Український математичний вісник
Том 1 **(2004)**, № 4, 485 – 500

Strongly Local Nonlinear Dirichlet Functionals

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(*Presented by I. V. Skrypnik*)

Abstract. We introduce a new notion of Markov functional and we prove that its properties allows to define a notion of capacity associated with the functional.

2000 MSC. 31C25, 35B65, 35J70.

Key words and phrases. Potential Theory, Dirichlet forms, Elliptic degenerate equations.

1. Introduction

Our goal in this paper is an extension of the results connected with the capacity associated to a (linear) Dirichlet form notion to the case of nonlinear Markov functionals.

For the notion of Dirichlet form we refer to the book of Fukushima-Oshima-Takeda, [13]. In [13] a purely analytical proof of fundamental properties of Dirichlet form is given, this type of proof firstly appeared in [18]; we recall also the papers [4], [7], where an analytical investigation of the properties of the harmonics relative to a a strongly local "Riemannian" Dirichlet forms is carried on. From Beuerling-Deny representation formula, [1], a Dirichlet form is represented as the sum of a strongly local part, of a "killing" part and of a global part. The Beuerling-Deny representation theorem is the fundamental tool allowing to prove that same properties of Dirichlet forms (in particular the Markov property) hold again for energy measures in the strong local (regular) case. Using the above mentioned properties of energy measure it can be proved that for the energy measure of a strongly local (regular) Dirichlet form a chain rule and a Leibnitz rule hold; those properties are the starting point for an investigation of local regularity of harmonics relative to a strongly local (regular) Dirichlet form, see in particular [4], [7]. The Beuerling-Deny

Received 15.12.2003
The Author has been supported by the MURST Research Project 2002012589

ISSN 1810 – 3200.

representation theorem is proved using Riesz theorem on representation of measures, which is an essentially linear tool, then it seems that in the proof of a nonlinear version of this result is difficult.

Previous work on a possible extension of the notion of Dirichlet form to the nonlinear case has been given by Benilan-Picard, [1], and Cipriani-Grillo, [10] [11]. In particular in [1] the relations between maximum principle and Markov property are investigated generalizing to nonlinear monotone case previous results obtained in [13] and [15] in the linear case. In [11] a notion of nonlinear Dirichlet form is given and the relations with a class of nonlinear semigroups (the order preserving contractions semigroups with a cyclically monotone generator) are investigated. The above papers deal with the general global case and are interested in the properties of the corresponding nonlinear semigroup; then the existence of an energy measure is not ensured and there is no proof of chain or Leibnitz rule for the energy measure, when such a measure exists.The first paper concerning local forms was [17], where a suitable chain rule for the energy measure connected with the form is assumed and Sobolev-Morrey inequalities are proved as a consequence of a Poincaré inequality. In [8], [7], [10], [11] some nonlinear forms on fractals are explicitely given and it is proved that the assumptions in [17] hold (see also the more recent papers [20], [14] on the p-Laplacian on the Sierpiski gasket).

In the paper [5] we have introduced the notion of nonlinear strongly local Dirichlet forms and we give our assumptions (in particular the Markov property) directly on the energy measure of the form, whose existence is assumed. We are able to prove in this framework (by purely analytical methods in the line of [18]) suitable Leibnitz and chain rule, which are the starting point for an investigation of local regularity of the harmonics relative to the form and in particular for a proof (under suitable assumptions) of an Harnack type inequality for positive harmonics (we observe that the chain rule proved here is the same assumed in [17] and that an Harnack nequality for positive harmonics in the linear case has been proved in [4], [7]). This last part will be developped in a forthcoming paper.

Here the notion of capacity relative to Markov (global) functional is introduced and we prove that a theory for this capacity can be developed essentially in connection with global assumptions in analogy with the linear case (see [13]). We finally observe that our framework contains the case of the subelliptic p-Laplacian, $p > 1$, related to some vector fields X_i, $i = 1, \dots, m$, which satisfy an Hörmander condition, considered on R^N endowed with the Lebesgue measure as well as the p-Laplacians on fractals considered in [5], [8], [10] (see also [19], [13], where the Authors give a construction of a p-Laplacian on the Sierpiski gasket and investigate the Hölder continuity of harmonics) or the global forms which arise in the theory of Sobolev spaces.

2. The capacity

We consider a locally compact separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that supp[m]$= X$. Let $\Phi : L^p(X,m) \to [0,+\infty]$, $1 < p$, be a l.s.c. convex functional with domain D, i. e. $D = \{v;\ \Phi(v) < +\infty\}$, with $\Phi(0) = 0$. We assume that D is dense in $L^p(X,m)$ and that the following conditions hold:
(H_1) D is a dense linear subspace of $L^p(X,m)$, which can be endowed with a norm $||.||_D$; moreover D has a structure of uniformly convex Banach space with respect to the norm $||.||_D$ and the following estimate holds: there exists $s \geq 0$ such that

$$c_1||v||_D^p \leq \Phi_s(v) = \Phi(v) + s\int_X |v|^p dm \leq c_2||v||_D^p$$

for every $v \in D$, where $c_1,\ c_2$ are positive constants.

