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Blow-up Phenomena Arising in a Reaction-absorption-diffusion Equation with Gradient Diffusivity

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Abstract. We study the blow-up phenomena arising in a p-laplacian equation with reaction and absorption terms. We show that there exists a unique blowing-up *approximate self-similar solution* which describe the asymptotic singular behaviour of a wide class of solutions. As a consequence, we conclude that in this class, the absorption became negligible in finite time in the competition between the reaction and the absorption terms.

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1. Introduction

In this paper we are concerned with the blowing-up behaviour of solutions of the reaction-diffusion-absorption equation with gradient diffusivity

$$u_t = (|u_x|^{m-1}u_x)_x + u^p - u^q, \quad m > 1,\ p > 1,\ q > 1, \tag{1.1}$$

in the range of parameters $1 < q < p < m$. We consider initial data $u(x,0) = u_0(x) \in C_0(\mathbf{R})$ such that blow-up in a finite time T occurs, in the sense that the solution $u(x,t)$ exists and is bounded for every $t < T$ and satisfies that

$$\sup \|u(x,t)\|_\infty \to \infty, \text{ as } t \to T.$$

In the study of the blow-up asymptotic properties, similarity solutions of the nonlinear partial differential equations (PDEs) are known

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to play a fundamental role. In particular, this holds for typical models from nonlinear diffusion and combustion/absorption theory, which was the origin of a general classification of types of self-similarities (of the first and second types) formulated in 1950s and various applications of group-theoretical techniques and the renormalization group analysis. We refer to G. I. Barenblatt's book [1] where a list of other references is available.

Such self-similar solutions are of special importance in the asymptotic analysis of blow-up singularity formation for quasilinear heat equations, i. e., those singularities which occur in finite time, see [22] and reference there in. However, in many models from the combustion/absorption theory, self-similar solutions of the related equation with the singular behaviour under study do not exist. In the analysis of such kind of singularities, it is shown that the behaviour of the solutions under study is somehow *hidden* and it becomes necessary, in order to understand the problem, to determine which terms of the equation become dominant and which ones negligible in the analyzed singularity formation. Therefore, a new *simplified* equation appears, which will play the key role in the asymptotic analysis. Then, the equation under study is considered as a perturbation of the main one, and an stability analysis in the context of perturbed dynamical systems becomes necessary. The study of blow-up problems has attracted a considerable attention during the last years. Concerning the asymptotic blow-up analysis, we refer the works [5], [13], [16], [19] and [15] where the asymptotic blow-up analysis has been done for different semilinear heat equations and [7], [8], [9], [3] and [4] for quasilinear equations. For an extensive list of other relevant results in the theory of blow-up, see [22] and [10].

1.1. Blow-up phenomena for the reaction-diffusion equation

Before dealing with solutions of the reaction-absorption-diffusion equation (1.1), we briefly comment the main properties of the solutions of the reaction-diffusion equation with gradient diffusivity

$$u_t = (|u_x|^{m-1}u_x)_x + u^p, \quad m > 1,\ p > 1, \tag{1.2}$$

in the range of parameters $1 < p < m$. It is known that solutions of (1.2) blow-up in a finite time T, and moreover, in the range of parameters under study, the blow-up is global and it is proved that the solution become unbounded for every $x \in \mathbf{R}$.

The equation is invariant under a group of scaling transformations and admits self-similar solutions of the form

$$u_*(x,t) = (T-t)^{-\alpha} f(\xi), \quad \xi = x/(T-t)^{\beta}, \ \alpha = 1/(p-1), \tag{1.3}$$

where $\beta = (p-m)/(p-1)(m+1)$ and f satisfies the ODE

$$A(f) \equiv (|f'|^{m-1} f')' - \beta \xi f' - \alpha f + f^p = 0, \tag{1.4}$$

with f positive and $f(\xi) \to 0$ as $\xi \to \infty$.

The problem of existence of such non-trivial similarity solutions is understood in greater detail. In fact, the existence result can be obtained without relevant modifications, following the ideas in [22], Chapter 4, where the analogous existence problem for the Porous medium equation is treated. The uniqueness proof can be obtained by using an ODE approach, [2] or by means of PDE techniques via the Sturm Theorem on zero sets for uniformly parabolic equations, see [9] for results concerning the PME equation and [3] for a general approach. Comparison intersection theory, which strongly relies on the Sturm Theorems for PDEs, cf. [21], has been widely used in the study of the classical properties of the solutions of parabolic PDEs, including regularity and asymptotic analysis. For the application of Sturmian Intersection Theory in the study of nonlinear parabolic equations with singularities, see [6] and the list of references there in.

