S; H_n) = L(D(S); H_n) \subset L(D(S); V_\omega^*) = D^*(S; V^*)$ the derivative of an element $y \in X_n$ it is possible to consider in sense of $D^*(S; V^*)$. From lemma 1 it follows that $W_n \subset W$.

5. Faedo-Galerkin method.

For every $n \geq 1$ let us enter $A_n := I_n^* A I_n : X_n \overset{\rightarrow}{\rightarrow} X_n^*$, $B_n := I_n^* B I_n : X_n \overset{\rightarrow}{\rightarrow} X_n^*$, $f_n := I_n^* f \in X_n^*$. We consider such sequence $\{y_{0n}\}_{n \geq 0} \subset H$ that

$$\forall n \geq 1 \quad H_n \ni y_{0n} \to y_0 \text{ in } H \text{ at } n \to +\infty. \tag{20}$$

With problem (3)–(4) we consider the following class of problems:

$$y_n' + A_n(y_n) + B_n(y_n) \ni f_n, \tag{21_n}$$

$$y_n(0) = y_{0n}. \tag{22_n}$$

DEFINITION 5. We say that the solution of (3)–(4) $y \in W$ turns out *by Faedo-Halerkin method*, if y is a weak limit of some subsequence $\{y_{n_k}\}_{k\ge1}$ form $\{y_n\}_{n\ge1}$ in W and

a) for every $n \ge 1$ $W_n \ni y_n$ is a solution of the problem $(21)_n - (22)_n$;

b) $y_{0n} \to y_0$ in H as $n \to \infty$;

c) $y_{n_k} \to y$ in $L_{r_i}(S;H)$ as $k \to \infty$, $i = 1,2$.

6. Choice of basic.

We say that the vector system $\{h_i\}_{i\ge1}$ from separable Hilbert space $(V;(\cdot,\cdot)_V)$, continuously and densely embedded in a Hilbert space $(H;(\cdot,\cdot)_H)$, is called *special basis* for the pair of spaces $(V;H)$, if it satisfies the following conditions:

i) $\{h_i\}_{i\ge1}$ is orthonormal in $(H,(\cdot,\cdot)_H)$;

ii) $\{h_i\}_{i\ge1}$ is orthogonal in $(V,(\cdot,\cdot)_V)$;

iii) $\forall i \ge 1$ $(h_i,v)_V = \lambda_i(h_i,v)_H$ $\forall v \in V$, where $0 \le \lambda_1 \le \lambda_2,...,\lambda_j \longrightarrow \infty$ at $j \longrightarrow \infty$.

LEMMA 2. *If V is a Hilbert space, compactly and densely embedded in a Hilbert space H, then there exists a special basis $\{h_i\}_{i\ge1}$ for $(V;H)$. Moreover, for an arbitrary such system, the triple $\Big(\{h_i\}_{i\ge1};V;H\Big)$ satisfies condition (γ) with constant $C=1$.*

Proof. From [11, page 54–58] under these assumptions it is well-known, that there exists a special basis $\{h_i\}_{i\ge1}$ for the pair $(V;H)$. So, in order to complete the proof it is enough to show that the triple $\Big(\{h_i\}_{i\ge1};V;H\Big)$ satisfies condition (γ) with constant $C=1$ for an arbitrary special basis $\{h_i\}_{i\ge1}$ for $(V;H)$. Therefore, let us take as H_n a linear span, stretched on $\{h_i\}_{i=1}^n$. We point out H_n is a finite-dimensional space. Thus, the norms $||\cdot||_H$ and $||\cdot||_V$ are equivalent on H_n (see [8]). From here it follows $\forall n \ge 1$ $\exists c_n > 0$, $\exists C > 0$: $\forall h \in H_n$ $||P_nh||_V \le c_n||P_nh||_H \le c_n||h||_H \le c_nC||h||_V$. It also means that $P_n \in L(V,V)$.

Further let us prove that $\forall n \ge 1$

$$||P_nh||_V \le ||h||_V \qquad \forall h \in \bigcup_{m\ge1} H_m. \tag{23}$$

Let $n \ge 1$ be fixed , then $\forall h \in \bigcup\limits_{m\ge1} H_m \Rightarrow \exists m_0 \ge n+1 : h \in H_{m_0}$. From here, taking into account i) and ii), we have $h = \sum\limits_{i=1}^{m_0}(h,h_i)_Hh_i$, $P_nh = \sum\limits_{i=1}^{n}(h,h_i)_Hh_i$. In order to obtain (23) it is necessary to show that P_nh is orthogonal to $(h-P_nh)$ in V. Because of $(P_nh, h - P_nh)_V = (\sum\limits_{i=1}^{n}(h,h_i)_Hh_i, \sum\limits_{j=n+1}^{m_0}(h,h_j)_Hh_j)_V = \sum\limits_{i=1}^{n}\sum\limits_{j=n+1}^{m}(h,h_i)_H(h,h_j)_H(h_i,h_j)_V = 0$, $\{h_i\}_{i\ge1}$ is orthogonal in V. So, in virtue of continuity of $||\cdot||_V$ and P_n on V $\forall n \ge 1$ we have that for all $n \ge 1$ and $v \in V$ $||P_nv||_V \le ||v||_V$.

The lemma is proved.

For interpolating pair A_0, A_1 (i.e. for Banach spaces A_0 and A_1, that are linearly and continuously embedded in some linear topological space) on a set $A_0 + A_1$ let us consider the functional

$$K(t,x) = \inf_{x=x_0+x_1:\ x_0\in A_0,\ x_1\in A_1}\Big(||x_0||_{A_0} + t||x_1||_{A_1}\Big), \qquad t \ge 0,\ x \in A_0 + A_1.$$

For fixed $x \in A_0 + A_1$, this map is monotone increasing, continuous and convex upwards function of the variable $t \geq 0$ (see [9, lemma 1.3.1]).

For $\theta \in (0,1)$ and $1 < p < +\infty$ let us consider the following space:

$$(A_0, A_1)_{\theta,p} = \Big\{ x \in A_0 + A_1 \ \Big| \ \int_0^{+\infty} \Big[t^{-\theta} K(t,x)\Big]^p \frac{dt}{t} < +\infty \Big\}. \tag{24}$$

$(A_0, A_1)_{\theta,p}$ with $||x||_{\theta,p} = (\int_0^{+\infty} [t^{-\theta} K(t,x)]^p \frac{dt}{t})^{\frac{1}{p}}$ is a Banach space (for more details see [9,1.3]) and it results in (see [9, theorem 1.3.3]):

$$A_0 \cap A_1 \subset (A_0, A_1)_{\theta,p} \subset A_0 + A_1 \qquad \forall \theta \in (0,1),\ \forall 1 < p < +\infty \tag{25}$$

with dense and continuous embedding.

Definition 6. Let it be $1 \leq r < 2$. We say that the filter of Banach spaces $\{Z_p\}_{p \geq r}$ and Hilbert space H satisfy *main conditions*, if

a) $\forall p_2 > p_1 > r$ $Z_{p_2} \subset Z_{p_1} \subset H$ with continuous and dense embedding;

b) $\forall p_2 > p > p_1 > r$ $(Z_{p_1}, Z_{p_2})_{\theta,p} = Z_p$, where $\theta = \theta(p) \in (0,1) : \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$;

c) Z_2 is a Hilbert space.