(H_2) We denote by D_0 the closure of $D \cap C_0(X)$ in D (with respect to the norm $||.||_D$) and we assume that $D \cap C_0(X)$ is dense in $C_0(X)$ for the uniform convergence on X, moreover we assume that that Φ_s is locally uniformly convex on D_0, i. e. if we have $\lim\limits_{n\to 0} \Phi_s(\frac{u_n+u}{2}) = \Phi_s(u)$ and $\lim\limits_{n\to 0} u_n = u$ weakly in D_0 then $\lim\limits_{n\to 0} u_n = u$ in D_0 (this last assumption is not necessary in the present paper, but simplify some proofs and will be used in forthcoming paper on the theory of capacity with respect to Φ_s).

Remark 2.1. We observe that, since Φ is convex, Φ is l.s.c. also with respect to the weak toplogy of $L^p(X,m)$. We remark that the assumption (H_1) substantially does not allow us to deal with the case $p = 1$ or with sublinear functionals. Moreover from the assumption (H_1) it follows that Φ is continuous on D for the norm $||.||_D$, [19] Ch. 1 Sec. 2 pg. 20, then from (H_2) the restriction of Φ to D_0 coincides with the relaxation of Φ defined on $D \cap C_0(X)$.

(H_3) For every $u, v \in D \cap C_0(X)$ we have $u \vee v \in D \cap C_0(X)$, $u \wedge v \in D \cap C_0(X)$ and

$$\Phi(u \vee v) + \Phi(u \wedge v) \leq \Phi(u) + \Phi(v).$$

Moreover for every $u \in D \cap C_0(X)$ we have that $u \wedge 1 \in D \cap C_0(X)$ and $\Phi(u \wedge 1) \leq \Phi(u)$. We observe that from (H_3), from Remark 2.1 and from the l.s.c. of our functional on $L^p(X,m)$ we have that the above inequalities hold again for every $u,\ v \in D_0$.

Remark 2.2. We observe, [12] pg. 15–19, that given an open set O whose closure is contained in an open relatively compact open set Ω, there exists a function $\tilde{u} \in C_0(X)$ such that $\tilde{u} \geq 1+\epsilon$, $\epsilon > 0$, on O and $\tilde{u} = 0$ on Ω^c, then from (H_2) and (H_3) there exists $u \in D \cap C_0(X)$ with $u \geq 1$ on O. Moreover we observe that, since $C_0(X)$ are dense in $L^p(X,m)$, we have that D_0 is dense in $L^p(X,m)$.

Remark 2.3. We observe that the assumption (H_3) is connected with the assumptions in [11], moreover if Φ has a subdifferential $\partial\Phi$ on D_0 with values in D_0' (the dual space of D_0), then the first inequality in (H_3) can be derived from the T-monotonicity of $\partial\Phi$.

If the functional Φ satisfies the assumptions $(H_1)(H_2)(H_3)$ we call Φ a (global) *Markov functional.*

The assumptions $(H_1)(H_2)$ and (H_3) allow us to define a capacity relative to the functional Φ (and the measure space(X,m)). The capacity of an open set O is defined as

$$\mathrm{cap}_{\Phi,s}(O) = \mathrm{cap}_\Phi(O) = \inf\{\Phi_s(v);\ v \in D_0,\ v \geq 1 \text{ a.e. on } O\}$$

if the set $\{v \in D_0\ , v \geq 1 \text{ a.e. on } O\}$ is not empty and

$$\mathrm{cap}_{\Phi,s}(O) = \mathrm{cap}_\Phi(O) = +\infty$$

if the set $\{v \in D_0\ , v \geq 1 \text{ a.e. on } O\}$ is empty (we drop out the index s from the notation of capacity when it is considered as fixed). Let E be a subset of X we define

$$\mathrm{cap}_\Phi(E) = \inf\{\mathrm{cap}_\Phi(O); O \text{ open set with } E \subset O\}.$$

We observe that from Remark 2.2 it follows that given an open set O whose closure is contained in an open relatively compact open set Ω we have $\mathrm{cap}_\Phi(O) < +\infty$.