Concerning the stability of solutions, it usually holds that the non-trivial profile $f(\xi) \geq 0$ (or the profile of the simplest geometric shape in several profiles are available) play a key role in the asymptotic analysis of general solutions. Such similarity solutions are known to be asymptotically stable as $t \to T^-$ in the corresponding rescaled variables,

$$\theta^*(\xi,\tau) = (T-t)^{-\alpha} u(\xi(T-t)^{\beta}, t), \ \tau = -\ln(T-t) \to \infty, \tag{1.5}$$

where the rescaled solution θ^* satisfies the rescaled parabolic equation

$$\theta^*_\tau = A(\theta^*) \quad \text{in} \ \ \mathbf{R} \times \mathbf{R}_+. \tag{1.6}$$

In [3] the authors stated a general stability result, which includes the analysis of the blow-up behaviour of solutions of (1.2) as a particular application. Next we deal with the analysis of the analogous stability problem for solutions of equation (1.1).

2. Preliminaries and main results

2.1. Global existence and Blow-up phenomena for (1.1).

We begin by showing that due to the presence of absorption and reaction terms in (1.1), there exist global solutions for small enough initial data, and also blowing-up solutions of the Cauchy problem related to (1.1). In fact, a straightforward calculation shows that the function

$$W_{k_1,k_2}(x,t) = (k_1 t + k_2)^{-1/(q-1)},\ t > 0,$$

is a supersolution of (1.1) if k_1 is small and k_2 is large enough. Therefore, it follows from the Maximum Principle that any solution with initial data satisfying $u_0(x) \le W_{k_1,k_2}(x,0)$ is bounded for every value of time. On the other hand, one can construct blowing-up subsolutions of (1.1) in the following way. For a fixed $0 < \varepsilon < 1$, consider $f_\varepsilon(\xi)$ the profile satisfying equation

$$A(f_\varepsilon) = (1-\varepsilon) f_\varepsilon^p,$$

with A defined in (1.4). Define

$$W_\varepsilon(x,t) = (h(t))^{-\alpha} f_\varepsilon(x(h(t))^{-\beta}),$$

where $h(t)$ is a positive function satisfying,

$$h'(t) = -1 + \frac{1}{\alpha}(h(t))^{(p-q)\alpha},\ \ h(0) < 1-\varepsilon.$$

It is not difficult to check that so defined $W_\varepsilon(x,t)$ is a subsolution of (1.1) which blows up at finite time and therefore, the same holds for any solution $u(x,t)$ with initial data verifying $u_0(x) \ge W_\varepsilon(x,0)$.

In the next sections, we show by developing the method described in [4], that the absorption term in (1.1) becomes negligible, in the blow-up analysis, in finite time. Therefore the blowing up solutions are also described by means of equation (1.2) and in the appropriate variables, the rescaled solution converges to the self-similarity profile f^* in (1.3). In order to prove the result we proceed as follows. We first establish the result for symmetric and decreasing (for $x > 0$) initial data. This allow us to construct a family of solutions of (1.1) with some appropriate geometric features such that the stability theorem in [4] applies and yields to the stability result for a general initial data in the class under study.

2.2. Rescaled variables and a priori estimates

Following the self-similar structure described above, we introduce the rescaled variables,

$$\theta(\xi,\tau) = (T-t)^{-\alpha} u(\xi(T-t)^{\beta}, t),\ \xi = x(T-t)^{-\beta},\ \tau = -\log(T-t).$$

By substituting in (1.2), we arrive at the rescaled perturbed equation

$$\theta_\tau = A(\theta) - e^{-(p-q)\alpha\tau}\theta^q, \quad \tau > 0 \tag{2.1}$$

to be compared with the corresponding autonomous one (1.6). The stability theorem is now stated in terms of the rescaled variables in the following way.

Theorem 2.1. *Let $\theta(\xi,\tau)$ be the rescaled solution of* (2.1) *with initial data $\theta_0(\xi) \in C_0(\mathbf{R})$, corresponding to the blowing-up solution of* (1.2) *$u(x,t)$, with blow-up time T. Then, there holds*

$$\theta(\tau) \to f^*, \text{ as } \tau \to \infty,$$

uniformly in $\mathbf{R}$.