Lemma 3. *Let us assume: $1 \leq r < 2$, filter of Banach spaces $\{Z_p\}_{p \geq r}$ and Hilbert space H satisfy main conditions, vector system $\{h_i\}_{i \geq 1} \subset Z_2$ such that the triple $\Big(\{h_i\}_{i\geq 1}; Z_2; H\Big)$ satisfies condition (γ) with constant $C \geq 1$ and $\{h_i\}_{i \geq 1} \subset Z_p$ for all $p > r$. Then, for all $p > r$ the triple $\Big(\{h_i\}_{i\geq 1}; Z_p; H\Big)$ satisfies condition (γ) with constant C.*

Remark 6. In the case $Z_2 \subset H$ with compact embedding, thanks to lemma, as a vector system $\{h_i\}_{i \geq 1}$ we can choose a special basis for the pair $(Z_2; H)$. In particular, the above result means that the special basis for $(Z_2; H)$ is a Schauder basis for an arbitrary space Z_p at $r < p \leq 2$.

Proof. For $1 \leq r < 2$ let $\{h_i\}_{i\geq 1} \subset Z_r$ be a vector system such that the triple $\Big(\{h_i\}_{i\geq 1}; Z_2; H\Big)$ satisfies condition (γ) with constant $C \geq 1$. Let us prove that $\forall p > r$ the triple $\Big(\{h_i\}_{i\geq 1}; Z_p; H\Big)$ satisfies condition (γ) with constant C.

At first we consider the case $p \geq 2$. Let $N > 2$ be an arbitrary fixed number. We check, that $\forall p \in [2, N)$ the triple $\Big(\{h_i\}_{i\geq 1}; Z_p; H\Big)$ satisfies condition (γ) with constant C. For the proof of this fact we benefit from transfinitary induction method. The set $W = [2, N)$ is well ordered by order $'' \prec '' := '' \leq ''$.

For an arbitrary $p \in W$ the statement $G(p)$ consists of the triple $\Big(\{h_i\}_{i\geq 1}; Z_p; H\Big)$ satisfies condition (γ) with constant C. So,

1) as $p = 2$ (for the first element of W) the statement $G(p)$ holds, thanks to conditions of this theorem;

2) let p be an arbitrary element in W. Assuming $G(q)$ is true for all $q \in I(p) = [2, p)$, we prove that from here the statement $G(p)$ follows. Let x be a fixed element in the space Z_a, dense in Z_p ($a \in (r, 2)$ is arbitrary). Then $\forall q \in [2, p]$, in virtue of (24) and the main

condition b) for $\{Z_p\}_{p\geq r}$ and H with $p=q$, $p_1=a$, $p_2=N$, it results in:

$$||x||_{Z_q}=||x||_{(Z_a,Z_p)_{\theta,q}}=\left(\int_0^{+\infty}\left[t^{-\theta}K(t,x)\right]^q\frac{dt}{t}\right)^{\frac{1}{q}}, \tag{26}$$

where $\theta=\theta(q)=\frac{\frac{1}{a}-\frac{1}{q}}{\frac{1}{a}-\frac{1}{N}}\in[\theta(2),\theta(p)]=\left[\frac{\frac{1}{a}-\frac{1}{2}}{\frac{1}{a}-\frac{1}{N}},\frac{\frac{1}{a}-\frac{1}{p}}{\frac{1}{a}-\frac{1}{N}}\right]\subset(0,1)$, i.e. $\frac{1}{q}=\frac{1-\theta}{a}+\frac{\theta}{N}$.

In the following we prove

$$||x||_{Z_q}\to||x||_{Z_p}\quad\text{as}\quad q\to p\ (q\in[2,p)). \tag{27}$$

Denoted by

$$f(t,q)=\left[t^{-\theta(q)}K(t,x)\right]^q\frac{1}{t},\qquad\forall(t,q)\in[0,+\infty)\times[2,p],$$

from (24) and (26) it obviously follows that for every $\forall q\in[2,p]$ $f(\cdot,q)\in L_1[0,+\infty)$; moreover for almost every $t\in[0,+\infty)$ $f(t,\cdot)\in C[2,p]$. Furthermore, pointing out that for every $t>0$ and $q\in[2,p]$

$$\left[t^{-\theta(q)}K(t,x)\right]^q\frac{1}{t}\leq\max\left\{\left[t^{-\theta(2)}K(t,x)\right]^2;\left[t^{-\theta(2)}K(t,x)\right]^p;\right.$$

$$\left.\left[t^{-\theta(p)}K(t,x)\right]^2;\left[t^{-\theta(p)}K(t,x)\right]^p\right\}\frac{1}{t}=:g(t);$$

having in mind (25) and $x\in Z_a=Z_a\cap Z_N$, we have:

$$\int_0^{+\infty}|g(t)|dt=\int_0^{+\infty}g(t)dt\leq\max\Big\{\int_0^{+\infty}\left[t^{-\theta(2)}K(t,x)\right]^2\frac{dt}{t};\int_0^{+\infty}\left[t^{-\theta(2)}K(t,x)\right]^p\frac{dt}{t};$$

$$\int_0^{+\infty}\left[t^{-\theta(p)}K(t,x)\right]^2\frac{dt}{t};\int_0^{+\infty}\left[t^{-\theta(p)}K(t,x)\right]^p\frac{dt}{t}\Big\}=$$

$$=\max\Big\{||x||^2_{(Z_a,Z_N)_{\theta(2),2}};||x||^p_{(Z_a,Z_N)_{\theta(2),p}};||x||^2_{(Z_a,Z_N)_{\theta(p),2}};||x||^p_{(Z_a,Z_N)_{\theta(p),p}}\Big\}<+\infty.$$

Thus, the theorem of continuous dependence of Lebesgue integral on parameter all conditions of the theorem on continuous association of an integral of Lebesgue on parameter [12, theorem 8.1.1] assures the convergence (27).

By using the induction assumption

$$\forall q\in[2,p)\quad\forall x\in Z_a\quad\forall n\geq1\qquad||P_nx||_{Z_q}\leq C||x||_{Z_q}.$$

and passing to the limit as $q\nearrow p$ in the last inequality, we obtain

$$||P_nx||_{Z_p}\leq C||x||_{Z_p}\qquad\forall x\in Z_a\quad\forall n\geq1.$$

Then from density Z_a in Z_p and continuity P_n on Z_p $\forall n\geq1$ the statement $G(p)$ follows. So, for all $p\in[2,N)$ the statement $G(p)$ is true. Because of N is arbitrary greater than 2, the triple $\left(\{h_i\}_{i\geq1};Z_p;H\right)$ satisfies condition (γ) with C for every $p\geq2$.

In order to conclude the proof of the theorem it is necessary to remark that the case $p \in (r, 2]$ can be proved similarly to the case $p \geq 2$, by replacing $'' \prec''$ with $'' >''$ and setting $W = (N, 2]$, where $N \in (r, 2)$ is arbitrary.