Proposition 2.1. *Consider an open set $O \subset X$ such that $\mathrm{cap}_\Phi(O) < +\infty$; there exists $e_O \geq 0$ in $\{v \in D_0,\ v \geq 1 \text{ a.e. on } O\}$, such that*

$$\mathrm{cap}_\Phi(O) = \Phi_s(e_O).$$

We say that $e_O \in D_0$ is a potential of O with respect to Ω. The potential e_O is unique up to sets of measure zero. Moreover if $O_1 \subset O_2$ are open sets in X we have $e_{O_1} \leq e_{O_2}$ a. e.

Proof. Let $M = \mathrm{cap}_\Phi(O)$. Denote $K = \{v \in D_0;\ v \geq\ 1$ a. e. on $O\}$. The set K is closed and convex in $L^p(X,m)$, then K is weakly closed in $L^p(X,m)$. Since Φ_s is l.s.c. on $L^p(X,m)$ for the strong and then for the weak topology, there is a minimum point e_O of Φ_s on K. Moreover we have $\Phi_s(e_O) = \inf\{\Phi_s(v);\ v \in D_0,\ v \geq 1$ a. e. on $O\} = cap_\Phi(O)$. The uniqueness of the potential in $L^p(X,m)$ follows from the strong convexity of Φ_s on $L^p(X,m)$.

The positivity of e_O follows from the inequality $\Phi_s(v \vee 0) \leq \Phi_1(v)$, which is a consequence of (H_3).

For the second and last part of the result we observe that from (H_3) $e_{O_1} \wedge e_{O_2}$ and $e_{O_1} \vee e_{O_2}$ are in D_0. Then again from (H_3) we have

$$\begin{aligned}\Phi_s(e_{O_1} \wedge e_{O_2}) &\leq \Phi_s(e_{O_1}) + \Phi_s(e_{O_2}) - \Phi_s(e_{O_1} \vee e_{O_2})\\ &= \mathrm{cap}_\Phi(O_1) + \mathrm{cap}_\Phi(O_2) - \Phi_s(e_{O_1} \vee e_{O_2}).\end{aligned}$$

Since $e_{O_1} \vee e_{O_2} \geq 1$ a. e. on O_2, we have $\Phi_s(e_{O_1} \vee e_{O_2}) \geq \mathrm{cap}_\Phi(O_2)$; then

$$\Phi_s(e_{O_1} \wedge e_{O_2}) \leq \mathrm{cap}_\Phi(O_1).$$

Since $e_{O_1} \wedge e_{O_2} \geq 1$ a. e. on O_1, we have also

$$\Phi_s(e_{O_1} \wedge e_{O_2}) = \mathrm{cap}_\Phi(O_1)$$

then $e_{O_1} \wedge e_{O_2} = e_{O_1}$ a. e., so $e_{O_1} \leq e_{O_2}$ a. e. □

Remark 2.4. The assumption (H_3) implies also that for an open set O with finite capacity we have $e_0 = 1$ a. e. (and then up to sets of zero capacity, see Proposition 2.3) on O.

We prove that our notion of capacity has all the set theoretic properties of a Choquet capacity:

Proposition 2.2. *The following properties hold:*
(a) *For every subset E of X we have $s\ m(E) \leq \mathrm{cap}_\Phi(E,\Omega)$.*
(b) *Let E_1 and E_2 be subsets of X with $E_1 \subset E_2$ then $\mathrm{cap}_\Phi(E_1) \leq \mathrm{cap}_\Phi(E_2)$ (monotonicity property).*
(c) *Let E_1 and E_2 be subsets of X, then*

$$\mathrm{cap}_\Phi(E_1 \cup E_2) + \mathrm{cap}_\Phi(E_1 \cap E_2) \leq \mathrm{cap}_\Phi(E_1) + \mathrm{cap}_\Phi(E_2).$$

(d) *Let E_n be an increasing sequence of subsets of X then*

$$\mathrm{cap}_\Phi(\cup_{n=1}^{+\infty} E_n) = \lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n).$$

$(e)x$ *Let E_n be a sequence of subsets of X then*

$$\mathrm{cap}_\Phi(\cup_{n=1}^{+\infty} E_n) \leq \sum_{n=1}^{+\infty} \mathrm{cap}_\Phi(E_n,\Omega).$$

Proof. The property (a) holds if $\mathrm{cap}_\Phi(E) = +\infty$ and if $\mathrm{cap}_\Phi(E,\Omega) < +\infty$ easily follows from the inequality

$$\Phi_s(v) \geq s \int_X |v|^p m(dx)$$

for every $v \in D_0$.

Consider now the property (b). Let E_1 and E_2 be open sets. The property holds if at least one of the sets $\{v \in D_0, v \geq 1 \text{ a.e. on } E_1\}$ or $\{v \in D_0, v \geq 1 \text{ a.e. on } E_2\}$ is empty. In the other cases the property follows from the relation

$$\{v \in D_0, v \geq 1 \text{ a.e. on } E_2\} \subset \{v \in D_0, v \geq 1 \text{ a.e. on } E_1\}.$$

In the general case the result follows from the fact that $E_2 \subset O$ with O open set implies $E_1 \subset O$.