We begin with some a priori bounds for the solutions to be used later on.

Lemma 2.1. *Let $u(x,t)$ a solution of* (1.2) *with initial datum $u_0 = \theta_0 \in C_0(\mathbf{R})$ and blowing up at time T. Then, the corresponding rescaled solution $\theta(\xi,\tau)$ satisfies:*

$$\|\theta(\tau)\| > c^* \quad \textit{for any } \tau \geq 0.$$

Proof. It is strongly based on the existence of the family of blowing-up homogeneous solutions of (1.2) of the type,

$$H_T(t) = c^*(T-t)^{-\alpha},\ c^* = \alpha^\alpha.$$

Assume that the result false and that $\|\theta(\tau_0)\| \leq c^*$ for some $\tau_0 \geq 0$. Then in the original variables we obtain that

$$u(x,t_0) \leq H_T(t_0) \equiv c^*(T-t_0)^{-\alpha}, \quad \text{with } t_0 = 1 - e^{\beta\tau_0}.$$

Hence, taking into account that $H_T(t)$ is a supersolution of equation (1.1) for every $T \in \mathbf{R}$ and by the strong Maximum Principle, we obtain that for a fixed positive $\delta \ll 1$ and arbitrarily small $\varepsilon > 0$,

$$u(x,t_0+\delta) \leq H_{T+\varepsilon}(t_0+\delta).$$

By comparison, the same inequality holds for $t \geq t_0+\delta$ and hence $u(x,t)$ does not blow-up at time T contradicting the assumption. □

Lemma 2.2. *Under the assumptions above, there exists a constant $M>0$ such that*

$$\theta(\xi,\tau) \leq M,\ \forall\, \tau > 0.$$

Proof. Let $\pm a$ be the interfaces of the symmetric equilibrium $f^*(\xi)$. We first prove that there exists $\tau_0 \geq 0$ such that

$$\theta(\xi,\tau) \leq f^*(0)+1, \quad \text{for all } \tau \geq \tau_0 \quad \text{and every } |\xi| \geq a. \tag{2.2}$$

Assume that (2.2) is false for certain sequences $\{\tau_j\} \to \infty$ and $\{\xi_j\}$ with $\xi_j \geq a$. Then, by the decreasing and symmetric hypotheses, the same holds for every $\xi \in [-\xi_0, \xi_0]$, with $\xi_0 = \liminf_{j\to\infty} \xi_j$. We construct a subsolution in a similar way as in Subsection 1.1. For a fixed $\varepsilon \sim 1^-$ we define

$$W_\varepsilon(\xi,\tau) = (h(\tau))^{-\alpha} f_\varepsilon(\xi(h(\tau))^{-\beta}),$$

where $h(\tau)$ is a positive function satisfying,

$$h'(\tau) = -1 + (h(\tau))^{(p-q)\alpha}, \;\; h(0) = m_0 \sim 1^-.$$

It is not difficult to check that so defined $W_\varepsilon(\xi,\tau)$ is a subsolution of (2.1) for every $\tau \geq \tau_{j_0} \gg 1$ which blows up at finite time. On the other hand, by the contradiction assumption, we have for ε and m_0 closed enough to 1 that $\theta(\xi,\tau_{j_0}) \geq W_\varepsilon(\xi,0)$. Then, by using invariance translation in time and the Maximum Principle, one can prove that the $\theta(\xi,\tau)$ also blows-up in finite time and a contradiction follows.

Finally we deal with the upper bound in $[-a,a]$. On the one hand, one has that for every $M \gg f^*(0)$, the symmetric and decreasing stationary profile f_M satisfying $f_M(0) = M$ is a supersolution of (2.1) in $[-a,a]\times\mathbf{R}_+$ and $f_M \to \infty$ in $[-a,a]$ as $M \to \infty$. Then the result follows by applying the Maximum Principle. □

Lemma 2.3. *Let $\theta(\xi,\tau)$ be the solution of* (2.1) *with initial datum $\theta_0(\xi)$ and interfaces $\pm a(\tau)$. There exists a constant $c > 0$ such that*

$$a(\tau) \leq c \quad \textit{for every} \;\; \tau \geq 0.$$

Proof. We first note that the a priori bound in (2.2) can be improved and give that

$$\theta(\xi,\tau) \leq f^*(0) \quad \text{for all } \tau \geq 0 \quad \text{and every } |\xi| \geq a. \tag{2.3}$$

In fact, by repeating the arguments in Lemma 2.2 with no significant modifications it follows that for every $\tau \geq 0$ there exists $\xi_0(\tau) \leq a$ such that $\theta(\xi_0(\tau),\tau) \leq f^*(\xi_0(\tau))$ whence (2.3) follows.