The lemma is proved.

COROLLARY 2. *Let V_1, V_2 be Banach spaces, continuously embedded in the Hilbert space H. Let us assume that for some filters of Banach spaces $\{Z_p^i\}_{p \geq r_i}$ ($r_i \in [1; 2)$, $i = 1, 2$), that together with H satisfy main conditions, there exist $p_i > r_i$ such that $V_i = Z_{p_i}^i$ ($i = 1, 2$), within to equivalence of norms. Moreover, there exist Hilbert space $Z \subset V_1 \cap V_2$, compactly embedded in H, such that for special basis $\{h_i\}_{i \geq 1}$ for pair $(Z; H)$, for some $0 \leq \mu_1 \leq \mu_2, ..., \mu_j \longrightarrow \infty$ at $j \longrightarrow \infty$ and $s_i > 0$ ($i = 1, 2$)*

$$Z_2^i = \left\{ u \in H \;\middle|\; \sum_{j=1}^{\infty} \mu_j^{s_i} (u, h_j)^2 < +\infty \right\}$$

be a Hilbert space with inner product

$$(u, v)_{Z_2^i} = \sum_{j=1}^{\infty} \mu_j^{2s_i} (u, h_j)(v, h_j). \tag{28}$$

Then triple $\Big(\{h_i\}_{i \geq 1}; V_i; H\Big)$ satisfies condition (γ) with constant $C = 1$ ($i = 1, 2$).

Proof. Having in mind lemma 2 and lemma 3, it is enough to show that $\{h_i\}_{i \geq 1}$ is a special basis for $(Z_2^i; H)$ $(i = 1, 2)$. Condition i) of definition 1 is obviously satisfied. Using (7) and condition i) we have

$$\forall i, j \geq 1 \qquad (h_i, h_j)_{Z_2^i} = \sum_{k=1}^{\infty} \mu_k^{2s} (h_i, h_k)(h_j, h_k) = \mu_i^{2s} \delta_{ij} = \mu_i^{2s} \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

so the condition ii) holds. Finally condition iii) follows from the last equality.

The lemma is proved.

REMARK 7. Further we shall consider, that the triple of spaces V_1, V_2 and H satisfies conditions of corollary 2! To obtain convergence c) in definition 5 we need to assume $V_1 \subset H$ or $V_2 \subset H$ with compact embedding.

7. The main result.

THEOREM. *Let $A : X_1 \to C_v(X_1^*)$ and $B : X_2 \to C_v(X_2^*)$ be such multi-valued maps that*

1) *A is λ_0-pseudomonotone on W_1, bounded and $+$-coercive on X_1;*

2) *B is λ_0-pseudomonotone on W_2, quasi-bounded, satisfies condition (κ) and $+$-coercivity condition on X_2, contraction of B on arbitrary finite-dimensional subspace $F \subset W$ is locally bounded.*

Moreover let $\{h_j\}_{j\ge1} \subset V_1 \cap V_2$ *is complete vector system, that exists by corollary 2, and* $\forall i = 1, 2$ *the triple* $\Big(\{h_j\}_{j\ge1}; V_i; H\Big)$ *satisfies condition* (γ). *Then for every* $f \in X^*$ *and* $y_0 \in H$ *the set*

$$K_H(f) := \Big\{ y \in W \;\big|\; y \;\; \text{the solution of (3)–(4), obtained by Faedo-Halerkin method} \Big\}$$

is non-empty and presentation

$$K_H(f) = \bigcup_{\{y_{0n}\}_{n\ge1}\subset H,\ \text{that satisfies (20)}} \ \bigcap_{n\ge1} \Big[\bigcup_{m\ge n} K_m(f_m)(y_{0m}) \Big]_{X_w}, \tag{29}$$

with $\forall n \ge 1 \quad K_n(f_n)(y_{0n}) = \Big\{ y_n \in W_n \mid y_n \;\; \text{solution of } (21)_n - (22_n) \Big\}$, *where* $[\,\cdot\,]_{X_w}$ *is closure operator in space X with weak topology, is true.*

Proof. Let us consider the map:

$$X \ni y \to C(y) := A(y) + B(y) \subset X^*.$$

Due to proposition 2 and proposition 3 the multi-valued map

$$\begin{aligned} &C : X \to C_v(X^*) \text{ is } \lambda_0\text{-pseudomonotone on } W, \\ &+\text{-coercitive and satisfies condition}(\kappa). \end{aligned} \tag{30}$$

Let $\{y_{0n}\}_{n\ge1} \subset H$ be an arbitrary sequence that satisfies (20). Then there exists such $\delta > 0$ that

$$\sup_{n\ge1} ||y_{0n}||_H \le \delta. \tag{31}$$

For every $n \ge 1$ we search such $x_n \in W_n \subset C(S; H)$ that $x_n(0) = y_{0n}$ and $||x_n||_W \le 1$. Now use the $+$-coercivity condition. Let us define $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}$ in such way: $\gamma(r) = \inf\limits_{||y||_X=r} ||y||_X^{-1}\Big([C(y), y]_+\Big)$. It is obvious that $\gamma(r) \longrightarrow +\infty$ as $r \longrightarrow +\infty$. Remark that for all $n \ge 1$ and $y \in X$ $[C(y) - f + x_n', y]_+ \ge$

$$\ge \Big(\gamma(||y||_X) - ||f||_{X^*} - ||x_n'||_{X^*}\Big)||y||_X \ge \Big(\gamma(||y||_X) - ||f||_{X^*} - 1\Big)||y||_X.$$

So, there exists such $r_0 > 0$ that

$$r_0 > 1 \ge 0,\ [C(y) - f + x_n', y]_+ \ge 0 \ \forall n \ge 1 \, \forall y \in X :\ ||y||_X \ge r_0. \tag{32}$$

Let us put $R = 3r_0$. Then for every $z \in \overline{B}_1(\overline{0}) \subset \overline{B}_{r_0}(\overline{0})$

$$\overline{B}_{r_0}(\overline{0}) \subset \overline{B}_{2r_0}(z) = \Big\{ y \in X \;\Big|\; ||y - z||_X \le 2r_0 \Big\} \subset \overline{B}_R(\overline{0}). \tag{33}$$

7.1. Resolvability of approximating problems.

LEMMA 4. *For all $n \geq 1$ there exists a solution of the problem $(21)_n - (22)_n$ $y_n \in W_n$ such that $\|y_n\|_X \leq R$.*

Proof. Let us for every $n \geq 1$ define $D_n(\cdot) := C_n(\,\cdot\, + x_n) : X_n \overset{\to}{\to} X_n^*$. Now prove that the given map satisfies the next properties:

i_1) $C_n, D_n : X_n \to C_v(X_n^*)$;

i_2) C_n, D_n is λ_0-pseudomonotone on W_n, locally finite-dimensional bounded;

i_3) $[D_n(y) - f_n + x_n', y]_+ \geq 0 \qquad \forall y \in X_n : \ ||y||_{X_n} = 2r_0$.