Consider the property (c). Let E_1 and E_2 be open sets, we observe that if $u \geq 1$ a.e. on E_1 and $v \geq 1$ a.e. on E_2 then $u \vee v \geq 1$ a.e. on $E_1 \cup E_2$ and $u \wedge v$ a.e. on $E_1 \cap E_2$; then, if the sets $\{v \in D_0, v \geq 1 \text{ a.e. on } E_1\}$ and $\{v \in D_0, v \geq 1 \text{ a.e. on } E_2\}$ are not empty, property (c) follows from the assumption (H_3). Moreover property (c) holds if one of the sets $\{v \in D_0, v \geq 1 \text{ a.e. on } E_1\}$ or $\{v \in D_0, v \geq 1 \text{ a.e. on } E_2\}$ is empty.

Consider now the general case. We have easily that the property holds if $\mathrm{cap}_\Phi(E_1) = +\infty$ or $\mathrm{cap}_\Phi(E_2) = +\infty$. Consider now the case where $\mathrm{cap}_\Phi(E_1)$ and $\mathrm{cap}_\Phi(E_2)$ are both finite. Then for every $\epsilon > 0$ there exists two open sets O_1 and O_2 such that $E_i \subset O_i$ and

$$\mathrm{cap}_\Phi(O_i) \leq \mathrm{cap}_\Phi(E_i) + \epsilon$$

for $i = 1, 2$. We have

$$\begin{aligned}\mathrm{cap}_\Phi(E_1 \cup E_2) + \mathrm{cap}_\Phi(E_1 \cap E_2) &\leq \mathrm{cap}_\Phi(O_1 \cup O_2) + \mathrm{cap}_\Phi(O_1 \cap O_2)\\ &\leq \mathrm{cap}_\Phi(O_1) + \mathrm{cap}_\Phi(O_2) \leq \mathrm{cap}_\Phi(E_1) + \mathrm{cap}_\Phi(E_2) + 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we have the result.

Consider now the property (d). Let E_n be open subsets of Ω and $E = \cup_{n=1}^{+\infty} E_n$; from the monotonicity property we have that

$$\mathrm{cap}_\Phi(E) \geq \lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) \tag{2.1}$$

then, if $\lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) = +\infty$, the property (d) holds.

Let now $\lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) < +\infty$.

Consider at first the case where $\mathrm{cap}_\Phi(E) < +\infty$). There exists e_{E_n} and e_E potentials of E_n and E; from Proposition 2.1 we have e_{E_n} is increasing with respect to n a. e. and that $e_{E_n} \leq e_E$ a. e., then e_{E_n} converges in $L^p(X, m)$ to $\tilde{e}_E$, with

$$\Phi_s(\tilde{e}_E) \leq \lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n)$$

(the limit in the right hand side exists finite since $\mathrm{cap}_\Phi(E_n)$ is increasing and bounded in n).
We observe that $\tilde{e}_E \geq 1$ a. e. on E and $\tilde{e}_E \in D_0$ then

$$\mathrm{cap}_\Phi(E) \leq \Phi_s(\tilde{e}_E) \leq \lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n).$$

Then from (2.1) we have the result. Consider now the case $\mathrm{cap}_\Phi(E) = +\infty$. Assume $\lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) < +\infty$. There exists e_{E_n} potentials of E_n and we have e_{E_n} is increasing with respect to n a. e. The sequence e_{E_n} is bounded in $L^p(X, m)$ then we can assume that e_{E_n} strongly converges in $L^p(X, m)$ to $\tilde{e}$ (we use here the monotone convergence property) and $\Phi(\tilde{e}) \leq \lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) = M < +\infty$, so we have that $\tilde{e}$ is in D_0 and $\tilde{e} \geq 1$ a. e. on E, then $cap_\Phi(E) \leq \Phi_s(\tilde{e}) < +\infty$ We have a contradiction, then the present case can not appear. Consider now the general case. from the monotonicity property we have that

$$\mathrm{cap}_\Phi(E) \geq \lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) \tag{2.2}$$

then, if $\lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) = +\infty$, the property (d) holds.
Let now $\lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) < +\infty$; for every $\epsilon > 0$ there exists an open set O_n such that $E_n \subset O_n$ and $\mathrm{cap}_\Phi(O_n) - \epsilon \leq \mathrm{cap}_\Phi(E_n) \leq \mathrm{cap}_\Phi(O_n)$, moreover we can assume the sequence O_n as increasing. We have

$$\mathrm{cap}_\Phi(E) \leq \mathrm{cap}_\Phi(\cup_n O_n) = \lim_{n\to+\infty} \mathrm{cap}_\Phi(O_n) \leq \lim_{n\to+\infty} \mathrm{cap}_\Phi(E_n) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary we have the result. The property (e) is an easy consequence of properties (c) and (d). □

