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# Group Classification of Systems of Non-Linear Reaction-Diffusion Equations

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**Abstract.** The completed group classification of systems of two coupled nonlinear reaction-diffusion equation with general diffusion matrix is carried out. The simple and convenient method for deduction and solution of classifying equations is presented.

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## 1. Introduction

Group classification of differential equations is one of corner stones of group analysis. Such classification specifies the origin of possible applications of powerful group-theoretical tools such as constructing of exact solutions, group generation of solution families starting with known ones, etc. A very important result of group classification consists in a priori description of mathematical models with a desired symmetry (e.g., relativistic invariance).

One of the most impressive results in group classification belongs to S. Lie who had completely classified second order ordinary differential equations [17]. It was Lie also who first presented the group classification of partial differential equations (PDE), namely, he had classified linear equations including two independent variables [18].

Using the classical Lie approach whose excellent presentation was given in [25] it is not difficult to derive determining equations for possible symmetries admitted by equations of interest. Moreover, to describe Lie symmetries for a fixed (even if very complicated) equation is a purely technical problem which is easily solved using special software packages. However, the situation is changing dramatically whenever we try to search

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for Lie symmetries for an equation including an arbitrary element which is not a priory specified, i.e., when we are interested in group classification of an entire class of differential equations.

The main problem of group classification of a substantially extended class of partial differential equations (PDEs) consists in effective solving of determining equations for coefficients of generators of symmetry group. In general the determining equations are rather complicated systems whose variables are not necessarily separable.

A nice result in group classification of PDEs belongs to Dorodnitsyn [26] who had classified nonlinear (but quasi linear) heat equations

$$u_t - u_{xx} = f(u) \tag{1.1}$$

where $f$ is an arbitrary function of the dependent variable $u$, the subscripts denote derivations w.r.t. the corresponding variables, i.e., $u_t = \partial u/\partial t$ and $u_{xx} = \partial^2 u/\partial x^2$. Moreover, in paper [26] more general equations $u_t - (Ku_x)_x = f(u)$ were classified. The related determining equations appears to be easily integrable, which made it possible to specify all non-equivalent non-linearities $f$ (which are power, logarithmic and exponential ones) which correspond to different symmetries of equation (1.1). The non-classical (conditional) symmetries of (1.1) were described by Fushchych and Serov [10] and Clarkson and Mansfield [6].

The results of group classification of equations (1.1) play an important role in constructing of their exact solutions and qualitative analysis of the nonlinear heat equation, refer, e.g. to [28].

In the present paper we perform the group classification of systems of the nonlinear reaction-diffusion equations

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \Delta(a_{11}u_1 + a_{12}u_2) = f^1(u_1, u_2), \\ \frac{\partial u_2}{\partial t} - \Delta(a_{21}u_1 + a_{22}u_2) = f^2(u_1, u_2) \end{aligned} \tag{1.2}$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ are function of $t, x_1, x_2, \dots, x_m$, symbols $a_{11}, a_{12}, a_{21}$, $a_{22}$ denote real constants and $\Delta$ is the Laplace operator in $R^m$. We shall write (1.2) also in the matrix form:

$$\frac{\partial u}{\partial t} - A\Delta u = f \tag{1.3}$$

where $A$ is a matrix whose elements are $a_{11}, \dots a_{22}$ and $f = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}$.

Mathematical models based on equations (1.2) are widely used in mathematical physics and mathematical biology. Some of these models are discussed in [23] and in Section 12 of the present paper, the entire collection of such models is presented in [20]. Thus the symmetry analysis of equations (1.2) has a large application value and can be used, e.g., to construct exact solutions for a very extended class of physical and biological systems. The comprehensive group analysis of systems (1.2) is also a nice "internal" problem of the Lie theory which admits exact general solution for the case of *arbitrary* number of independent variables $x_1, x_2, \ldots, x_m$.

Symmetries of equation (1.3) for the case of a *diagonal* (and invertible) matrix $A$ were investigated by Yu. A. Danilov [7]. Unfortunately, the results presented in [7] and cited in the handbook [14] are neither complete nor correct. We discuss these results in detail in Section 12.

Symmetry classification of equations (1.3) with a diagonal diffusion matrix was presented in paper [3], then some results missing in [3] were added in Addendum [4] and paper [5]. However, we shall demonstrate that the results given in [3]–[5] are still incomplete, and add the list of non-equivalent equations given in these papers.

We notice that symmetries of equations (1.3) with a diagonal diffusion matrix was partly described in paper [15] were symmetries of more general class of diffusion equations where studied.

Equations (1.3) with *arbitrary* invertible matrix $A$ were investigated in paper [23], the related results were announced in [24]. Unfortunately, mainly due to typographical errors made during publishing procedure, presentation of classification results in [23] was not satisfactory [1].

In the present paper we give the completed group classification of coupled reaction-diffusion equations (1.3) with an arbitrary diffusion matrixes $A$. Moreover, we present a straightforward and easily verified procedure of solution of the determining equations which guarantees the completeness of the obtained results. We also indicate clearly the equivalence relations used in the classification procedure. In addition, we extend the results obtained in [23] to the case of *non-invertible* matrix $A$.

The additional aim of this paper is to present a rather straightforward and conventional algorithm for investigation of symmetries of a class of partial differential equations which includes (1.3) as a particular case. We will show that the classical Lie approach (refer, for example, [11],[25]) when applied to systems (1.3) admits a rather simple formulation which can be used even by such investigators which are not experts in group analysis of differential equations. Furthermore the algorithm may be

[1] The tables presenting the results of group classification have been deformed and cut off. It is necessary to stress that it was the authors fault, one of whom signed the paper proofs without careful reading.

used to search for conditional symmetries of (1.3) [23] (for definition of conditional symmetries see [12]).

There exist two non-equivalent $2\times 2$ matrices with zero determinant, namely, the diagonal matrix with the only non-zero element and the Jordan cell. We will consider the following generalized versions of the related equation (1.2)

$$\begin{aligned} \partial_t u_1 - \Delta u_1 &= f^1(u_1, u_2), \\ \partial_t u_2 - p_\mu \partial_\mu u_1 &= f^2(u_1, u_2) \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \partial_t u_1 - p_\mu \partial_\mu u_2 &= f^1(u_1, u_2), \\ \partial_t u_2 - \Delta u_1 &= f^2(u_1, u_2). \end{aligned} \tag{1.5}$$

Here $p_\mu$ are arbitrary constants and summation is imposed over repeating $\mu = 1, 2, \cdots, m$ . Moreover, without loss of generality we set

$$p_1 = p_2 = \cdots = p_{m-1} = 0, \; p_m = p. \tag{1.6}$$

In the case $p \equiv 0$ equations (1.4) and (1.5) are nothing but particular cases of (1.2), which include such popular models of mathematical biology as the FitzHung-Naguno [9] and Rinzel-Keller [27] ones. In addition, (1.5) can serve as a potential equation for the nonlinear D'alembert equation.

The determining equations for symmetries of equations (1.2) are rather complicated systems of PDE including two arbitrary elements, i.e., unknown functions $f^1$ and $f^2$. To handle them we use the approach developed in paper [29] , whose main idea is to make *a priori* classification of realizations of the related Lie algebras. In fact this method has roots in works of S. Lie who used his knowledge of vector field representations of Lie algebras in space of two variables to classify second order ordinary equations [17]. In the case of partial differential equations we have no hope to classify all related realizations of vector fields . However, for some fixed classes of PDEs it appears to be possible to make this classification restricting ourselves to realizations which are compatible with equations of interest [29].

We notice that analogous technique was used earlier [13] to classify the nonlinear Schrödinger equations with cubic nonlinearity and variable coefficients.

In Section 2 we present the general equivalence transformations for equations (1.3) which are valid for arbitrary nonlinearities $f^1$ and $f^2$, and give the list of additional equivalence transformations which are valid for some fixed nonlinearities.