Let us prove i_1). The convexity and completeness of $C_n(y)$ and $D_n(y)$ are evident. Now prove that for every $\omega, y \in X_n$ and $n \geq 1$

$$[C_n(\omega) - f_n + x_n', y]_+ = [C(\omega) - f + x_n', y]_+.$$

In virtue of definitions of $[\cdot,\cdot]_+$ and I_n^* we have

$$[C_n(\omega) - f_n + x_n', y]_+ = [I_n^*(C(\omega) - f + x_n'), y]_+ = \sup_{d \in C(\omega)} \langle I_n^*(d - f + x_n'), y\rangle =$$

$$= \sup_{d \in C(\omega)} \langle d - f + x_n', y\rangle = [C(\omega) - f + x_n', I_n y]_+ = [C(\omega) - f + x_n', y]_+.$$

Let us put in last $\omega = y + x_n'$. Due to (33) and (32) we receive i_3).

Let us consider i_2). Because of boundedness of $I_n \in L(X_n; X)$, $A : X_1 \to C_v(X_1^*)$, $I_n^* \in L(X^*; X_n^*)$ and locally finite-dimensional boundedness of $B : X_2 \to C_v(X_2^*)$ follows the locally finite-dimensional boundedness of C_n.

Now prove the λ_0-pseudomonotony of C_n on W_n. Let $\{y_m\}_{m \geq 0} \subset W_n$ be an arbitrary such sequence that $y_m \rightharpoonup y_0$ in W_n, $I_n^* C(y_m) = C_n(y_m) \ni d_m \rightharpoonup d \in X_n^*$ as $m \to +\infty$ and inequality (7) is holds. As $W_n \subset W$ with continuous embedding then

$$y_m \rightharpoonup y_0 \text{ in } W \quad \text{as } \ m \to +\infty. \tag{35}$$

From (34) follows that for every $m \geq 1$ there exists such $g_m \in C(y_m)$ that $I_n^* g_m = d_m$. So, for all $m \geq 1$

$$g_m \in C(y_m) \text{ and } I_n^* g_m = d_m. \tag{36}$$

Inasmuch as $\forall m \geq 1$

$$\langle d_m, y_m - y_0\rangle = \langle I_n^* g_m, y_m - y_0\rangle = \langle g_m, y_m - y_0\rangle,$$

then

$$\overline{\lim_{m \to \infty}} \langle d_m, y_m - y_0\rangle = \overline{\lim_{m \to \infty}} \langle g_m, y_m - y_0\rangle \leq 0. \tag{37}$$

In virtue of boundedness of A and quasi-boundedness of B we have that the sequence $\{g_m\}_m$ is bounded in X^*. Consequently from (35)–(37) and (30) the existence of such subsequences $\{y_{m_k}\}_{k \geq 1} \subset \{y_m\}_{m \geq 1}$ and $\{g_{m_k}\}_{k \geq 1} \subset \{g_m\}_{m \geq 1}$ and $g \in X^*$ that $\forall w \in X$

$$\underline{\lim_{k \to \infty}} \langle g_{m_k}, y_{m_k} - w\rangle \ \geq \ [C(y_0), y_0 - w]_- \text{ and } g_{m_k} \rightharpoonup g \text{ in } X^* \text{ as } k \to \infty \tag{38}$$

is follows. Remark that for every $k \geq 1$ and $w \in X_n$

$$\langle g_{m_k}, y_{m_k} - w\rangle = \langle I_n^* g_{m_k}, y_{m_k} - w\rangle = \langle d_{m_k}, y_{m_k} - w\rangle; \tag{39}$$

$$[C(y_0), y_0 - w]_- = \inf_{g \in C(y_0)} \langle g, y_0 - w\rangle = \inf_{g\in C(y_0)} \langle I_n^* g, y_0 - w\rangle =$$
$$= \inf_{I_n^* g \in C_n(y_0)} \langle I_n^* g, y_0 - w\rangle = [C_n(y_0), y_0 - w]_-. \qquad (40)$$

From (38)–(40) follows that $\forall w \in X_n$

$$\varliminf_{k\to\infty} \langle d_{m_k}, y_{m_k} - w\rangle \ \geq\ [C_n(y_0), y_0 - w]_- \text{ and } d_{m_k} \rightharpoonup I_n^* g \text{ in } X^* \text{ as } k \to \infty.$$

So, C_n is λ_0-pseudomonotone on W_n. Due to properties of C_n and from $x_n \in W_n$ it follows that D_n is finite-dimensional locally bounded and λ_0-pseudomonotone on W_n.

Simultaneously with the problem $(21)_n$–$(22)_n$ let us consider the next

$$\begin{cases} z_n' + D_n(z_n) \ni f_n - x_n', & (41)_n \\ z_n(0) = \bar{0}. & (42)_n \end{cases}$$

problem on searching solution z_n in W_n.

Let $L_n : D(L_n) \subset X_n \to X_n^*$ with $D(L_n) = \{y \in W_n \mid y(0) = \bar{0}\} = W_n^0$ be such operator that for every $y \in W_n^0$ $L_n y = y'$. The derivative y' of an element y we understand in sense of scalar distributions space $D^*(S; H_n)$.

Lemma 5. *For every $n \geq 1$ the operator L_n satisfies next properties:*
$i_4)$ L_n *is linear;*
$i_5)$ *for every $y \in W_n^0$ $\langle L_n y, y\rangle \geq 0$;*
$i_6)$ L_n *is maximal monotone.*

Proof. Property $i_4)$ is obvious. Now prove $i_5)$. Due to (6) for all $y_n \in W_n^0$ we have

$$\langle L_n y_n, y_n\rangle = \langle y_n', y_n\rangle = \frac{1}{2}(\|y_n(T)\|_H^2 - \|y_n(0)\|_H^2) = \frac{1}{2}\|y_n(T)\|_H^2 \geq 0. \qquad (43)$$

In virtue of the linearity of L_n on W_n^0 and (43) the monotony of L_n on W_n^0 is follows.

Let us prove the maximal monotony of L_n on W_n^0. For such $v \in X_n$, $w \in X_n^*$ that for every $u \in W_n^0$ $\langle w - L_n u, v - u\rangle \geq 0$ is true let us prove that $v \in W_n$ and $v' = w$. If we take $u = h\varphi x \in W_n^0$ with $\varphi \in D(S)$, $x \in H_n$ and $h > 0$ we get

$$0 \leq\ \langle w - \varphi' h x, v - \varphi h x\rangle = \langle w, v\rangle -$$
$$-\left(\int_S (\varphi'(s) v(s) + \varphi(s) w(s)) ds, hx\right) + \langle \varphi' h x, \varphi h x\rangle =$$
$$= \langle w, v\rangle + h\langle v'(\varphi) - w(\varphi), x\rangle,$$

where $v'(\varphi)$, $w(\varphi)$ are values of distributions v' and w on $\varphi \in D(S)$. So, for every $\varphi \in D(S)$ and $x \in X_n$ $\langle v'(\varphi) - w(\varphi), x\rangle \geq 0$ is true. Thus we obtain $v'(\varphi) = w(\varphi)$ for all $\varphi \in D(S)$. It means that $v' = w \in X^*$. Now prove $v(0) = \bar{0}$. Due to (6) with $u(t) = v(T)\cdot\frac{t}{T} \in W_n^0$ we receive $0 \leq\ \langle v' - L_n u, v - u\rangle = \langle v' - u', v - u\rangle = \frac{1}{2}\Big(||v(T) - u(T)||_H^2 - ||v(0) - u(0)||_H^2\Big) = -||v(0)||_H^2 \leq 0$ and $||v(0)||_{X_n} = 0$.