We give now the notion of quasi-continuity:

Definition 2.1. *Let u be a function defined on X, we say that u is quasi-continuous (with respect to Φ) if for every $\epsilon > 0$ there exists a set $E_\epsilon \subset \Omega$ such that $\mathrm{cap}_\Phi(E_\epsilon) \leq \epsilon$ and the restriction of u to E^c_ϵ is continuous, moreover we can assume E_ϵ open.*

We also have to deal with the notion of quasi-uniform convergence.

Definition 2.2. *Let u_n be a sequence of functions defined on X we say that u_n converges to a function u quasi-uniformly (with respect to Φ) if for every $\epsilon > 0$ there exists a set E_ϵ such that $\mathrm{cap}_\Phi(E_\epsilon) \leq \epsilon$ and the restriction of the sequence u_n to E_ϵ^c converges uniformly to u on E_ϵ^c, moreover we can assume E_ϵ open.*

Proposition 2.3. *Let $u \in D_0$, then there is $\tilde{u}$ quasi-continuous such that $\tilde{u} = u$ a. e., moreover $\tilde{u}$ is uniquely determined up to sets of zero capacity.*

Proof. Let $u \in D_0$ there exists a sequence $u_n \in D_0 \cap C_0(X)$ such that u_n converges in D_0 to u. We can choose u_n such that u_n converges to u a. e. and

$$||u_n - u_{n+1}||_{D_0} \leq 2^{-n}.$$

From (H_3) we have $|u_n - u_{n+1}| \in D_0$, then

$$\mathrm{cap}_\Phi\Big(\big\{|u_n - u_{n+1}| > 2^{-\frac{n}{2p}}\big\}\Big) \leq \Phi_1\left(\frac{|u_n - u_{n+1}|}{2^{-\frac{n}{2p}}}\right)$$
$$\leq c_3 \frac{||u_n - u_{n+1}||_{D_0}^p}{2^{-\frac{n}{2}}} \leq c_3 2^{-n(p-\frac{1}{2})}.$$

Denote

$$O_q = \cup_{n\geq q}\big\{|u_n - u_{n+1}| > 2^{-\frac{n}{2p}}\big\}.$$

From Proposition 2.2 (e) we have

$$\mathrm{cap}_\Phi(O_q) \leq \sum_{n\geq q} 2^{-n(p-\frac{1}{2})} \leq c_4 2^{-q(p-\frac{1}{2})}$$

and

$$|u_m - u_n| \leq c_5 2^{-\frac{n}{2p}}$$

on O_q^c, where $m \geq n$. The sets O_q are decreasing in q. Then u_n converges uniformly to $\tilde{u}$, which coincides with u a.e., on O_q^c, so the restriction of u to O_q^c is continuous. The quasi-continuity of u easily follows. Moreover $\tilde{u}$ is defined on the set $\cup_q O_q^c$, which is such that $\mathrm{cap}_\Phi(X - \cup_q O_q^c) = 0$, moreover $\tilde{u} = u$ a. e. on $\cup_q O_q^c$ and then on X. □

We say that $\tilde{u}$ is the quasi-continuous representative of u and in the following we identify $u \in D_0$ with his quasi-continuous representative considering u as defined up to sets of zero capacity.

Lemma 2.1. *Let u be in D_0. We have*

$$\mathrm{cap}_\Phi(\{u > \epsilon\}) \leq c \frac{||u||_{D_0}^p}{\epsilon^p},$$

where $\epsilon > 0$ is arbitrary and the set $\{u > \epsilon\}$ is defined up to sets of capacity zero.

Proof. Let $u_n \in D \cap C_0(X)$ such that the sequence u_n converges to u in D_0. Let $\epsilon,\ \sigma > 0$ be arbitrary; as in Proposition 2.3 there exists E_σ with $\mathrm{cap}_\Phi(E_\sigma) \leq \sigma$ such that (at least after extraction of subsequences) we have that u_n converges to u uniformly on $X - E_\sigma$. Then we there exists $n_{\epsilon,\sigma}$ such that for $n \geq n_{\epsilon,\sigma}$ we have $|u_n - u| \leq \frac{\epsilon}{2}$ on $X - E_\sigma$ and $||u_n - u||_{D_0} \leq \sigma$. We have

$$\{u > \epsilon\} \subset \big\{u_n > \frac{\epsilon}{2}\big\} \cup E_\sigma,$$

where $n \geq n_{\epsilon,\sigma}$. Then from Proposition 2.2 we obtain

$$\begin{aligned}\mathrm{cap}_\Phi\big(\{u > \epsilon\}\big) &\leq \sigma + \frac{\Phi_s(u_n)}{(\frac{\epsilon}{2})^p}\\ &\leq \sigma + c_2^p\,\frac{||u_n||_{D_0}^p}{(\frac{\epsilon}{2})^p} \leq \sigma + c_2^p\,\frac{\big(||u||_{D_0} + \sigma\big)^p}{(\frac{\epsilon}{2})^p}.\end{aligned}$$