Concerning the interfaces, consider the function $f^*(\xi - A)$ with shifted argument, where $A > 0$ is large enough and such that

$$\theta(A,\tau) < f^*(0) \;\text{ for every } \tau \geq 0, \;\text{ and }\; \theta(\xi,0) \leq f^*(\xi - A) \text{ for } \xi \geq A.$$

Since $f^*(\xi - A)$ is a supersolution of (2.1) in $[A,\infty) \times \mathbf{R}_+$, the result follows by comparison. □

3. The stability S-Theorem

In order to apply the main Stability Theorem in [4], we first remain the main notions, properties and hypotheses and setting the appropriate frame for the application above.

Let X be a complete metric space with the distance function $d(\cdot,\cdot)$. In the application to parabolic blow-up problems, where the Sturm Theorem on zero sets plays a key role, the space $X = C^1(I)$, where $I \subset \mathbf{R}$ is a bounded closed interval, is a natural metric for using Sturmian intersection properties. However, due to a special geometric structure of solutions in this application and the standard regularity theory, we set $X = C(I)$.

We deal with a bounded class L of solutions $\theta \in C([0,\infty) : X)$ of (2.1) defined for every $\tau > 0$ with values in X and by U we denote the corresponding bounded subset of admissible initial data. Actually, the analysis is based on metric-topology arguments applied to families of curves $\{\theta(\tau)\}$ and $\{\theta^*(\tau)\}$, which are formally treated as solutions of the abstract equations (2.1) and (1.6) respectively.

Denote by $\omega(u_0)$ the ω-limit set of an orbit $\{\theta(\tau), \tau > 0\} \subset L$ of equation (2.1) with initial data $\theta_0 \in U$

$$\omega(\theta_0) = \{f \in X : \text{ there exists a sequence } \{\tau_j\} \to \infty \text{ such that } \theta(\tau_j) \to f\},$$

which is assumed to be compact subset of X.

By $\{\varphi^*_\tau\}$ we denote a continuous semigroup induced by the autonomous equation (1.6), globally defined on a bounded subset $\bar{U}^*$ of admissible initial data. The corresponding bounded class of solutions $\theta^*(\tau) = \varphi^*_\tau(\theta^*_0) \in C([0,\infty) : X)$ with $\theta^*_0 \in \bar{U}^*$ is denoted by $\bar{L}^*$. However, we only deal with a "restricted" class $L^* \subseteq \bar{L}^*$ and its corresponding subset U^* of initial data. Both subsets are characterized later on. By $\omega^*(\theta^*_0)$ with $\theta^*_0 \in U^*$, we denote the corresponding ω-limit set. Let f^* be an equilibrium

$$\varphi^*_\tau(f^*) \equiv f^*.$$

Let us present the main hypotheses.

(H1) Compactness of the orbits of (2.1). We assume that, for any data $\theta_0 \in U$, orbit $\{\theta(\tau),\ \tau > 0\}$ is relatively compact in X, and if

$$\theta^s(\tau) \equiv \theta(\tau + s), \quad \tau, s > 0,$$

then the sets $\{\theta^s\}$ are relatively compact in $L^\infty_{\text{loc}}([0,\infty) : X)$.

(H2) Convergence of equations. This means that $B(\cdot,\tau)$ is a small perturbation of $A(\cdot)$ in the sense that given a solution $\theta(\tau) \in L$ of

(2.1), if for a sequence $\{\tau_j\} \to \infty$ the sequence $\{\theta(\tau_j + \tau)\}$ converges in $L^\infty_{\rm loc}([0,\infty) : X)$ as $j \to \infty$ to a function $\theta^*(\tau)$, then $\theta^*(\tau) \in L^*$ is a solution of (1.6).

Next, we introduce key hypotheses including a topological (oriented intersection, in applications) S-relation of partial ordering induced by the non-perturbed evolution driven by equation (1.6).

We present first the hypotheses related to the autonomous equation.