Lemma 5 is proved.

Now let us continue the proof of lemma 4. In virtue of [13, theorem 2.1] with $V = W = X = X_n$, $A = D_n$, $B \equiv \overline{0}$, $L = L_n$, $D(L) = W_n^0$, $f = f_n - x_n'$, $r = 2r_0$ and properties $i_1)$–$i_6)$ the problem $(41)_n$–$(42)_n$ has such solution $z_n \in W_n$ that $||z_n||_X \le 2r_0$. Remark that under boundedness condition on A_n and quasi-boundedness condition on B_n it is easy to find the estimate for selectors (similar to (55)) to apply the λ_0-pseudomonotony for A and B on W_n. Because of (33) and $z_n \in W_n$ it follows that $y_n := z_n + x_n \in W_n$ is such solution of $(21)_n$–$(22)_n$ that $||y_n||_X \le R$.

Lemma 4 is proved.

7.2. Boundary transition.

Due to lemma 4 we have a sequence of Halerkin approximate solutions $\{y_n\}_{n\ge1}$ that satisfies next conditions

$$\text{a) } \forall n \ge 1: \quad ||y_n||_X \le R; \tag{44}$$

$$\text{b) } \forall n \ge 1: \quad y_n \in W_n \subset W, \quad y_n' + C_n(y_n) \ni f_n; \tag{45}$$

$$\text{c) } \forall n \ge 1: \quad y_n(0) = y_{0n} \to y_0 \text{ in } H \text{ as } n \to \infty. \tag{46}$$

From (45) it follows

$$\forall n \ge 1 \quad \exists d_n \in C(y_n): \quad I_n^* d_n =: d_n^1 = f_n - y_n' \in C_n(y_n) = I_n^* C_n(y_n). \tag{47}$$

Lemma 6. *In virtue of* (44)–(47) *there exist such subsequences* $\{y_{n_k}\}_{k\ge1} \subset \{y_n\}_{n\ge1}$ *and* $\{d_{n_k}\}_{k\ge1} \subset \{d_n\}_{n\ge1}$ *that for some* $y \in W$, $d \in X^*$, $z \in H$ *the next*

$$1)\ y_{n_k} \rightharpoonup y \ \textit{in} \ X \qquad \textit{as} \ \ k \to \infty; \tag{48}$$

$$2)\ y_{n_k} \to y \ \textit{in} \ L_{p_0}(S;H) \ \textit{as} \ \ k \to \infty; \tag{49}$$

$$3)\ y_{n_k}' \rightharpoonup y' \ \textit{in} \ X^* \qquad \textit{as} \ \ k \to \infty; \tag{50}$$

$$4)\ d_{n_k} \rightharpoonup d \ \textit{in} \ X^* \qquad \textit{as} \ \ k \to \infty; \tag{51}$$

$$5)\ y_{n_k}(T) \rightharpoonup z \ \textit{in} \ H \qquad \textit{as} \ \ k \to \infty \tag{52}$$

is true. Moreover, in (48)–(52):

$$\text{(i) } y(0) = y_0, \quad \text{(ii) } z = y(T), \quad \text{(iii) } d = f - y'. \tag{53}$$

Proof. 1°. At first we prove that $\{d_n\}_{n\ge1}$ is bounded in X^*. In virtue of (47) and definition of C it follows that for every $n \ge 1$ there exists such $d_n' \in A(y_n)$ and $d_n'' \in B(y_n)$ that $d_n' + d_n'' = d_n$. Due to boundedness of A there exists such $c_1 > 0$ that

$$\forall n \ge 1 \qquad ||d_n'||_{X_1^*} \le c_1. \tag{54}$$

From (47), (6), (54) and (31) it results in for all $n \ge 1$

$$+\infty > ||f||_{X^*} R \ge ||f||_{X^*} ||y_n||_X \ge \langle f, y_n\rangle = \langle f_n, y_n\rangle = \langle y_n', y_n\rangle +$$

$$+\langle d_n^1, y_n\rangle = \langle y_n', y_n\rangle + \langle d_n', y_n\rangle + \langle d_n'', y_n\rangle \ \ge$$

$$\geq \frac{1}{2}\Big(||y_n(T)||_H^2 - ||y_n(0)||_H^2\Big) - c_1R + \langle d_n'', y_n\rangle \geq$$
$$\geq -\delta^2/2 - c_1R + \langle d_n'', y_n\rangle.$$

So, for every $n \geq 1$

$$\langle d_n'', y_n\rangle \leq ||f||_{X^*}R + \delta^2/2 + c_1R =: c_2 < +\infty.$$

From here, taking into account $X_1 \subset X$ with continuous embedding, estimation (44) and quasi-boundedness of B it follows that

$$\exists c_3 > 0 : \ \forall n \geq 1 \quad ||d_n''||_{X_2^*} \leq c_3.$$

With help of (54) we have:

$$\exists c_4 > 0 : \quad \forall n \geq 1 \quad ||d_n||_{X^*} \leq c_4. \tag{55}$$

2°. Now let us prove the boundedness of $\{y_n'\}_{n\geq 1}$ in X^*. From (47) it follows that for every $n \geq 1$ $y_n' = I_n^*(f - d_n)$ and so, with help of (55) and (19), for all $n \geq 1$

$$||y_n'||_{X^*} = ||I_n^*(f - d_n)||_{X^*} \leq C \cdot ||f - d_n||_{X^*} \leq C(||f||_{X^*} + c_4) =: c_5 < +\infty, \tag{56}$$

where $C \geq 1$ is the constant from condition (γ).

3°. Here we prove the precompactness of $\{y_n\}_{n\geq 1}$ in $L_{r_i}(S; H)$ $(i = 1, 2)$. Without loss of generality it is enough to prove the existence of such subsequence $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ that converges in $L_{r_i}(S; H)$.