Let $\sigma \to 0$, then

$$\mathrm{cap}_\Phi\big(\{u > \epsilon\}\big) \leq 2^p c_2^p\,\frac{||u||_{D_0}^p}{\epsilon^p}$$

□

Proposition 2.4. *Let u_n be a sequence in D_0 converging in D_0 (with the norm $||.||_{D_0}$) to u; then there exists a subsequence converging quasi-uniformly. Moreover there exists a subsequence converging to u up to a set of zero capacity.*

Proof. Let u be the limit of u_n in D_0. We observe that there exists a subsequence, again denoted by u_n, which converges a. e. to u. Moreover up to extraction of subsequences we may assume

$$||u_n - u_{n+1}||_{D_0} \leq 2^{-n}.$$

We observe that from (H_3) we have $|u_n - u_{n+1}| \in D_0$. From Lemma 2.1 we obtain

$$\mathrm{cap}_\Phi\Big(\big\{|u_n - u_{n+1}| > 2^{-\frac{n}{2p}}\big\}\Big) \leq c_3\frac{||u_n - u_{n+1}||_{D_0}^p}{2^{-\frac{n}{2}}} \leq c_3 2^{-n(p-\frac{1}{2})}.$$

Denote

$$E_q = \cup_{n\geq q}\big\{|u_n - u_{n+1}| > 2^{-\frac{n}{2p}}\big\}.$$

From Proposition 2.2 (e) we have

$$\mathrm{cap}_\Phi(E_q) \leq \sum_{n\geq q} 2^{-n(p-\frac{1}{2})} \leq c_4 2^{-q(p-\frac{1}{2})}$$

and

$$|u_m - u_n| \leq c_5 2^{-\frac{n}{2p}}$$

on E_q^c, where $m \geq n \geq q$. The sets E_q is decreasing in q, then u_n converges uniformly to u on E_q^c. We observe that there is O_q open containing E_q such that

$$\mathrm{cap}_\Phi(O_q) \leq c_4 2^{-q(p-\frac{2}{3})}$$

and we have that u_n converges uniformly to u on O_q^c. □

We say that a property holds quasi-everywhere (q. e.) if the property holds up to sets of zero capacity.

Proposition 2.5. *Let $u \in D_0$ then u is a measurable function with respect to every positive Radon measure ν, which does not charge sets of zero capacity.*

Proof. There exists a sequence $u_n \in D \cap C_0(X)$ converging to u in D_0. The functions u_n are measurable with respect to ν and by Proposition 2.4 u_n converges to u q. e. (at least after extraction of subsequences). Then we obtain the result. □

The following property follows immediately from the definition of capacity.

Proposition 2.6. *The capacities $\mathrm{cap}_{\Phi,s}$, $s > 0$, are mutually equivalent; moreover if*

$$\Phi(u) \geq c \int |u|^p m(dx)$$

for a constant $c > 0$, then $\mathrm{cap}_{\Phi,0}$ is equivalent to every capacity $\mathrm{cap}_{\Phi,s}$ with $s > 0$.

We are now in position to give the definition of quasi-open set:

Definition 2.3. *A set E is quasi-open (for the capacity cap_Φ) if for every $\epsilon > 0$ there exists a set A_ϵ such that $cap_\Phi(A_\epsilon) \leq \epsilon$ and $E \cup A_\epsilon$ is open.*

The following result is an immediate consequence of Proposition 2.3:

Proposition 2.7. *Let $u \in D_0$; the set $E_s = \{u > s\}$ (defined up to sets of zero capacity) is quasi-open.*

3. The potentials and the capacity measure

We fix in this section $s = 1$ but the results hold for any $s > 0$.