In Section 3 the simplified algorithm for investigation of symmetries of systems of reaction-diffusion equation is presented.

In Section 4 we deduce determining equations for symmetries admitted by equations (1.3) and specify the general form of the related group generators.

In Section 5 we present the kernel of symmetry group for equations (1.3) and give definitions of main and extended symmetries.

In Sections 6–8 the results of group classification of equations (1.4) and (1.5) are presented. Equations (1.3) with invertible diffusion matrix are classified in Sections 9 and 10, the case of nilpotent diffusion matrix is studied in Section 11.

In Section 12 we discuss the results of group classification and present some important model equations which appear to be particular subjects of our analysis. The Appendix includes *a priori* classification of realizations of low dimension Lie algebras which are used in the main text to solve the determining equations.

## 2. Equivalence Transformations

The problem of group classification of equations (1.2)–(1.5) will be solved up to equivalence transformations.

We say the equation

$$\tilde{u}_t - \tilde{A}\Delta\tilde{u} = \tilde{f}(\tilde{u}) \tag{2.1}$$

be equivalent to (1.3) if there exist an invertible transformation $u \to \tilde{u} = G(u,t,x)$, $t \to \tilde{t} = T(t,x,u)$, $x \to \tilde{x} = X(t,x,u)$ and $f \to \tilde{f} = F(u,t,x,f)$ which connects (1.3) with (2.1). In other words the equivalence transformations should keep the general form of equation (1.3) but can change concrete realizations of matrix $A$ and non-linear terms $f^1$ and $f^2$.

Let us note that there are six ad hoc non-equivalent classes of equations (1.3) corresponding to the following forms of matrices $A$

$$\begin{gathered} I.\ A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad I^*.\ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad II.\ A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}, \\ III.\ A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad IV.\ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad V.\ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{gathered} \tag{2.2}$$

where $a$ is an arbitrary parameter. Indeed any $2\times 2$ matrix $A$ can be reduced to one of the forms (2.2) using linear transformations of dependent variables and scaling independent variables in (1.3). For matrices $I$ and $III$ it is possible to restrict ourselves to the cases $a \neq 0, 1$ and $a \neq 0$

respectively, but we prefer to reserve the possibility to treat version $I^*$ as a particular case of versions $I$ and $III$.

The group of equivalence transformations for equation (1.3) can be found using the classical Lie approach and treating $f^1$ and $f^2$ as additional dependent variables. In addition to the obvious symmetry transformations

$$t \to t' = t + a, \quad x_\mu \to x'_\mu = R_{\mu\nu} x_\nu + b_\mu \tag{2.3}$$

where $a, b_\mu$ and $R_{\mu\nu}$ are arbitrary parameters satisfying $R_{\mu\nu} R_{\mu\lambda} = \delta_{\mu\lambda}$, this group includes the following transformations

$$\begin{aligned} &u_a \to K^{ab} u_b + b_a, \quad f^a \to \lambda^2 K^{ab} f^b, \\ &t \to \lambda^{-2} t, \quad x_a \to \lambda^{-1} x_a \end{aligned} \tag{2.4}$$

where $K^{ab}$ are elements of an invertible constant matrix $K$ commuting with $A$, $\lambda \neq 0$ and $b_a$ are arbitrary constants.

Let us specify the form of matrices $K$. By definition, $K$ commutes with $A$, so for the versions I–V present in (2.2) we have

$$I^*: \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad K_{11}K_{22} - K_{21}K_{12} \neq 0; \tag{2.5}$$

$$I, IV: \quad K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \quad K_1 K_2 \neq 0; \tag{2.6}$$

$$II: \quad K = \begin{pmatrix} K_1 & -K_2 \\ K_2 & K_1 \end{pmatrix}, \quad K_1^2 + K_2^2 \neq 0; \tag{2.7}$$

$$III, V: \quad K = \begin{pmatrix} K_1 & 0 \\ K_2 & K_1 \end{pmatrix}, \quad K_1 \neq 0. \tag{2.8}$$

In addition, for the Case I there is one more transformation (2.4) with

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda^2 = a. \tag{2.9}$$

Such transformations reduce to the change $a \to \frac{1}{a}$ in the related matrix $A$, i.e., to scaling the parameter $a$.

Equivalence transformations (2.4) are valid also for equations (1.4) and (1.5) . The related matrices $K$ are given in (2.6) and (2.8).

It is possible to show that there is no more extended equivalence relations valid for arbitrary nonlinearities $f^1$ and $f^2$. However, if functions $f^1, f^2$ are fixed, the class of equivalence transformations is more extended. In addition to transformations (2.4) it includes symmetry transformations which does not change the form of equation (1.3). Moreover, for some classes of functions $f^1, f^2$ equation (1.3) admits additional equivalence transformations (AET). The corresponding set of equivalence transformations for equation (1.3) can be found using the classical Lie approach and treating $f^1$ and $f^2$ as additional dependent variables constrained by the relations specifying the dependence of $f^1, f^2$ on $u_1$ and $u_2$.

In spite of the fact that we search for AET *after* description of symmetries of equations (1.3) and specification of functions $f^1, f^2$, for convenience we present the list of the additional equivalence transformations in the following formulae:

$$
\begin{aligned}
&1. \quad u_1 \to \exp(\omega t)u_1, \quad u_2 \to \exp(\rho t)u_2,\\
&2. \quad u_1 \to u_1 + \omega t + \lambda_a x_a + \mu x^2, \; u_2 \to u_2,\\
&3. \quad u_1 \to u_1, \; u_2 \to u_2 + \rho t + \lambda_a x_a + \mu x^2,\\
&4. \quad u_1 \to u_1 + \rho t, \; u_2 \to u_2 \exp(\rho t),\\
&5. \quad u_1 \to \exp(\omega t)u_1, \; u_2 \to u_2 + \omega t,\\
&6. \quad u_1 \to u_1, \; u_2 \to u_2 + \rho t u_1,\\
&7. \quad u_1 \to \exp(\omega t)u_1, \; u_2 \to u_2 + \omega \frac{t^2}{2},\\
&8. \quad u_1 \to \exp(\omega t)u_1, \; u_2 \to u_2 + \kappa t u_1 + \rho \frac{t^2}{2},\\
&9. \quad u_1 \to u_1, \; u_2 \to u_2 - \rho t u_1 + \rho\lambda \frac{t^2}{2},\\
&10. \quad u_1 \to \exp(\rho t)u_1, \; u_2 \to u_2 - \kappa\rho t,\\
&11. \quad u_1 \to \exp(\rho t)u_1, \; u_2 \to \exp(\rho t)\Big(u_2 + \varepsilon\rho \frac{t^2}{2} u_1\Big),\\
&12. \quad u_1 \to u_1 + \rho t + \nu x^2, \; u_2 \to u_2 - \rho t - \nu x^2,\\
&13. \quad u_1 \to u_1 + \rho t, \; u_2 \to e^{-\frac{\rho}{\nu}t} u_2,\\
&14. \quad u_1 \to u_1 + \rho t, \; u_2 \to u_2 + \rho t u_1 + \rho \frac{t^2}{2},\\
&15. \quad u_1 \to u_1 \cos\omega t - u_2 \sin\omega t, \; u_2 \to u_2 \cos\omega t + u_1 \sin\omega t,\\
&16. \quad u_1 \to \exp(\omega t)u_1, \; u_2 \to \exp(\omega t)(u_2 - \omega t u_1)\\
&17. \quad \text{Transformations (11.2) valid for equations with matrix } A\\
&\qquad \text{of type } V \text{ only.}
\end{aligned}
\tag{2.10}
$$

Here the Greek letters denote parameters which are either arbitrary or specified in the tables presented below. We stress once more that in contrast with (2.4), equivalence transformations (2.10) are admitted by some particular equations (1.3), which will be specified in the following.