Due to estimates (44) and (56) the boundedness of $\{y_n\}_{n\geq 1}$ in W is holds, i.e.

$$\exists\, c_6 > 0 : \quad ||y_n||_W = ||y_n||_X + ||y_n'||_{X^*} \leq c_6 < +\infty \ \ \forall n \geq 1. \tag{57}$$

In virtue of $W \subset C(S; H)$ with continuous embedding the existence of such $c_7 > 0$ that

$$\forall n \geq 1 \quad \text{for a.a. } t \in S \qquad ||y_n(t)||_H \leq c_7 < +\infty \tag{58}$$

is follows. Now use [1, theorem I.5.1] with $r_i = p_1$, $p_1 = \min\{q_0, q_1, q_2\}$, $B_0 = V_1$ or $B_0 = V_2$, $B = H$ and $B_1 = V^* = V_1^* + V_2^*$. Because of $X \subset L_{p_1}(S; V_1)$ and $X^* \subset L_{\min\{q_0,q_1,q_2\}}(S; V^*)$ with continuous embedding, thanks to (57), it follows that $\{y_n\}_{n\geq 1}$ is precompact in $L_{p_1}(S; H)$. Let $\{y_m\}_m$ be such subsequence from $\{y_n\}_{n\geq 1}$ that tends to some y in $L_{p_1}(S; H)$. Setting $\psi_m(t) = ||y_m(t) - y(t)||_H : S \to \mathbb{R}_+$ for all m and $t \in S$ ($\psi_m \to \bar{0}$ in $L_{p_1}(S)$) it follows the existence of such $\{y_{n_k}\}_{k\geq 1} \subset \{y_m\}$ that for almost all $t \in S$ $|\psi_{n_k}(t)|^{p_1} \to 0$ (see [14]). Consequently, $\psi_{n_k}(t) \to 0$ a.e. in S. Due to (58) we have $||y||_{C(S;H)} \leq c_7$. So, the sequence $\psi_{n_k}^{r_i}(\cdot)$ satisfies the conditions of the Lebesgue theorem with the integrable majorant $g(\cdot) \equiv (2c_7)^{r_i}$ $(i = 1, 2)$. Therefore, the sequence

$$\{y_n\}_{n\geq 1} \quad \text{is precompact in} \quad L_{r_i}(S; H) \quad i = 1, 2. \tag{59}$$

4°. In virtue of inequalities (6), (57) and a priory estimates (44), (55) and (31), the boundedness of $\{y_n(T)\}_{n\geq 1}$ in H is follows. For every $n \geq 1$ $\langle y_n', y_n\rangle + \langle d_n^1, y_n\rangle = \langle f_n, y_n\rangle$. Thus,

$$||y_n(T)||_H^2 \leq ||y_{0n}||_H^2 + 2\langle f - d_n, y_n\rangle \leq \delta^2 + 2(||f||_{X^*} + c_4)R =: c_8 < +\infty, \tag{60}$$

where $c_8 > 0$ is not depends on $n \geq 1$.

5°. Due to (57), (59), (55), (60) and to the Banach-Alaoglu theorem it follows the existence of such $\{y_{n_k}\}_{k\geq 1}$ from $\{y_n\}_{n\geq 1}$, $\{d_{n_k}\}_{k\geq 1}$ from $\{d_n\}_{n\geq 1}$, y from W, d from X^* and z from H that (48)–(52) are true.

6°. Let us prove (iii). For $\varphi \in D(S)$, $n \geq 1$ and $h \in H_n$ let $\psi(\cdot) = h \cdot \varphi(\cdot) \in X_n \subset X$. Then for every such $k \geq 1$ that $n_k \geq n$ due to lemma 1 b) we have

$$\Big(\int\limits_S \varphi(\tau)(y'_{n_k}(\tau) + d_{n_k}(\tau))d\tau, h\Big) = \int\limits_S \Big(\varphi(\tau)(y'_{n_k}(\tau) + d_{n_k}(\tau)), h\Big)d\tau =$$

$$= \int\limits_S \Big(y'_{n_k}(\tau) + d_{n_k}(\tau), \varphi(\tau)h\Big)d\tau = \langle y'_{n_k} + d_{n_k}, \psi\rangle = \langle y'_{n_k} + d_{n_k}, I_{n_k}\psi\rangle =$$

$$= \langle I^*_{n_k}(y'_{n_k} + d_{n_k}), \psi\rangle = \langle y'_{n_k} + d^1_{n_k}, \psi\rangle = \langle f_{n_k}, \psi\rangle = \langle f, I_{n_k}\psi\rangle =$$

$$= \int\limits_S (f(\tau), \varphi(\tau)h)d\tau = \int\limits_S (\varphi(\tau)f(\tau), h)d\tau = \Big(\int\limits_S \varphi(\tau)f(\tau)d\tau, h\Big).$$

So, for all such $k \geq 1$ that $n_k \geq n$

$$\Big(\int\limits_S \varphi(\tau)y'_{n_k}(\tau)d\tau, h\Big) = \Big(\int\limits_S \varphi(\tau)(f(\tau) - d_{n_k}(\tau))d\tau, h\Big) =$$

$$= \int\limits_S \big((f(\tau) - d_{n_k}(\tau)), \varphi(\tau)h\big)d\tau = \langle f - d_{n_k}, \psi\rangle \to \langle f - d, \psi\rangle =$$

$$= \Big(\int\limits_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\Big) \text{ as } k \to \infty. \tag{61}$$

It follows from $d_{n_k} \rightharpoonup d$ in X^*. In virtue of (50) we have

$$\Big(\int\limits_S \varphi(\tau)y'_{n_k}(\tau)d\tau, h\Big) \to \Big(\int\limits_S \varphi(\tau)y'(\tau)d\tau, h\Big) = \big(y'(\varphi), h\big) \text{ as } k \to +\infty. \tag{62}$$

Due to (61)–(62) we obtain

$$\forall \varphi \in D(S)\ \forall h \in \bigcup_{n\geq 1} H_n \quad \big(y'(\varphi), h\big) = \Big(\int\limits_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h\Big).$$

The set $\bigcup\limits_{n\geq 1} H_n$ is dense in V, then

$$\forall \varphi \in D(S) \quad y'(\varphi) = \int\limits_S \varphi(\tau)(f(\tau) - d(\tau))d\tau.$$

So, $y' = f - d$ and $d = f - y'$.

7°. Now prove (i). Let us for every $n \geq 1$ and $h \in H_n$ define $\psi(\cdot)$ as $(T - \cdot)h \in X_n$. From (51) it follows:

$$\langle y', \psi \rangle = \int\limits_S (y'(\tau), \psi(\tau)) d\tau = \int\limits_S (f(\tau) - d(\tau), \psi(\tau)) d\tau = \langle f - d, \psi \rangle =$$

$$= \lim_{k\to\infty} \langle f - d_{n_k}, I_{n_k}\psi \rangle = \lim_{k\to\infty} \langle I^*_{n_k}(f - d_{n_k}), \psi \rangle = \lim_{k\to\infty} \langle f_{n_k} - d^1_{n_k}, \psi \rangle = \lim_{k\to\infty} \langle y'_{n_k}, \psi \rangle.$$

Now let us use (5). Remark that $\psi'(\tau) = -h$ a.e. in S. Then, taking into account $y_{n_k} \rightharpoonup y$ in X and $y_{n_k 0} \to y_0$ in H as $k \to \infty$, we obtain:

$$\lim_{k\to\infty} \langle y'_{n_k}, \psi \rangle = \lim_{k\to\infty} \big\{ -\langle \psi', y_{n_k} \rangle + (y_{n_k}(T), \psi(T)) - (y_{n_k 0}, \psi(0)) \big\} =$$

$$= \lim_{k\to\infty} \left\{ \int\limits_S (y_{n_k}(\tau), h) d\tau - (y_{n_k 0}, Th) \right\} = \lim_{k\to\infty} \int\limits_S (y_{n_k}(\tau), h) d\tau -$$

$$- \lim_{k\to\infty} (y_{n_k 0}, Th) = \int\limits_S (y(\tau), h) d\tau - (y_0, Th) = -\langle \psi', y \rangle - (y_0, Th)\,.$$

Let us use (5) again: $-\langle \psi', y \rangle - (y_0, Th) = \langle y', \psi \rangle - (y(T), \psi(T)) +$

$$+ (y(0), \psi(0)) - (y_0, Th) = \langle y', \psi \rangle + T(y(0) - y_0, h).$$

So, for every $h \in \bigcup\limits_{n\geq 1} H_n$ $\langle y', \psi \rangle = \langle y', \psi \rangle + T(y(0) - y_0, h)$. Hence, $(y(0) - y_0, h) = 0$. From density $\bigcup\limits_{n\geq 1} H_n$ in H it follows that $y(0) = y_0$.