Theorem 3.1. *Let E be a set in X then*

$$\mathrm{cap}_\Phi(E) = \inf\{\Phi_1(v);\ v \in D_0 \quad v \geq 1 \ \ q.e.\ on\ E\}.$$

Proof. Denote

$$\mathrm{cap}'_\Phi(E) = \inf\{\Phi_1(v);\ v \in D_0 \quad v \geq 1 \ \ q.e.\ on\ E\}.$$

We prove at first that

$$\mathrm{cap}'_\Phi(E) \leq \mathrm{cap}_\Phi(E). \tag{3.1}$$

If $\mathrm{cap}_\Phi(E) = +\infty$ the relation (3.1) holds. Otherwise for every $\epsilon > 0$ there exists an open set O containing E such that $\mathrm{cap}_\Phi(E)+\epsilon \geq \mathrm{cap}_\Phi(O)$. Let e_O be the potential of O; we have

$$\mathrm{cap}_\Phi(E) + \epsilon \geq \mathrm{cap}_\Phi(O) = \Phi_1(e_O) \tag{3.2}$$

and $e_O \geq 1$ a. e. then q. e. on O. Since $e_O \geq 1$ q. e. on E we have

$$\Phi_1(e_O) \geq \mathrm{cap}'_\Phi(E). \tag{3.3}$$

We now prove that

$$\mathrm{cap}_\Phi(E) \leq \mathrm{cap}'_\Phi(E). \tag{3.4}$$

If $\mathrm{cap}'_\Phi(E) = +\infty$ the relation (3.4) holds. Otherwise for every $\epsilon > 0$ there exists $u \in D_0$ such that $\mathrm{cap}'_\Phi(E) + \epsilon \geq \Phi_1(u)$ and ≥ 1 q. e. on E . Since u is quasi-continuous and $u \geq 1$ q. e. on E, for every $\sigma > 0$ there exists an open set O such that the restriction of u to $X \vee O$ is continuous and $\mathrm{cap}_\Phi(O) \leq \sigma$. Denote

$$U = \{x; u(x) \geq 1 - \epsilon\} \cup O.$$

The set U is open; moreover, since $\mathrm{cap}_\Phi(O) \leq \epsilon$, there exists $w \in D_0$ such that $w \geq 1$ a. e. on O and $\Phi_1(w) \leq 2\sigma$. Let

$$z = \Big(\frac{1}{1-\sigma}u\Big) \vee w.$$

We have $z \geq 1$ q. e. on U then on E and $z \in D_0$; we obtain

$$\mathrm{cap}_\Phi(E) \leq \mathrm{cap}_\Phi(U) \leq \Phi_1(z) \leq \Phi_1\Big(\frac{1}{1-\epsilon}u\Big) + \Phi_1(w).$$

Since $\sigma > 0$ is arbitrary and since Φ_1 is continuous on D_0, we obtain

$$\mathrm{cap}_\Phi(E) \leq \Phi_1(u) \leq \mathrm{cap}'_\Phi(E) + \epsilon.$$

Since $\sigma > 0$ is arbitrary , we obtain (3.4). $\square$

We now prove that the inf in the Theorem 3.1 is really a minimum:

Theorem 3.2. *Let E be a set of finite capacity in X then*

$$\mathrm{cap}_\Phi(E) = \min\{\Phi_1(v);\ v \in D_0 \quad v \geq 1 \ \ q.e. \ on \ E\}.$$

The minimum point $e_E \in D_0$ is unique; we call e_E the potential of E. Assume that Φ has a subdifferential $\partial\Phi : D_0 \to D_0'$, where D_0' denotes the dual of D_0; then e_E is the unique solution of the variational inequality

$$\langle \partial\Phi(u), v-u\rangle + \int_X |u|^{p-1}\mathrm{sign}(u)(v-u)\ m(dx) \geq 0$$

$$\forall\, v \in K,\ \ u \in K$$

where $\langle .,.\rangle$ denotes the duality between D_0' and D_0 and

$$K = \{v \in D_0 \quad v \geq 1 \ \ q.e. \ on \ E\} \subset D_0.$$

Proof. It is enough to prove that the convex set K is closed in D_0. Let v_n be a sequence in K such that $v_n \to v_0$ in D_0. From Proposition 2.4 we have, at least after extraction of subsequences, that $v_n \to v_0$ q. e. so we have also $v_0 \geq 1$ q. e. on E then $v_0 \in K$. □

Lemma 3.1. *Let v be a function in $C_0(X)$ with support K; then there exists a sequence $v_n \in D \cap C_0(X)$ such that the support of every v_n is contained in K and the sequence v_n converges to v uniformly on X.*

Proof. We can assume, without loss of generality v positive. Let O be the set where $v > 0$, then O is open and K is the closure of O. By Remark 2.2. and the assumption (H_3) there exists a positive function v_O such that $v_O \in D \cap C_0(X)$ and $v_O = 1$ on O, $0 \leq v_O \leq 1$ everywhere. From (H_2)there exists a sequence of positive functions v_n in $D \cap C_0(X)$ uniformly convergent to v. We can assume without loss of generality that $|v_n - v| \leq \frac{1}{n}$. Let $\tilde{v}_n = (v_n - \frac{1}{n}v_O)^+$ then $\tilde{v}_n$ has support contained in K, moreover the sequence $\tilde{v}_n$ converges uniformly to v on X. □

Proposition 3.1. *Let g be a positive functional in D_0'; then there exists a positive Radon measure γ (that does not charge sets of zero capacity) such that*

$$\langle g, v\rangle = \int v\ \gamma(dx)$$

for every $v \in D_0$.