(H3) Ordered invariant one-parametric family from domain of stability. Let $W^s(f^*)$ be the domain of attraction (asymptotic stability) of the equilibrium f^*

$$W^s(f^*) = \{\theta_0^* \in U^* :\ \varphi_\tau^*(\theta_0) \to f^* \ \text{ as } \ \tau \to \infty\}.$$

We assume that there exists a one-parametric continuous set $F^* = \{f_\mu,\ \mu \in (\mu_1 \mu_2)\} \subset W^s(f^*)$ such that $f_{\mu^*} = f^*$ for some $\mu^* \in (\mu_1, \mu_2)$. Each closed subinterval $\{f_\mu,\ \mu_1 < a \le \mu \le b < \mu_2\}$ is relatively compact in X.

The family F^* is one-parametric and we assume that it admits a total ordering denoted by $\preceq$ in the sense that $f_\mu \preceq f_\nu$ (or $f_\nu \succeq f_\mu$) for all $\mu \le \nu$. Moreover, $f_\mu \prec f_\nu$ for all $\mu < \nu$, i. e., $f_\mu \ne f_\nu$.

For any $\mu \in (\mu_1, \mu_2)$, denote

$$F_\mu(\tau) \equiv \varphi_\tau^*(f_\mu) \to f^* \quad \text{as} \quad \tau \to \infty. \tag{3.1}$$

The *invariance* of the family means that for every $\mu \in (\mu_1, \mu_2)$ and $\tau > 0$, $F_\mu(\tau) \in F^*$. i. e., $F_\mu(\tau) \equiv f_{\rho(\tau)}$ for some continuous function $\rho(\tau)$ with $\rho(0) = \mu$. This implies that for the autonomous equation (1.6) we need to specify two orbits $\{F_\pm(\tau),\ \tau \in \mathbf{R}\}$, $F_\pm(\tau) \to f^*$ as $\tau \to \infty$ satisfying $F_-(\tau_1) \prec F_-(\tau_2)$ and $F_+(\tau_1) \succ F_+(\tau_2)$ for any $\tau_1 < \tau_2$ and $F_-(\tau) \prec f^* \prec F_+(\tau)$ for any $\tau \in \mathbf{R}$.

(H4) Asymptotic structural properties and intersection S-relation for (1.6)**.** (i) *Asymptotic transversality of* F^*: there exist $\mu_1 < \nu < \mu < \mu_2$ such that for any $\theta_0^* \in U^*$,

$$f_\nu \prec \theta_0^* \prec f_\mu. \tag{3.2}$$

(ii) *S-relation and S-semigroup.* We assume that the total ordering in F^* can be extended as a binary relation for solutions in L^*. As we have mentioned, in the applications this S-relation is induced by the Sturmian intersection property. It can be classified as a "restricted partial ordering" of solutions $\theta^*(\tau) \in L^*$ with data from U^* and elements of $F^* = \{f_\mu\}$.

The S-relation $\preceq$ satisfies two properties of partial ordering for any $v, v_1, v_2 \in U^*$:

(i) (reflexivity) $v \preceq v$, and

(ii) (antisymmetry) $v_1 \preceq v_2$ and $v_2 \preceq v_1$ imply $v_1 = v_2$.

The constraint of S-relation induced by the subset F^* of particular elements does not satisfy the transitivity property, i. e., $v_1 \preceq f_\mu \preceq v_2$ does not imply that $v_1 \preceq v_2$ for any v_1, $v_2 \in U^*$. Actually, such relation is defined *relative* to the elements of F^* only, and we do not define any partial ordering in U^* or L^*.

Later on we will use $\preceq$ as a standard ordering relation, so that $v \prec f_\mu$ means that $v \preceq f_\mu$ and $v \neq f_\mu$. The S-relation is assumed to be *closed* meaning that for any convergent sequence $\{v_n\} \subset U^*$ there holds

$$v_n \preceq f_\mu \quad \text{and} \quad v_n \to \bar{f} \implies \bar{f} \preceq f_\mu. \tag{3.3}$$

Let us present the main hypothesis on the autonomous evolution: the semigroup φ^*_τ on U^* induced by equation (1.6) preserves the S-relation relative to the set F^* (and is called an S-semigroup) in the following sense: given a $\mu > 0$,

$$v_0 \prec f_\mu \ (v_0 \succ f_\mu) \implies v(\tau) \prec F_\mu(\tau) \ (v(\tau) \succ F_\mu(\tau)) \ \text{ for all } \ \tau > 0. \tag{3.4}$$