## 3. An Algorithm for Description of Symmetries for the Systems (1.3)–(1.5)

Let us investigate Lie symmetries of systems (1.3)–(1.5), i.e., find all continuous groups of transformations for $u, t, x$ which keep these equations invariant. In contrast with the equivalence transformations, symmetry transformations do not change functions $f^1$ and $f^2$.

In as much as any term in (1.3) does not depend on $t$ and $x$ explicitly, this equation *with arbitrary functions* $f^1$ *and* $f^2$ admits obvious symmetry w.r.t. translations of all independent variables and rotations of spatial variables present in (2.3). For equations (1.4) and (1.5) such symmetries also have the form (2.3) where the indices of $R_{\mu\nu}$ runs over the values $1, 2, \ldots, m-1$.

To find all Lie symmetries we require form-invariance of the systems of reaction diffusion equations with respect to the one-parameter groups of transformations:

$$t \to t'(t, x, \varepsilon), \qquad x \to x'(t, x, \varepsilon), \qquad u \to u'(t', x', \varepsilon), \tag{3.1}$$

where $\varepsilon$ is a group parameter. In other words, we require that $u'(t', x', \varepsilon)$ satisfies the same equation as $u(t, x)$:

$$L'u' = f(u'), \tag{3.2}$$

where $L$ are the linear differential expressions involved into equations (1.3)–(1.5), i.e.,

$$L = \frac{\partial}{\partial t} - A\sum_i \frac{\partial_i^2}{\partial x_i^2} - B\frac{\partial}{\partial x_m}, \quad L' = \frac{\partial}{\partial t'} - A\sum_i \frac{\partial_i^2}{\partial x_i'^2} - B\frac{\partial}{\partial x_m'}.$$

Here $B$ is the zero matrix for equation (1.3), $B = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}$ for equations (1.4) and $B = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$ for equation (1.5).

Starting with the infinitesimal transformations:

$$\begin{aligned} &t \to t' = t + \Delta t = t + \varepsilon\eta, \qquad x_a \to x_a' = x_a + \Delta x_a = x_a + \varepsilon\xi^a, \\ &u_a \to u_a' = u_a + \Delta u_a = u_a + \varepsilon\pi_a \end{aligned} \tag{3.3}$$

we obtain the following representation for the operator $L'$:

$$L' = \left[1 + \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] L \left[1 - \varepsilon\left(\eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a}\right)\right] + O(\varepsilon^2). \tag{3.4}$$

Using the Lie algorithm one can find find the determining equations for the functions $\eta$, $\xi_a$ and $\pi_a$ which specify the generator $X$ of the symmetry group:

$$X = \eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a} - \pi^b\frac{\partial}{\partial u_b} \tag{3.5}$$

where a summation from 1 to $m$ and from 1 to $n$ is assumed over repeated indices $a$ and $b$ respectively. We will obtain these determining equations directly.

First we notice that without loss of generality it is possible to restrict ourselves to such functions $\eta, \xi^a, \pi^a$ which satisfy the conditions

$$\frac{\partial\eta}{\partial u_a} = 0, \quad \frac{\partial\xi^a}{\partial u_b} = 0, \quad \frac{\partial^2\pi^a}{\partial u_c\partial u_b} = 0. \tag{3.6}$$

This is nothing but a consequence of results of paper [2] were PDE are classified whose symmetries satisfy (3.6). These results admit a straightforward generalization to the case of systems (2.2) with invertible matrix $A$.

Substituting (3.3), (3.4) into (3.2), using (1.3)–(1.5) and neglecting the terms of order $\varepsilon^2$ we find that:

$$[Q, L]\, u - L\omega = \pi f + \frac{\partial f}{\partial u_a}\left(-\pi^{ab}u_b - \omega^a\right), \tag{3.7}$$

where

$$Q = \eta\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x_a} + \pi$$

$[Q, L] = QL - LQ$ is a commutator of operators $Q$ and $L$ and $\pi$ is a matrix whose elements are $\pi^{ab}$, so that [23] $\pi^a = \pi^{ab}u_b + \omega^a$, with $\pi^{ab}$ and $\omega^a$ being functions of independent variables $t, x$.

Equation (3.7) is compatible with (1.3)–(1.5) and does not impose new nontrivial conditions for $u$ if the commutator $[Q, L]$ admits the representation:

$$[Q, L] = \Lambda L + \varphi \tag{3.8}$$

where $\Lambda$ and $\varphi$ are $2 \times 2$ matrices dependent on $t, x$.

Substituting (3.8) into (3.7) the following classifying equations for $f$ are obtained:

$$\left(\Lambda^{kb} + \pi^{kb}\right) f^b + \varphi^{kb}u^b + (L\omega)^k = (\omega^a + \pi^{ab}u_b)\frac{\partial f^k}{\partial u^a}. \tag{3.9}$$

Thus, to find all non-linearities $f^k$ generating Lie symmetries for equations (1.3)–(1.5) it is necessary to solve operator equations (3.8) and find the general form of matrices $\Lambda$, $\pi$, $\varphi$ and functions $\eta$, $\xi$. In the second step we find the non-linearities $f^a$ solving the system (3.9) with its known coefficients.

We stress that the described procedure of group classification of equations (1.3)–(1.5) is equivalent to the standard Lie algorithm but is more straightforward. In addition, it is rather convenient, and till an appropriate moment all equations (1.3) with non-singular matrices $A$ can be analyzed in a parallel way.

## 4. Determining Equations

Evaluating the commutator in (3.8) and equating the coefficients for linearly independent differential operators we obtain the determining equations:

$$\left(\frac{\partial\xi^a}{\partial x_b}+\frac{\partial\xi^b}{\partial x_a}\right)A=-\delta_{ab}(\Lambda A+[A,\pi]),\quad \frac{\partial^2\eta}{\partial t\partial x_a}=0,\quad \frac{\partial\eta}{\partial t}=\Lambda, \tag{4.1}$$

$$\frac{\partial\xi^a}{\partial t}-2\frac{\partial}{\partial x_a}A\pi-\Delta A\xi^a=0,\qquad \varphi=\frac{\partial\pi}{\partial t}-\Delta A\pi \tag{4.2}$$

where $\delta_{ab}$ is the Kronecker symbol.

The general expressions for coefficient functions $\eta,\xi^a$ and $\pi$ of symmetry $X$ (3.5) can be obtained evaluating determining equations (4.1) and (4.2). We shall not reproduce this procedure here but present the general form of the related generator (3.5) found in [23]:

$$\begin{aligned}X=\lambda K+\sigma_\alpha G_\alpha+\omega_\alpha\widehat{G}_\alpha+\mu D-2(C^{ab}u_b+B^a)\frac{\partial}{\partial u_a}&\\ +\Psi^{\mu\nu}x_\mu\partial_\nu+\nu\partial_t+\rho_\mu\partial_\mu&\end{aligned} \tag{4.3}$$

where the Greek letters denote arbitrary constants moreover $\Psi^{\mu\nu}=-\Psi^{\nu\mu}$, $B^a$ are functions of $t,x$, and $C^{ab}$ are functions of $t$ satisfying

$$C^{ab}A^{bk}-A^{ab}C^{bk}=0 \tag{4.4}$$

and

$$\begin{aligned}
K&=2t\Big(t\frac{\partial}{\partial t}+x_\mu\frac{\partial}{\partial x_\mu}\Big)-\frac{x^2}{2}(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a}-tm\Big(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}\Big),\\
G_\alpha&=t\partial_\alpha+\frac{1}{2}x_\alpha(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a},\\
\widehat{G}_\alpha&=e^{\gamma t}\Big(\partial_\alpha+\frac{1}{2}\gamma x_\alpha(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a}\Big),\\
D&=2t\frac{\partial}{\partial t}+x_\mu\frac{\partial}{\partial x_\mu}.
\end{aligned} \tag{4.5}$$

If $a = 0$ then the related generator $X$ again has the form (4.3) where however $\lambda = \sigma_\mu = \omega_\mu = C^2 = 0$ and $B^2$ is a function of $t, x$ and $u$.