Proof. Consider a positive function $v \in D \cap C_0(X)$ with support contained in the compact set K. Let e_K be the potential of K we have $ve_K = v$ then

$$0 \le \langle g, v\rangle \ \le \langle g, ve_K\rangle \ \le \langle g, e_K\rangle M,$$

where $M = \sup\ v$. Then if $v \in D \cap C_0(X)$ (without assumptions on positivity) we have

$$|\langle g, v\rangle| \le 2\langle g, e_K\rangle M$$

Using the previous lemma we have that there exists a measure γ such that

$$\langle g, v\rangle = \int v\ \gamma(dx) \qquad (3.5).$$

for every $v \in D \cap C_0(X)$.

Let O be a relatively compact open set by Remark 2.2 there exists a sequence $v_n \in D \cap C_0(X)$ such that $\mathrm{supp}(v_n) \subset \bar{O}$, $0 \le v_n \le 1$ and $\lim_{n\to+\infty} v_n = 1$ everywhere on O. Let e_O be the potential of O, we have $v_n e_O = v_n$ then

$$\int v_n\ \gamma(dx) = \langle g, v_n\rangle = \langle g, v_n e_O\rangle \le \langle g, e_O\rangle \le c_2||g||_{D_0'}\mathrm{cap}_\Phi(O) \quad (3.6).$$

Passing to the limit in (3.6) as $n \to +\infty$ (by the dominated convergence theorem) we obtain

$$\gamma(O) \le c_2||g||_{D_0'}\mathrm{cap}_\Phi(O) \qquad (3.7).$$

From (3.7) it follows that every set of zero capacity contained in a relatively compact open set has zero γ measure. The space X can be covered by a numerable union of relatively compact open sets; then by (e) Proposition 2.2 we obtain that γ does not charge sets of zero capacity.

Let now $v \in D_0$; there exists a sequence v_n in $D \cap C_0(X)$ such that v_n converges to v in D_0. We have that, at least after extraction of subsequences, v_n converges to v q. e. then γ a. e. By the Fatout lemma we have

$$\int v\ \gamma(dx) \le \liminf_{n\to+\infty} \int v_n\ \gamma(dx) = \liminf_{n\to+\infty}\langle g, v_n\rangle = \langle g, v\rangle.$$

We have also

$$v_n \le v + |v_n - v| \qquad (3.8)$$

q. e., so γ a. e. Then

$$\langle g, v\rangle = \liminf_{n\to+\infty}\langle g, v_n\rangle = \liminf_{n\to+\infty}\int v_n\ \gamma(dx)$$

$$\le \int v\ \gamma(dx) + \liminf_{n\to+\infty} \int |v_n - v|\ \gamma(dx)$$

$$\le \int v\gamma(dx) + \liminf_{n\to+\infty} \langle g, |v_n - v| rangle = \int v\gamma(dx),$$

where we use the previous inequality. So

$$\langle g, v\rangle = \int v\gamma(dx).$$

□

An easy consequence of Proposition is the following result:

Theorem 3.3. *Let the assumptions of Theorem hold and let E be a set of finite capacity and e_E its potential; then there exists a positive Radon measure $\gamma_E \in D_0'$ such that*

$$\partial\Phi(u) + |u|^{p-1}\mathrm{sign}(u) = \gamma_E.$$

The measure γ_E is called the capacitary measure of E and its support is contained in E.

Assume now that

$$\Phi(u) = \int \alpha(u)(dx),$$

where α is a positive Radon measure defined for $u \in D_0$ and assume that for every $u, v \in D_0$ we have

$$\lim_{t\to 0} \frac{\alpha(u+tv) - \alpha(u)}{t} = \mu(u, v)$$

in the weak* topology of $\mathcal{M}$, where μ is linear in v. Then the functional Φ has a Gateaux derivative on D_0 with values in D_0' defined by

$$\langle \Phi'(u), v\rangle = \int \mu(u, v)(dx).$$

Assume also that the following locality assumption holds: let $u = cst$ on $\mathrm{supp}(v)$, $u, v \in D_0$, then $\mu(u, v) = 0$.

Proposition 3.2. *Let the above assumptions hold and that the conditions in Section 2 hold for $s = 0$. Denote by e_E the potential of the set E for the capacity $\mathrm{cap}_{\Phi,0}$; then we have $\gamma_E = 0$ on the interior of E (where γ_E is the capacitary measure of E with respect to the capacity $\mathrm{cap}_{\Phi,0}$).*

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