Moreover, although is not always necessary for the asymptotic analysis, in main applications the semigroup φ^*_τ is *strong* S-semigroup, i. e., for any given $\mu > 0$, $\tau_0 \geq 0$ and arbitrarily small $\tau > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} &v(\tau_0) \prec F_\mu(\tau_0) \ (v(\tau_0) \succ F_\mu(\tau_0)) \\ &\qquad \implies v(\tau_0 + \tau) \prec F_{\mu-\delta}(\tau_0 + \tau) \ (v(\tau_0 + \tau) \succ F_{\mu+\delta}(\tau_0 + \tau)). \end{aligned}$$

Dynamical systems generating order-preserving semigroups satisfy a number of fundamental properties, and their asymptotic behaviour is well understood, see books [14] ,[20] and the papers [17], [18] and [10]. For the study of perturbed dynamical systems see the book [12] where a list of relevant references in this subject is available.

Our applications to blow-up singularities in reaction-diffusion or reaction-absorption equations deal with dynamical systems admitting no partial ordering between solutions having the same blow-up time. One can see that the S-relation of restricted partial ordering in (H4) which mimics the Sturm Theorem on zero sets for linear parabolic equations, expresses purely geometric intersection properties of curves and cannot be extended to any partial ordering of solutions. Moreover, we note that even intersection comparison theory do not apply when comparing solutions of different equations, difficulty solved with the approach in [4].

We now state the main stability theorem (the S-Theorem, for short).

Theorem 3.1. *Under hypotheses* (H1)–(H4)*, for any* $\theta_0 \in U$ *there hold*

$$\omega(\theta_0) = f^*.$$

3.1. Application of the S-Theorem. Blowing-up behaviour for equation (1.2).

As we mention above, we consider $X = C(I)$ be the space of bounded and continuous compactly supported functions defined on sufficiently large closed symmetric interval $I = [-k, k]$, $k \gg 1$ defined taking into account the analysis of the interfaces of the solutions in Lemma 2.3. We consider the distance function given by the L^∞-norm $\|\cdot\|$. The convenient solution class L is given by

$$L = \{\theta(\tau) \in X \text{ satisfying (2.1) and } c_1 \leq \|\theta(\xi,\tau)\| \leq c_2 \ \forall\, \tau \geq \tau_0\}, \tag{3.5}$$

where $c_1 = c_1(k) > 0$ is a sufficiently small constant and the constant $c_2 = c_2(k) > c_1$ is large enough, both defined taking into account the a priori bounds of the solution stated in Lemmas 2.1 and 2.2. Then, we denote by U the class of such smooth compactly supported initial data θ_0 such that the corresponding solution $\theta \in L$. Finally, the class L^* of solutions $\theta^*(\tau)$ of the unperturbed equation (1.6) is defined in the same way, and by U^* we denote the bounded set of the corresponding initial data θ_0^*.

We next prove via the S-Theorem that $L \subset W^s(f^*)$ and hence, the result stated in Theorem 2.1 follows. We need to check hypotheses (H1)–(H4). The proof is similar to that in [4] for the Porous medium equation. We include it for the convenience of the reader. To begin with, we note that interior regularity results for parabolic equations guarantee the compactness and convergence hypotheses (H1) and (H2). To see that $\omega(\theta_0) \in U^*$ for every $\theta_0 \in U$ we argue as follows. Assume for contradiction that there exists $f = \lim_{j\to\infty} \theta(\tau_j) \in \omega(\theta_0)$ which does not belong to U^*. This means that one of the bounds in the definition of L^* is not true for the corresponding solution $\theta^*(\xi,\tau)$ and some $\tau_1 > \tau_0$. By the convergence $\theta(\xi, \tau + \tau_j) \to \theta^*(\xi,\tau)$ uniformly on $[\tau_0, \tau_1] \times \mathbf{R}$ we deduce that the assumption holds for $\theta(\xi, \tau_0 + \tau_j)$ for any $\tau_j \gg 1$ contradicting the definition of L.