Formula (4.3) presents a symmetry operator for equation (1.2) iff the related classifying equations (3.9) for $f^1$ and $f^2$ are satisfied, i.e.,

$$
\begin{aligned}
(\lambda t(m+4)+\mu)f^a + \Big(\frac{1}{2}\lambda x^2 + \sigma_\mu x_\mu + \gamma e^{\gamma t}\omega_\mu x_\mu\Big)(A^{-1})^{ab} f^b \\
+ C^{ab} f^b + C^{ab}_t u_b + B^a_t - \Delta A^{ab} B^b \\
= \Big(B^s + C^{sb} u_b + \lambda t m u_s + \Big(\frac{1}{2}\lambda x^2 + \sigma_\mu x_\mu \\
+ \gamma e^{\gamma t}\omega_\mu x_\mu\Big)(A^{-1})^{sk} u_k\Big)\frac{\partial}{\partial u_s} f^a. \quad (4.6)
\end{aligned}
$$

Thus the group classification of equations (1.3) with a non-singular matrix $A$ reduces to solving equation (4.6) where $\lambda, \mu, \sigma_\nu, \omega_\nu, \gamma$ are arbitrary parameters, $B^a$ and $C^{ab}$ are functions of $(t, x)$ and $t$ respectively. Moreover, matrix $C$ with elements $C^{ab}$ should commute with $A$.

We notice that relations (4.3)–(4.6) are valid for group classification of systems (1.3) of coupled reaction-diffusion equations including *arbitrary* number $n$ of dependent variables $u = (u_1, u_2, \dots u_n)$ provided the related $n \times n$ matrix $A$ be invertible [23]. In this case indices $a, b, s, k$ in (4.3)–(4.6) run over the values $1, 2 \dots n$.

Consider now equation (1.4) and the related symmetry operator (3.5). The determining equations for $\eta$, $\xi^\mu$ and $\pi^a$ are easily obtained using (3.8), (3.9) and have the following form

$$
\begin{aligned}
&\eta_{tt} = \eta_{x_\mu} = \frac{\partial \eta}{\partial u_a} = 0, \quad \xi^\mu_t = \frac{\partial \xi^\mu}{\partial u_a} = 0,\\
&\frac{\partial^2 \pi^a}{\partial u_b \partial u_c} = 0, \quad \frac{\partial \pi^a_{x_\mu}}{\partial u_b} = 0, \quad \frac{\partial \pi^1}{\partial u_2} = \frac{\partial \pi^2}{\partial u_1} = 0;\\
&\frac{\partial \pi^1}{\partial u_1} - \frac{\partial \pi^2}{\partial u_2} = \frac{1}{2}\eta_t, \quad \text{if} \quad p \neq 0;\\
&\xi^\mu_{x_\nu} + \xi^\nu_{x_\mu} = -\delta^{\mu\nu}\eta_t, \ \mu \neq m
\end{aligned}
\qquad (4.7)
$$

where subscripts denote derivatives w.r.t. the corresponding independent variable, i.e., $\eta_t = \frac{\partial \eta}{\partial t}$, $\xi^\mu_{x_\nu} = \frac{\partial \xi^\mu}{\partial x_\nu}$, etc.

Integrating system (4.7) we obtain the general form of operator $X$:

$$
X = \nu\partial_t + \rho_\nu\partial_\nu + \Psi^{\mu\nu}\partial_\nu x_\mu + \mu D - 2B^a\frac{\partial}{\partial u_a} - 2Fu_1\frac{\partial}{\partial u_1} - 2Gu_2\frac{\partial}{\partial u_2}; \quad (4.8)
$$

$$
\mu = 2(F - G) \ \text{ if } p \neq 0 \qquad (4.9)
$$

where $B^a$ are functions of $(t,x)$, $F$ and $G$ are functions of $t$ and summation over the indices $\mu,\nu$ is assumed with $\mu,\nu = 1,2,\cdots,n-1$.

The classifying equations (3.9) reduce to the following ones

$$\begin{aligned}
&(\mu+F)f^1 + F_t u_1 + (\partial_t - \Delta)B^1 \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + Gu_2\frac{\partial}{\partial u_2}\Big)f^1, \\
&(\mu+G)f^2 + G_t u_2 + B^2_t - pB^2_{x_m} \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + Gu_2\frac{\partial}{\partial u_2}\Big)f^2.
\end{aligned} \tag{4.10}$$

Relations (4.8)–(4.10) are valid for $p \neq 0$ and $p = 0$ as well (in the last case condition (4.9) should be omitted). Solving (4.10) we specify both the coefficients of infinitesimal operator (4.8) and the related nonlinearities $f^1$ and $f^2$.

For equations (1.5) we obtain in analogous way that generator (3.5) reduces to

$$X = \mu\left(3t\partial_t + 2x_\nu\partial_\nu - u_2\frac{\partial}{\partial u_2}\right) - F\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right) - B^a\frac{\partial}{\partial u_a} \tag{4.11}$$

while the classifying equations are

$$\begin{aligned}
&(3\mu+F)f^1 + F_t u_1 + B^1_t - pB^2_{x_m} \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + (F+\mu)u_2\frac{\partial}{\partial u_2}\Big)f^1, \\
&(4\mu+F)f^2 + F_t u_2 + B^2_t - \Delta B^1 \\
&\qquad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + Fu_1\frac{\partial}{\partial u_1} + (F+\mu)u_2\frac{\partial}{\partial u_2}\Big)f^2
\end{aligned} \tag{4.12}$$

where $F$ and $B^1$, $B^2$ are unknown functions of $t$ and $t,x$ respectively.

The determining equations for symmetries of equation (1.5) with $p = 0$ are qualitatively different for the cases, when the number $m$ of spatial variables $x_1, x_2, \ldots x_m$ is $m = 1$, $m = 2$ and $m > 2$. The related generator (3.5) has the form

$$\begin{aligned}
X = \alpha D &+ \left(\int (N-M)dt\right)\frac{\partial}{\partial t} + 2mH_a\frac{\partial}{\partial x_a} \\
&- \left(N + (m-2)\frac{\partial H_a}{\partial x_a}\right)u_1\frac{\partial}{\partial u_1} - \left(M + (m+2)\frac{\partial H_a}{\partial x_a}\right)u_2\frac{\partial}{\partial u_2} \\
&\qquad - B^1\frac{\partial}{\partial u_1} - B^2\frac{\partial}{\partial u_2} - B^3 u_1\frac{\partial}{\partial u_2}
\end{aligned} \tag{4.13}$$

where summation from 1 to $m$ is imposed over repeating indices, the Greek letters denote arbitrary parameters, $M, N$ are functions of $t$, $B^1, B^2$ are functions of $t, x$, $B^3$ is a function of $t, x, u_1$ and $H_a = 2\lambda_b x_b x_a - x^2\lambda_a$ for $m > 2$ . For $m = 2$ $H_a$ are arbitrary functions satisfying the Caushy-Rieman conditions $\frac{\partial H_1}{\partial x_1} = \frac{\partial H_2}{\partial x_2}$, $\frac{\partial H_1}{\partial x_2} = -\frac{\partial H_2}{\partial x_1}$; for $m = 1$ $H_1$ is a function of $x$ and the sums with respect to $a$ in (4.13) are degenerated to one terms.