8°. It is remain to prove (ii). The proof is similar to 7°. Let us take $\psi(\cdot) \equiv h \in \bigcup\limits_{n\geq 1} H_n$. Hence, $\psi \in X_{n_0}$ for some n_0. Due to (5), (52) and (i) from (53)

$$(y(T) - y(0), h) = \int\limits_S (y'(\tau), h) d\tau = \lim_{k\to\infty} \int\limits_S (y'_{n_k}(\tau), h) d\tau =$$

$$= \lim_{k\to\infty} (y_{n_k}(T) - y_{n_k}(0), h) = (z - y(0), h).$$

Thus, for every $h \in \bigcup\limits_{n\geq 1} H_n$ $(y(T) - z, h) = 0$. Hence, $y(T) = z$.

The lemma is proved.

To prove y is a solution of problem (3)–(4), obtained by FG method, it is remain to show (due to lemma 6) that $d \in C(y)$. At first we make sure that

$$\overline{\lim_{k\to\infty}} \langle d_{n_k}, y_{n_k} - y \rangle \leq 0. \tag{63}$$

In virtue of (6) and (iii) for all $k \geq 1$

$$\langle d_{n_k}, y_{n_k} - y \rangle = \langle d_{n_k}, y_{n_k} \rangle - \langle d_{n_k}, y \rangle = \langle d^1_{n_k}, y_{n_k} \rangle - \langle d_{n_k}, y \rangle =$$

$$= \langle f_{n_k} - y'_{n_k}, y_{n_k} \rangle - \langle d_{n_k}, y \rangle = \langle f_{n_k}, y_{n_k} \rangle - \langle y'_{n_k}, y_{n_k} \rangle - \langle d_{n_k}, y \rangle =$$

$$= \langle f, y_{n_k}\rangle - \langle d_{n_k}, y\rangle + \frac{1}{2}\left(\|y_{n_k}(0)\|_H^2 - \|y_{n_k}(T)\|_H^2\right). \tag{64}$$

Due to lemma 6, (6), [2, lemma I.5.3], (52), (53) and (64)

$$\overline{\lim_{k\to\infty}}\langle d_{n_k}, y_{n_k} - y\rangle \le \overline{\lim_{k\to\infty}}\langle f, y_{n_k}\rangle + \overline{\lim_{k\to\infty}}\langle d_{n_k}, -y\rangle +$$

$$+\overline{\lim_{k\to\infty}}\frac{1}{2}\left(\|y_{n_k}(0)\|_H^2 - \|y_{n_k}(T)\|_H^2\right) \le \langle f, y\rangle - \langle d, y\rangle +$$

$$+\frac{1}{2}\left(\|y(0)\|_H^2 - \|y(T)\|_H^2\right) = \langle f - d, y\rangle - \langle y', y\rangle = \langle y' - y', y\rangle = 0.$$

The inequality (63) is proven.

In virtue of (48), (50), (51), (63) and due to λ_0-pseudomonotony of C on W it follows that there exist such subsequences $\{d_m\} \subset \{d_{n_k}\}_{k\ge 1}$ and $\{y_m\} \subset \{y_{n_k}\}_{k\ge 1}$ that $\forall \omega \in X$

$$\underline{\lim}_{m\to\infty} \langle d_m, y_m - \omega\rangle \ \ge\ [C(y), y - \omega]_- . \tag{65}$$

To finish the proof of the theorem it is enough to show

$$\langle d, y\rangle \ \ge\ \overline{\lim_{m\to\infty}}\langle d_m, y_m\rangle. \tag{66}$$

Because of (65), (66), (51) and (48) we obtain that for every $\omega \in X$ $[C(y), y-\omega]_- \le \langle d, y - \omega\rangle$. It is equivalent to $[C(y), \omega]_+ \ge \langle d, \omega\rangle$ $\forall \omega \in X$. It means that $d \in C(y)$. So, y is the solution of (3)–(4).

Now prove (66): $\overline{\lim_{m\to\infty}}\langle d_m, y_m\rangle = \overline{\lim_{m\to\infty}}\langle d_m, I_m y_m\rangle = \overline{\lim_{m\to\infty}}\langle d_m^1, y_m\rangle =$

$$= \overline{\lim_{m\to\infty}}\langle f_m - y_m', y_m\rangle \le \overline{\lim_{m\to\infty}}\langle f_m, y_m\rangle + \overline{\lim_{m\to\infty}}(-\langle y_m', y_m\rangle) = \overline{\lim_{m\to\infty}}\langle f, y_m\rangle +$$

$$+\frac{1}{2}\overline{\lim_{m\to\infty}}\left(\|y_m(0)\|_H^2 - \|y_m(T)\|_H^2\right) \le \ \langle f, y\rangle - \frac{1}{2}\left(\|y(T)\|_H^2 - \|y(0)\|_H^2\right) =$$

$\langle f, y\rangle - \langle y', y\rangle = \langle d, y\rangle$. So, we proved that $y \in W$ is the solution of (3)–(4).

Remark that (29) is immediately follows from boundary transition, [15, property 2.29. IV.8] and definition 5.

The theorem is proved.

8. Example.

Let us consider bounded domain $\Omega \subset \mathbb{R}^n$ with rather smooth boundary $\partial\Omega$; $S = [0, T]$, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$; N_1^i(correspondingly N_2^i) be a number of differentiations by x of order $\le m_i - 1$ (correspondingly m_i) and let $A_\alpha^i(x, t, \eta, \xi)$ be a family of real functions ($|\alpha| \le m_i$) defined in $Q \times R^{N_1^i} \times R^{N_2^i}$ $(i = 1, 2)$. Let

$$D^k u = \{D^\beta u, |\beta| = k\} \text{ be differentiation by } x,$$

$$\delta u = \{u,\ \ Du,\ \ ...,\ \ D^{m-1}u\},$$

$$A_\alpha^i(x, t, \delta u, D^m v):\ \ x, t \to A_\alpha^i(x, t, \delta u(x,t), D^m v(x,t)), \quad i = 1, 2.$$

Moreover let $\psi : \mathbb{R} \to \mathbb{R}$ be convex lower semicontinuous coercive functional, $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}$ be its subdifferential.