Consider the crucial hypotheses (H3)–(H4) dealing with the class of solutions L^* for the unperturbed equation (1.6) (cf.[3]).
(H3) Let us introduce the family $F^* \subset W^s(f^*)$. We define functions f_μ for every $\mu \in \mathbf{R}$ by translation

$$f_\mu(\xi) = f^*(\xi - \mu) \quad \text{in } \mathbf{R}.$$

The corresponding solution $F_\mu(\tau)$ of the unperturbed parabolic equation (1.6) is given explicitly

$$F_\mu(\tau) = \varphi_t^*(f_\mu) \equiv f^*(\xi - \mu e^{\beta\tau}) \quad \text{in } \mathbf{R} \times \mathbf{R}_+.$$

Since $\beta < 0$, we have that for any $\mu \in \mathbf{R}$, $F_\mu(\tau) \to f^*$ as $\tau \to \infty$ uniformly on compact subsets in $\mathbf{R}$, so that $F^* \subset W^s(f^*)$. Thus,

$$F^* = \{f_\mu,\, \mu \in \mathbf{R}\}$$

is a continuous one-parametric family of functions satisfying $f_{\mu^*} \equiv f^*$ for $\mu^* = 0$ and

$$f_\mu(\cdot) \to 0 = f_\infty \quad \text{as } \mu \to \pm\infty. \tag{3.6}$$

The total ordering $\preceq$ in F^* is straightforward. It characterizes the number and the character of intersection of profiles from F^*. We say that $f_\lambda \prec f_\nu$ if these profiles intersect each other exactly once and the difference $f_\nu(\xi) - f_\lambda(\xi)$ has a change of sign from $-$ to $+$ at the intersection. Then $f_\lambda \preceq f_\nu$ means that either $f_\lambda \prec f_\nu$ or $f_\lambda = f_\nu$.

(H4) (i) **Asymptotic transversality of** F^*. The property (3.2) follows from obvious geometric structure of the family F^*. In particular, it suffices to observe that the interfaces $a_\mu < b_\mu$ of the function $f_\mu(\xi)$ satisfy

$$a_\mu \to \pm\infty, \;\; b_\mu \to \pm\infty \;\text{ as }\; \mu \to \pm\infty.$$

This implies that the asymptotic transversality assumption holds for any initial data with compact support.

(ii) *S-relation and S-semigroup.* We keep the same definition of $\prec$ and $\preceq$ as in (H3) for functions $\theta_0^* \in U^*$. Obviously, both the reflexivity and antisymmetry properties of the S-relation hold. In order to show that the S-relation is closed in the sense of (3.3) and that the S-semigroup property (3.4) holds, we need to prove special intersection properties characterizing solutions in L^*. The proof relies on the existence of a two-parametric family of functions $G^* = \{f_\mu^\lambda\}$ from the local domain of unstability $W^u(f^*)$ of the equilibrium f^*. These functions are defined as follows:

$$f_\mu^\lambda(\xi) = \lambda^{-\alpha} f^*((\xi - \mu)\lambda^{-\beta}), \quad \mu \in \mathbf{R}, \;\; \lambda \in \mathbf{R}.$$

The corresponding solutions of the PDE (1.6) are given explicitly

$$F_\mu^\lambda(\xi, \tau) = (1 - (1 - \lambda)e^\tau)^{-\alpha} f^*((\xi - \mu e^{\beta\tau})(1 - (1 - \lambda)e^\tau)^{-\beta}). \tag{3.7}$$

These rescaled solutions are obtained from the self-similar ones for the original PDE by translations in both independent variables x and t. We have that for any fixed μ, solutions (3.7) stabilize as $\tau \to \infty$ to the trivial one $f_\infty \equiv 0$ if $\lambda > 1$ and blow-up in finite time if $\lambda < 1$. For $\lambda = 1$ we are given the family F^* from the domain of stability of f^*.

Next we introduce the main intersection property of the evolution in L^*. Given a solution $\theta \in L^*$, we denote by $I_\mu(\tau) \equiv \text{Int}(\theta(\tau), f_\mu)$ the number of intersections of $\theta(\xi,\tau)$ and the function $f_\mu(\xi)$.

Lemma 3.1. *Let $\theta(\tau) \in L^*$ and $\theta_0 \notin F^*$. Then*
(i) for any $\mu \in \mathbf{R}$, $I_\mu(\tau) > 0$ for $\tau \geq 0$, and
(ii) $\text{Int}(g, f^*) > 0$ *for any $g \in \omega(\theta_0)$.*

Proof. The proof of (i) is quite similar to the stated for Lemma 2.2 by using special subsolutions of the type (3.7) and we omit the details. For (ii), given a sequence $\{\tau_k\} \to \infty$, we pass to the limit $\tau_k + \tau \to \infty$ so that $\theta(\tau_k + \tau) \to \tilde{\theta}(\tau)$ and repeat the same argument applied to the solution $\tilde{\theta} \in L^*$ of the equation (1.6) with initial data $g \in \omega(\theta_0)$.