The corresponding classifying equations have the form

$$\begin{aligned}
&\Big(\frac{\alpha}{2} + 2N - M + (m-2)\frac{\partial H_a}{\partial x_a}\Big)f^1 + N_t u_1 + B^1_t \\
&\quad = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} + B^3 u_1\frac{\partial}{\partial u_2} + \Big(N + (m-2)\frac{\partial H_a}{\partial x_a}\Big)u_1\frac{\partial}{\partial u_1} \\
&\qquad + \Big(M + (m+2)\frac{\partial H_a}{\partial x_a}\Big)u_2\frac{\partial}{\partial u_2}\Big)f^1, \\
&\Big(\frac{\alpha}{2} + N + (m+2)\frac{\partial H_a}{\partial x_a}\Big)f^2 + B^3 f^1 + M_t u_2 + B^3_t u_1 + B^2_t \\
&\qquad - \Delta B_1 + (2-m)\Big(\Delta\frac{\partial H^a}{\partial x^a}\Big)u_1 = \Big(B^1\frac{\partial}{\partial u_1} + B^2\frac{\partial}{\partial u_2} \\
&\qquad + B^3 u_1\frac{\partial}{\partial u_2} + \Big(N + (m-2)\frac{\partial H_a}{\partial x_a}\Big)u_1\frac{\partial}{\partial u_1} \\
&\qquad + \Big(M + (m+2)\frac{\partial H_a}{\partial x_a}\Big)u_2\frac{\partial}{\partial u_2}\Big)f^2.
\end{aligned} \tag{4.14}$$

We notice that in this case symmetry classification appears to be rather complicated and cumbersome. Nevertheless, the classifying equations can be effectively solved using the approach outlined in the following sections.

Thus the group classification of equations (1.3), (1.4) and (1.5) reduces to searching for general solutions of equations (4.6), (4.10), (4.12) and (4.14). To solve these equation it is necessary to make an effective separation of independent variables. To do this we will use an approach which includes *a priori* specification and simplification of possible forms of generators $X$ (4.3), (4.8), (4.11) and (4.13) using the condition that $X$ belong to $n$-dimensional Lie algebra with $n = 1, 2, \ldots$. This specification will be based on classification of algebras of $3\times 3$ matrices of special form.

## 5. Basic, Main and Extended Symmetries

Let us start with equation (1.3). The general form for the related symmetries and the classifying equation for nonlinearities $f^1, f^2$ are given by relation (4.3) and (4.6) respectively.

Equation (4.6) does not include parameters $\Psi^{\mu\nu}, \nu$ and $\rho_\nu$ present in (4.3) thus for any $f^1$ and $f^2$ equation (1.3) admits symmetries generated by the following operators

$$P_0 = \partial_t, \quad P_\lambda = \partial_\lambda, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu. \tag{5.1}$$

For some classes of nonlinearities $f^1$ and $f^2$ the invariance algebra of equation (1.3) is more extended but includes (5.1) as a subalgebra. We will refer to (5.1) as to *basic symmetries*.

Operators (5.1) generate the maximal local Lie group which is admitted by equations (1.3) for *any* functions $f^1$ and $f^2$. In other words the basic symmetries generate the kernel of the invariance group of equation (1.3).

Let us specify *main symmetries* for equation (1.3), whose generator $\tilde{X}$ has the form (4.3) with $\Psi^{\mu\nu} = \nu = \rho_\nu = \sigma_\nu = \omega_\nu = 0$, i.e.,

$$\tilde{X} = \mu D + C^{ab} u_b \frac{\partial}{\partial u_a} + B^a \frac{\partial}{\partial u_a}. \tag{5.2}$$

The classifying equation for symmetries (5.2) can be obtained from (4.6) by setting $\mu = \sigma^a = \omega^a = 0$. As a result we get

$$(\mu\delta^{ab} + C^{ab}) f^b + C^{ab}_t u_b + B^a_t - \Delta A^{ab} B^b = (C^{nb} u_b + B^n) \frac{\partial f^a}{\partial u_n}. \tag{5.3}$$

Operator (5.2) is a particular case of (4.3). Moreover, it is easily verified that operators (5.2) and (5.1) form a Lie algebra which is a subalgebra of symmetries for equation (1.3). On the other hand, if equation (1.3) admits a more general symmetry (4.3) with $\sigma_a \neq 0$ or (and) $\lambda \neq 0, \ \omega^\mu \neq 0$ then it has to admit symmetry (5.2) also. To prove this we will calculate multiple commutators of (4.3) with the basic symmetries (5.1) and use the fact that such commutators have to belong to symmetries of equation (1.3).

Let equation (1.3) admits extended symmetry (4.3) with $\sigma_\nu \neq 0, \ \Psi^{\mu\nu} = \rho_\mu = \nu = \lambda = \omega^k = 0$, i.e.,

$$X = \sigma_\alpha G_\alpha + \mu D + (C^{ab} u_b + B^a) \frac{\partial}{\partial u_b}. \tag{5.4}$$

Commuting $Y$ with $P_\alpha$ we obtain one more symmetry

$$Y_\alpha = -\frac{\sigma_\alpha}{2} (A^{-1})^{ab} u_b \frac{\partial}{\partial u_a} + B^a_{x_\alpha} \frac{\partial}{\partial u_a} + \mu P_{x_\alpha}. \tag{5.5}$$

The latest term belongs to the basic symmetry algebra (5.1) and so can be omitted. The remaining terms are of the type (5.2).

Thus supposing the extended symmetry (5.4) is admissible we conclude that equation (1.3) has to admit the main symmetry also.

Commuting (5.5) with $P_0$ and $P_\lambda$ we come to the following symmetries:

$$Y_{\mu\nu} = B^a_{x_\mu x_\nu}\frac{\partial}{\partial u_a}, \quad Y_{\mu t} = B^a_{x_\mu t}\frac{\partial}{\partial u_a}. \tag{5.6}$$

Any symmetry (5.4)–(5.6) generates this own system (4.6) of classifying equations. After straightforward but rather cumbersome calculations we conclude that all these systems are compatible provided the following condition is satisfied

$$(A^{-1})^{ab} f^b = (A^{-1})^{nb} u_b \frac{\partial f^a}{\partial u_n}. \tag{5.7}$$

If (5.7) is satisfied equation (1.3) admits symmetry (5.4) with $\mu = C^{ab} = B^a = 0$, i.e., Galilei generators $G_\alpha$ of (4.5).

In analogous way, supposing that equation (1.3) admits extended symmetry (4.3) with $\lambda \neq 0$ and $\omega^a = 0$ we prove that it has to admit also symmetry (5.4) with $\mu \neq 0$ and $\sigma_\nu \neq 0$. The related functions $f^1$ and $f^2$ should satisfy relations (5.7) and (5.3). Moreover, analyzing possible dependence of $C^{ab}$ and $B^a$ in the corresponding relations (4.6) on $t$ we conclude that they should be ether scalars or linear in $t$, i.e., $C^{ab} = \mu^{ab}t + \nu^{ab}$. Moreover, up to equivalence transformations (2.4) we can choose $B^a = 0$, and reduce (5.3) to the following system:

$$\begin{aligned} &(m+4)f^a + \mu^{ab} f^b = (\mu^{kb} u_b + m u_k)\frac{\partial f^a}{\partial u_k}, \\ &\nu^{ab} f^b + \mu^{ab} u_b = \nu^{kb} u_b \frac{\partial f^a}{\partial u_k} \end{aligned} \tag{5.8}$$

where the parameters $\nu^{ab}$ and $\mu^{ab}$ are distinct from zero in the case of the diagonal matrix $A$ only.