Let us consider the next problem with Dirichlet boundary conditions:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha|\le m_1} (-1)^{|\alpha|} D^\alpha (A^1_\alpha(x,t,\delta y, D^m y)) + \sum_{|\alpha|\le m_2} (-1)^{|\alpha|} D^\alpha (A^2_\alpha(x,t,\delta y, D^m y)) +$$

$$+\Phi(y(x,t)) \ni f(x,t) \qquad \text{a.e. in } Q, \tag{67}$$

$$y(x,0) = y_0(x) \qquad \text{a.e. in } \Omega, \tag{68}$$

$$y(x,t) = 0 \qquad \text{a.e. in } \Gamma_T. \tag{69}$$

On some conditions on coefficients A^i_α the given problem is equivalent to the next differential-operator inclusion

$$y' + A_1(y) + A_2(y) + \partial\varphi(y) \ni f, \qquad y(0) = y_0, \tag{70}$$

where $f \in X^* = L_2(S; L_2(\Omega)) + L_{q_1}(S; W^{-m_1,q_1}(\Omega)) + L_{q_2}(S; W^{-m_2,q_2}(\Omega))$, $y_0 \in L_2(\Omega)$ are some fixed elements, $\partial\varphi$ is subdifferential from the integral functional

$$\varphi(y) = \int_Q \psi(y(x,t))dxdt$$

in space $L_2(S; L_2(\Omega))$. The element $y \in X$ that satisfies (70) is called the generous solution of the problem (67)–(69).

Let us also take $(H, (\cdot,\cdot)) = \big(L_2(\Omega), (\cdot,\cdot)_{L_2(\Omega)}\big)$, $V_i := W_0^{m_i,p_i}(\Omega)$ $(i = 1,2)$. It follows that V_i $(i = 1,2)$ is a reflexive separable Banach space. Further, we consider that $p_i > 1$ and $m_i \in \mathbb{N}$.

Choice of basis. Due to the corollary 2 and [9, theorem 4.3.1.2] as complete vector system in spaces $W_0^{m_i,p_i}(\Omega)$ we can take the special basis for the pair $(H_0^{\max\{m_1;m_2\}+\varepsilon}(\Omega); L_2(\Omega))$ with some $\varepsilon > 0$.

Definition of operators A_i. Let the family of real functions $A^i_\alpha(x,t,\eta,\xi)$ $(|\alpha| \le m_i)$ defined in $Q \times R^{N^i_1} \times R^{N^i_2}$ satisfies next conditions

for almost all $x,t \in$ Q the map $\eta,\xi \to A^i_\alpha(x,t,\eta,\xi)$ is continuous on $R^{N^i_1} \times R^{N^i_2}$;

$$\text{for all } \eta,\xi \text{ the map } x,t \to A^i_\alpha(x,t,\eta,\xi) \text{ is measurable on } Q. \tag{71}$$

$$\text{for all } u,v \in L^{p_i}(0,T;V_i(\Omega)) =: V_i \quad A^i_\alpha(x,t,\delta u, D^m u) \in L^{q_i}(Q). \tag{72}$$

Then for every $u \in V_i$

$$w \to a_i(u,w) = \sum_{|\alpha|\le m} \int_Q A^i_\alpha(x,t,\delta u, D^m u) D^\alpha w dxdt, \tag{73}$$

is continuous and

$$\text{for every } u \in V_i \text{ there exists such } A_i(u) \in V_i' \text{ that } a_i(u,w) = \langle A_i(u), w\rangle. \tag{74}$$

Conditions on A_i. Similarly to [1, sections 2.2.5, 2.2.6, 3.2.1] we have

$$A_i(u) = A_i(u,u), \quad A_i(u,v) = A_{i1}(u,v) + A_{i2}(u),$$

where

$$\langle A_{i1}(u,v),w\rangle = \sum_{|\alpha|=m_i}\int_Q A^i_\alpha(x,t,\delta u, D^{m_i}v)D^\alpha w dxdt, \tag{75}$$

$$\langle A_{i2}(u),w\rangle = \sum_{|\alpha|\le m_i-1}\int_Q A^i_\alpha(x,t,\delta u, D^{m_i}u)D^\alpha w dxdt \quad (i=1,2). \tag{76}$$

Now consider the next conditions:

$$\langle A_{i1}(u,u),u-v\rangle - \langle A_{i1}(u,v),u-v\rangle \ \ge 0 \ \ \forall u,v\in V_i; \tag{77}$$

$$\text{if } u_j \rightharpoonup u \text{ in } V_i,\ \ u'_j \rightharpoonup u' \text{ in } V_i^* \quad \langle A_{i1}(u_j,u_j)-A_{i1}(u_j,u),u_j-u\rangle\to 0, \text{ then}$$
$$A^i_\alpha(x,t,\delta u_j,D^{m_i}u_j)\rightharpoonup A^i_\alpha(x,t,\delta u,D^{m_i}u) \text{ in } L^{q_i}(Q); \tag{78}$$

coercivity.

Remark 8. Similarly to [1, theorem 2.2.8] the sufficient conditions of (77), (78) are:

$$\sum_{|\alpha|=m_i} A^i_\alpha(x,t,\eta,\xi)\xi_\alpha \frac{1}{|\xi|+|\xi|^{p_i-1}} \to +\infty \ \texttt{as}\ |\xi|\to\infty \tag{79}$$

for almost all x,t from Q and bounded η;

$$\sum_{|\alpha|=m_i} (A^i_\alpha(x,t,\eta,\xi)-A^i_\alpha(x,t,\eta,\xi^*))(\xi_\alpha-\xi^*_\alpha)>0 \text{ as } \xi\ne\xi^* \tag{80}$$

for almost all $x,t\in Q$ and $\forall\eta$.

The next condition guarantees the coercivity:

$$\sum_{|\alpha|=m_i} A^i_\alpha(x,t,\eta,\xi)\xi_\alpha \ge c|\xi|^{p_1} \text{ for rather large } |\xi|. \tag{81}$$

The sufficient condition of (72) (see [1, p. 336]) is:

$$|A^i_\alpha(x,t,\eta,\xi)| \le c[|\eta|^{p_1-1}+|\xi|^{p_1-1}+k(x,t)], \quad k\in L_{q_i}(Q). \tag{82}$$

By analogy with the proof of [1, theorem 3.2.1] and [1, statement 2.2.6] for $i=1,2$ we can receive the next

Proposition 4. *Let operator $A_i : V_i \to V_i'$ $(i=1,2)$, defined in* (74), *satisfies* (71), (72), (77) *and* (78). *Then A_i is pseudomonotone on W_i and bounded operator.*

Due to last statement and to the theorem it follows that under listed above conditions for all $f\in X^*$ there exists such $R>0$ that $K_H(f) := \{y\in W \mid y$ is a generous solution of the problem (67)–(69), turned out by FG method $\}$ is non-empty weakly compact in $(\overline{B}_R, ||\cdot||_X)$ set with representation (29).

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