Let us show that the S-relation is closed in the sense of (3.3). Indeed, if $\bar{f} \preceq f_\mu$ does not hold, then, by construction, the only possibility is that $\bar{f} \not\equiv f_\mu$ does not intersect f_μ. Since $\bar{f} \in U^*$, this contradicts Lemma 3.1, (i).

Finally, we prove that $\{\varphi_t^*\}$ is an S-preserving semigroup preserving the S-relation. This is a consequence of the Sturm Theorem and Lemma 3.1. Assume that $\theta_0^* \preceq F_\mu(0)$. This means that $I_\mu(0) = +1$, i. e., there exists a unique intersection of the profiles and the difference $F_\mu(\xi, 0) - \theta_0^*(\xi)$ changes sign from $-$ to $+$ at the intersection. For degenerate equations admitting weak solutions, intersections can be a point or an interval. By the Sturm Theorem and Lemma 3.1, it follows that $I_\mu(\tau) \equiv +1$ for every $\tau > 0$ and hence, the same local character of the intersection is preserved in time. Therefore $\theta^*(\tau) \prec F_\mu(\tau)$ for any $\tau > 0$. By repeating the arguments relative to the profile f_ν and the solution $F_\nu(\tau)$, we obtain the opposite estimate $\theta^*(\tau) \succ F_\nu(\tau)$ for all $\tau > 0$. □

Hence, (H1)–(H4) hold and this provide us with the following conclusion.

Theorem 3.2. *Any bounded and compactly supported rescaled solution of the parabolic equation* (2.1), *$\theta(\xi,\tau) \in L$, satisfies*

$$\theta(\tau) \to f^* \text{ as } \tau \to \infty$$

uniformly in $\mathbf{R}$.

As a straightforward consequence of the analysis, we can also conclude that any bounded and compactly supported solution $\theta(\tau)$ of (2.1) stabilize either to the trivial solution or to the profile f^*. The analysis performed shows how the invariance properties of the semigroup φ_t can be used in order to get estimates for a “good" set of initial data, namely symmetric and decreasing initial data. This set provides us with a family $\hat{F}^*$ satisfying (H3) and (H4) with respect to the perturbed equation (2.1). Hence, although this equation is not autonomous, the classical intersection comparison arguments apply to any solution of (2.1) and the solutions with initial data in the family $\hat{F}^*$. We note that now we are comparing solutions of the same equation (2.1). Then, the stability theory in [3] applies without changes, and the convergence result follows for a wide class of initial data. Such an approach allows to avoid rather delicate calculations and estimates for general orbits.

End of the Proof of Theorem 2.1 Theorem 3.2 guarantees that any solution $\theta \in L$ of the perturbed equation (2.1) has the asymptotic self-similar behaviour with the unique rescaled similarity profile f^*. It is clear from the a priori bounds stated in Lemmas 2.1, 2.2 and 2.3, that any solution with symmetric and decreasing initial data belongs to L and therefore the result in Theorem 2.1 follows.

On the other hand, consider an initial datum θ_0 satisfying the previous hypotheses. Define a one-parametric family $\hat{F}^* = \{\hat{f}_\lambda(\xi) = \theta_0(\xi+\lambda),\ \lambda \in \mathbf{R}\}$ belonging to the domain of stability of f^*. Then, following the main lines of the stability analysis of unperturbed equations in [2], Section 4, we see that this family $\hat{F}^*$ satisfies the crucial hypotheses (H3)–(H4) relative to the perturbed equation (2.1) and the sets L and U. Therefore, the stability result [3] applies and proves the convergence of any rescaled solution from L (with blow-up time T) to the unique profile f^*. The proof is completed.

We note that the same stability analysis applies to more general perturbations under the assumptions that the associated semigroup admits translations in both x and t, as well as spatial symmetry and monotonicity properties. It could be also an interesting problem to analyze the behaviour of the global bounded solutions. Our conjecture is that also in this analysis *asymptotic simplification* occurs and the reaction term becomes negligible in the asymptotic behaviour.

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