Finally for general symmetry (4.3) it is not difficult to show that the condition $\omega_\nu \neq 0$ leads to the following equation for $f^a$

$$(A^{-1})^{kb}(f^b + \gamma u^b) = (A^{-1})^{ab} u_b \frac{\partial f^k}{\partial u_a}. \tag{5.9}$$

We notice that relations (5.7) and (5.9) are particular cases of (5.3) for $\mu = 0$, $C^{ab} = (A^{-1})^{ab}$ and $\mu = 0$, $C^{ab} = e^{\gamma t}(A^{-1})^{ab}$ respectively. Thus if relation (5.7) is valid then, in addition to $G_\alpha$ (4.5) equation (1.3) admits the symmetry

$$X = (A^{-1})^{ab} u_b \frac{\partial}{\partial u_a}. \tag{5.10}$$

Alternatively, if (5.9) is satisfied, equation (1.2) admits symmetry $\widehat{G}_\alpha$ (2.6) and also the following one

$$X = e^{\gamma t}(A^{-1})^{ab}u_b\frac{\partial}{\partial u_a}, \quad \gamma \neq 0. \tag{5.11}$$

Thus it is reasonable first to classify equations (1.3) which admit main symmetries (5.2) and then specify all cases when these symmetries can be extended.

The conditions when system (1.3) admits extended symmetries are given by relations (5.7)–(5.9).

Concerning equations (1.4) and (1.5) we notice that in accordance with (4.8) and (4.11) they admit basic symmetries only.

Now we are ready to search for solutions to classifying equations (4.10), (4.12) and (5.3). To present clearly main details of our approach we start with group classification of systems (1.4), because this problem appears to be essentially more simple than other ones considered here.

## 6. Symmetry Algebras of Equations (1.4)

Consider equations (1.4) and suppose that parameter $p$ is nonzero. Then scaling dependent and independent variables we can reduce its value to $p = 1$.

To solve rather complicated classifying equations (4.10), (4.12) and (5.3) we use the main algebraic property of the related symmetries, i.e., the fact that they should form a Lie algebra. In other words, instead of going throw all non-equivalent possibilities arising via separation of variables in the classifying equations we first specify all non-equivalent realizations of the invariance algebra for our equations whose elements are defined by relations (5.2), (4.8) and (4.11) up to arbitrary constants and arbitrary functions. Then using the one-to-one correspondence between these algebras and classifying equations (4.10), (4.12), (5.3) we easily solve the group classification problems for equations (1.3)–(1.5).

Let us start with classifying equations (4.10) and the related symmetries (4.8). For any functions $f^1$ and $f^2$ equations (1.4) admit symmetries (5.1) where the indices $\mu$, $\nu$ and $\lambda$ run over the values $1, 2, \ldots m-1$ and $1, 2, \ldots m$ respectively.

In accordance with (4.8) any symmetry generator extending algebra (5.1) has the following form

$$X = \mu D - 2B^a\frac{\partial}{\partial u_a} - 2Fu_1\frac{\partial}{\partial u_1} + (\mu - 2F)u_2\frac{\partial}{\partial u_2}. \tag{6.1}$$

Let $X_1$ and $X_2$ be operators of the form (6.1) then the commutator $[X_1, X_2]$ is also a symmetry whose general form is given by (6.1). Thus operators (6.1) form a Lie algebra which we denote as $\mathcal{A}$.

Let us specify algebras $\mathcal{A}$ which can appear in our classification procedure. First consider one-dimensional $\mathcal{A}$ , i.e., suppose that equation (1.4) admits the only symmetry of the form (6.1). Then any commutator of operator (5.1) with (6.1) should be reducible to a linear combination of operators (5.1) and (6.1). This condition presents us the following possibilities only:

$$\begin{aligned}
&X = X_1 = \mu D - 2\alpha_a \frac{\partial}{\partial u_a} - 2\beta u_1 \frac{\partial}{\partial u_1} - (2\beta - \mu) u_2 \frac{\partial}{\partial u_2},\\
&X = X_2 = e^{\nu t}\Big(\alpha_a \frac{\partial}{\partial u_a} + \beta u_1 \frac{\partial}{\partial u_1} + \beta u_2 \frac{\partial}{\partial u_2}\Big),\\
&X = X_3 = e^{\nu t + \rho\cdot x}\alpha^a \frac{\partial}{\partial u_a}
\end{aligned} \tag{6.2}$$

where the Greek letters again denote arbitrary parameters and $\rho \cdot x = \rho_\mu x_\mu$.

All the other choices of arbitrary functions $F$ and $B^a$ in (6.1) correspond to algebras $\mathcal{A}$ whose dimension is larger than one.

The next step is to specify all non-equivalent sets of arbitrary constants in (6.2) using the equivalence transformations (2.4).

If the coefficient for $u_a \frac{\partial}{\partial u_a}$ ($a$ is fixed) is non-zero then translating $u_a$ we reduce to zero the related coefficient $\alpha_a$ in $X_1$ and $X_2$; then scaling $u_a$ we can reduce to $\pm 1$ all non-zero $\alpha_a$ in (6.2). In addition, all operators (6.2) are defined up to constant multipliers. Using these simple arguments we come to the following non-equivalent versions of operators (6.2) belonging to one-dimensional algebras $\mathcal{A}$:

$$\begin{aligned}
&X_1^{(1)} = \mu D - u_1 \frac{\partial}{\partial u_1} + (\mu - 1) u_2 \frac{\partial}{\partial u_2},\\
&X_1^{(2)} = D + u_2 \frac{\partial}{\partial u_2} + \nu \frac{\partial}{\partial u_1},\\
&X_1^{(3)} = D - u_1 \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2},\\
&X_2^{(\nu)} = e^{\nu t + \rho_2\cdot x}\Big(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\Big);\\
&X_3^{(1)} = e^{\sigma_1 t + \rho_1\cdot x}\Big(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\Big),\\
&X_3^{(2)} = e^{\sigma_2 t + \rho_2\cdot x}\frac{\partial}{\partial u_1},\ X_3^{(3)} = e^{\sigma_3 t + \rho_3\cdot x}\frac{\partial}{\partial u_2}.
\end{aligned} \tag{6.3}$$

To describe *two-dimensional* algebras $\mathcal{A}$ we represent one of the related basis element $X$ in the general form (6.1) and calculate the commutators

$$Y = [P_0, X] - 2\mu P_0, \quad Z = [P_0, Y], \quad W = [X, Y]$$

where $P_0$ is operator given in (5.1), $Y, Z$ and $W$ are symbols denoting the terms in the r.h.s.. After simple calculations we obtain

$$\begin{aligned} Y &= F_t\Big(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\Big) + B_t^a\frac{\partial}{\partial u_a}, \\ Z &= F_{tt}\Big(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\Big) + B_{tt}^a\frac{\partial}{\partial u_a}, \\ W &= 2\mu t Z + \mu x_b B_{tx_b}^a\frac{\partial}{\partial u_a}. \end{aligned} \tag{6.4}$$

By definition, $Y$, $Z$ and $W$ belong to $\mathcal{A}$. Let $F_t \neq 0$ than it follows from (6.4) that

$$\mu \neq 0: \; B_{tt}^a = F_{tt} = B_{tb}^a = 0, \tag{6.5}$$

$$\mu = 0: \; F_{tt} = \alpha F_t + \gamma^a B_t^a, \quad B_{tt}^a = \gamma^a F_t + \beta^{ab} B_t^b, \tag{6.6}$$

otherwise the dimension of $\mathcal{A}$ is larger than 2. The Greece letters in (6.5) and (6.6) denote arbitrary parameters.

Starting with (6.5) we conclude that up to translations of $t$ the coefficients $F$ and $B_a$ have the following form

$$F = \sigma t \text{ or } F = \beta; \; B^a = \nu^a t + \alpha^a \text{ if } \mu \neq 0.$$

If $F = \sigma t$ then the change

$$u_a \to u_a e^{-\sigma t} - \frac{\nu_a}{\mu} t \tag{6.7}$$

reduces the related operator (4.8) to the following form:

$$X = \mu\left(D + u_2\frac{\partial}{\partial u_2}\right) - 2\alpha_a\frac{\partial}{\partial u_a}, \tag{6.8}$$

i.e., $X$ coincides with $X_1$ of (6.2) for $\beta = 0$. Moreover it is possible to show that (6.7) gives the equivalence transformation for the related equations (1.4) (i.e., for equations (1.4) which admit symmetry (6.8)).

The choice $F = \beta$ corresponds to the following operator (6.1)

$$X = X_4 = X_1 - 2t\alpha^a \frac{\partial}{\partial u_a} \tag{6.9}$$

where $X_1$ is given in (6.2).

Thus if one of basis elements of two dimension algebra $\mathcal{A}$ is of general form (6.1) with $\mu \neq 0$ then it can be reduced to (6.8) or (6.9). We denote such basis element as $e_1$. Without loss of generality the second basis element $e_2$ of $\mathcal{A}$ is a linear combination of operators $X_2^{(\nu)}$ and $X_3^{(a)}$ (6.3). Going over possible pairs $(e_1, e_2)$ and requiring $[e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2$ we come to the following two dimensional algebras

$$\begin{gathered}
A_1 = \left\langle D + u_2 \frac{\partial}{\partial u_2}, X_2^{(0)} \right\rangle, \quad A_2 = \langle X_1^{(2)}, X_3^{(3)} \rangle, \\
A_3 = \langle X_1^{(3)}, X_3^{(3)} \rangle, \quad A_4 = \langle X_1^{(1)}, X_3^{(3)} \rangle, \quad A_5 = \langle X_1^{(1)}, X_3^{(3)} \rangle, \\
A_6 = \left\langle D + 2u_2 \frac{\partial}{\partial u_2} + u_1 \frac{\partial}{\partial u_1} + \nu t \frac{\partial}{\partial u_2}, X_3^{(2)} \right\rangle \\
A_7 = \left\langle D + 2u_1 \frac{\partial}{\partial u_1} + 3u_2 \frac{\partial}{\partial u_2} + 3\nu t \frac{\partial}{\partial u_1}, \quad X_3^{(1)} \right\rangle.
\end{gathered} \tag{6.10}$$

The form of basis elements in (6.10) is defined up to transformations (6.7) (2.4).

If $\mathcal{A}$ does not include operators (6.1) with non-trivial parameters $\mu$ then in accordance with (6.7) its elements are of the following form

$$e_a = F_{(a)} \left( u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right) + B^b_{(a)} \frac{\partial}{\partial u_b}, \ a = 1, 2 \tag{6.11}$$

where $F_{(\alpha)}$ and $B^b_{(a)}$ are solutions of (6.6).

Formulae (6.10), (6.11) define all non-equivalent two-dimensional algebras $\mathcal{A}$ which have to be considered as possible symmetries of equations (1.4). We will see that asking for invariance of (1.4) w.r.t. these algebras the related arbitrary functions $f^a$ are defined up to arbitrary constants, and it is impossible to make further specification of these functions by extending algebra $\mathcal{A}$.

## 7. Group Classification of Equations (1.4)

To classify equations (1.4) which admit one- and two- dimension extensions of the basis invariance algebra (5.1) it is sufficient to solve classifying equations (4.10) for $f^a$ with *known* coefficient functions $B^a$ and $F$ of symmetries (6.1). These functions are easily found comparing (4.8) with (6.3), (6.10) and (6.11).

Let us present an example of such calculation which corresponds to algebra $A_1$ whose basis elements are $X_1 = 2t\partial_t + x_a\partial_a + u_2\frac{\partial}{\partial u_2}$ and $X_2^{(0)} = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}$, refer to (6.10). Operator $X_2^{(0)}$ generates the following form of equation (4.10):

$$f^a = \left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right) f^a, \quad a = 1, 2$$

whose general solution is

$$f^1 = u_1 F_1\left(\frac{u_2}{u_1}\right), \quad f^2 = u_1 F_2\left(\frac{u_2}{u_1}\right). \tag{7.1}$$

Here $F_1$ and $F_2$ are arbitrary functions of $\frac{u_2}{u_1}$.

Equations (1.4) with non-linearities (7.1) admit symmetry $X_2^{(0)}$. In order this equation be invariant w.r.t. $X_1$ also, functions $f^1$, $f^2$ have to satisfy equation (4.10) with $F = 0$, i.e.,

$$f^1 = -u_1\frac{\partial f^1}{\partial u_1}; \quad f^2 = -\frac{1}{2}u_1\frac{\partial f^2}{\partial u_1}. \tag{7.2}$$

It follows from (7.1), (7.2) that

$$f^1 = \alpha u_1^3 u_2^{-2}, \quad f^2 = \lambda u_1^2 u_2^{-1}. \tag{7.3}$$

Thus equation (1.4) admits symmetries $X_0^{(2)}$ and $X_1$ which form algebra $A_1$ (6.10) provided $f^1$ and $f^2$ are functions given in (7.3). These symmetries are defined up to arbitrary constants $\alpha$ and $\lambda$. If one of them is nonzero, than it can be reduced to $+1$ or $-1$ by scaling independent variables.

In analogous way we solve equations (4.10) corresponding to other symmetries indicated in (6.3) and (6.10). For one-dimension algebras (6.3) the related non-linearities $f^1$ and $f^2$ are defined up to arbitrary functions $F_1$ and $F_2$ while for two dimension algebras (6.10) functions $f^1$ and $f^2$ are defined up to two integration constants. We shall not reproduce the related rather routine calculations but present their results in Table 1.

**Table 1. Non-linearities and symmetries for equation (1.4) with p=1**

| No | Non-linearities | Arguments of $F_1$ $F_2$ | Symmetries |
|---|---|---|---|
| 1. | $f^1 = u_1^{2\mu+1}F_1,$ $f^2 = u_1^{\mu+1}F_2$ | $u_2u_1^{\mu-1}$ | $\mu D - u_1\frac{\partial}{\partial u_1} + (\mu-1)u_2\frac{\partial}{\partial u_2}$ |
| 2. | $f^1 = F_1u_2^{-2},$ $f^2 = F_2u_2^{-1}$ | $u_1 - \nu\ln u_2$ | $D + u_2\frac{\partial}{\partial u_2} + \nu\frac{\partial}{\partial u_1}$ |
| 3. | $f^1 = u_1(F_1 + \lambda\ln u_1),$ $f^2 = u_2(F_2 + \lambda\ln u_1)$ | $\frac{u_2}{u_1}$ | $e^{\lambda t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right)$ |
| 4. | $f^1 = u_1^3F_1,$ $f^2 = u_1^2F_2$ | $u_2 - \ln u_1$ | $D - u_1\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}$ |
| 5. | $f^1 = F_1,$ $f^2 = F_2 + \nu u_2$ | $u_2$ | $e^{\nu t}\Psi(x)\frac{\partial}{\partial u_2}$ |
| 6. | $f^1 = \alpha u_1 + F_1,$ $f^2 = \lambda u + F_2$ | $u_1$ | $e^{\lambda t+\nu x_m}\tilde{\Psi}_\mu(\tilde{x})\frac{\partial}{\partial u_1},$ $\mu = \lambda - \nu^2 - \alpha$ |
| 7. | $f^1 = \sigma u + F_1,$ $f^2 = \lambda v + F_2$ | $u - v$ | $e^{\lambda t}\, e^{\frac{x_m+t}{2}}\Psi_\mu(\tilde{x}, x_m + t)\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right),\ \mu = \lambda - \sigma + \frac{1}{4}$ |
| 8. | $f^1 = \alpha u_1^3u_2^{-2},$ $f^2 = \beta u_1^2u_2^{-1}$ | | $D + u_2\frac{\partial}{\partial u_2},\quad u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}$ |
| 9. | $f^1 = \alpha e^{-2u_1},$ $f^2 = \lambda e^{-u_1}$ | | $D + u_2\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_1},\quad \Psi(x)\frac{\partial}{\partial u_2}$ |
| 10. | $f^1 = \lambda e^{3u_2},$ $f^2 = \alpha e^{2u_2}$ | | $D - u_1\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2},$ $\tilde{\Phi}_0(t, \tilde{x})\frac{\partial}{\partial u_1}$ |
| 11. | $f^1 = \alpha u_1^{2\mu+1},$ $f^2 = \lambda u_1^{\mu+1}$ | | $\mu D - u_1\frac{\partial}{\partial u_1} + (\mu-1)u_2\frac{\partial}{\partial u_2},$ $\Psi(x)\frac{\partial}{\partial u_2}$ |
| 12. | $f^1 = \lambda u_2^{3\mu-2},$ $f^2 = \alpha u_2^{2\mu-1}$ | | $(\mu-1)D - \mu u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2},$ $\tilde{\Phi}_0(t, \tilde{x})\frac{\partial}{\partial u_1}$ |
| 13. | $f^1 = \frac{\alpha}{u_1},\quad f^2 = \ln u_1$ | | $D + 2u_2\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_1} + t\frac{\partial}{\partial u_2},$ $\Psi(x)\frac{\partial}{\partial u_2}$ |
| 14. | $f^1 = \ln u_2,\ f^2 = \alpha u_2^{\frac{1}{3}}$ | | $D + 2u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2} + 3t\frac{\partial}{\partial u_1},$ $\tilde{\Phi}_0(t, \tilde{x})\frac{\partial}{\partial u_1}$ |

Here $D$ is the dilatation operator given in (4.5), $\tilde{x}=(x_1, x_2, \dots, x_{m-1})$, $\Psi(x)$ is an arbitrary function of spatial variables; $\tilde{\Psi}_\mu(\tilde{x}), \Psi_\mu(\tilde{x}, x_m+t)$ and $\tilde{\Phi}_\mu(t,\tilde{x})$ are solutions of the Laplace and linear heat equations

$$\begin{aligned}
&\tilde{\Delta}\tilde{\Psi}_\mu = \mu\tilde{\Psi}_\mu, \quad \Delta\Psi_\mu = \mu\Psi_\mu, \quad (\frac{\partial}{\partial t} - \tilde{\Delta})\tilde{\Phi}_0 = 0, \\
&\tilde{\Delta} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_{m-1}^2}, \quad \Delta = \tilde{\Delta} + \frac{\partial^2}{\partial x_m^2}.
\end{aligned} \tag{7.4}$$

We notice that equations (1.4) with non-linearities 5, 6, 9-14 of Table 1 admit infinite-dimension algebras $\mathcal{A}$ because the related symmetries are defined up to arbitrary functions $\Psi(x)$ or arbitrary solutions of equations (7.4). Nevertheless, the form of these non-linearities was fixed requiring invariance w.r.t. one- and two-dimension algebras enumerated in (6.3), (6.10).

The second note is that equations (1.4) with non-linearities given in Item 8 of Table 1 admit additional equivalence transformations $u_\alpha \to e^{\sigma t}u_\alpha$ while for Items 9, 11, 13 and 10, 12, 14 we have in our disposal transformations 3 and 2 respectively from the list (2.10).

## 8. Group Classification of Equations (1.5)

Like (1.4), equations (1.5) with arbitrary functions $f^1$ and $f^2$ admit the basic symmetries (5.1) were $\mu, \nu = 1, 2, \dots, m-1$. To classify equations admitting other symmetries it is sufficient to find the general solution for equations (4.12).

We will solve (4.12) using the technique applied in Sections 5 and 6. Comparing (4.11) and (6.1) we conclude that generators of extended symmetry for equations (1.4) and (1.5) are rather similar and so we can essentially exploit the algebra classification scheme used in Section 5. As a result we easily come to the following list of one-dimension algebras $A$ (compare with (6.3))

$$\begin{aligned}
&\tilde{X}_1^{(1)} = \mu\tilde{D} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}, \\
&\tilde{X}_1^{(2)} = \tilde{D} - \nu\frac{\partial}{\partial u_1}, \quad \tilde{X}_2^{(\nu)} = e^{\nu t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right), \\
&\tilde{X}_1^{(3)} = \tilde{D} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + \nu\frac{\partial}{\partial u_2}, \\
&\tilde{X}_3^{(3)} = e^{\sigma_3 t + \rho_3 x}\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right), \\
&\tilde{X}_3^{(j)} = e^{\sigma_i t + \rho_i\cdot x}\frac{\partial}{\partial u_j}, \quad j = 1, 2
\end{aligned} \tag{8.1}$$

where $\tilde{D} = 3t\partial_t + 2x_\nu\partial_\nu - u_2\frac{\partial}{\partial u_2}$. The two-dimension algebras are given by the following relations (compare with (6.10)):

$$\begin{aligned}
&\tilde{A}_1 = \langle \tilde{D}, \tilde{X}_2^{(0)}\rangle, \quad \tilde{A}_2 = \langle \tilde{X}_1^{(2)}, X_3^{(3)}\rangle,\\
&\tilde{A}_3 = \langle \tilde{X}_1^{(3)}, \tilde{X}_3^{(1)}\rangle, \quad \tilde{A}_4 = \langle \tilde{X}_1^{(1)}, \tilde{X}_3^{(2)}\rangle, \quad \tilde{A}_5 = \langle \tilde{X}_1^{(1)}, \tilde{X}_3^{(1)}\rangle,\\
&\tilde{A}_6 = \left\langle \tilde{D} + 4\Big(u_2\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_1} + t\frac{\partial}{\partial u_2}\Big), X_3^{(2)}\right\rangle,\\
&\tilde{A}_7 = \left\langle \tilde{D} + 3\Big(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + t\frac{\partial}{\partial u_1}\Big), X_3^{(1)}\right\rangle.
\end{aligned} \tag{8.2}$$

Using (8.1), (8.2) and solving the related classifying equations (4.12) we find non-linearities $f^1$, $f^2$ which are given in Table 2. In six cases enumerated in the table the corresponding equations (1.5) admit infinite dimension symmetry algebras whose generators are defined up to arbitrary functions, see Items 5–7, 9–14 here.

**Table 2. Non-linearities and symmetries for equation (1.5) with $p = 1$**

| No | Non-linearities | Arguments of $F_1$ $F_2$ | Symmetries |
|---|---|---|---|
| 1. | $f^1 = u_1^{1+3\mu}F_1$, $f^2 = u_1^{1+4\mu}F_2$ | $u_2u_1^{-\mu-1}$ | $\mu\tilde{D} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$ |
| 2. | $f^1 = u_2^3F_1$, $f^2 = u_2^4F_2$ | $u_1 - \nu\ln u_2$ | $\tilde{D} - \nu\frac{\partial}{\partial u_1}$ |
| 3. | $f^1 = u_1(F_1 + \nu\ln u_1)$, $f^2 = u_2(F_2 + \nu\ln u_1)$ | $\frac{u_2}{u_1}$ | $e^{\nu t}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right)$ |
| 4. | $f^1 = u_1^{-2}F_1$, $f^2 = u_1^{-3}F_2$ | $u_2 - \nu\ln u_1$ | $\tilde{D} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + \nu\frac{\partial}{\partial u_2}$ |
| 5. | $f^1 = \lambda u_1 + F_1$, $f^2 = -\mu u_1 + F_2$ | $u_2$ | $e^{\lambda t}\Psi_\mu(x)\frac{\partial}{\partial u_1}$ |
| 6. | $f^1 = \nu u_2 + F_1$, $f^2 = \lambda u_2 + F_2$ | $u_1$ | $e^{\lambda t - \nu x_m}\Psi(\tilde{x})\frac{\partial}{\partial u_2}$ |
| 7. | $f^1 = \alpha u_1 + F_1$, $f^2 = \sigma u_2 + F_2$ | $u_1 - u_2$ | $e^{\lambda t}\, e^{\frac{x_m - t}{2}}\Psi_\mu(\tilde{x}, x_m + t)\Big(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\Big)$, $\mu = \lambda - \sigma + \frac{1}{4}$ |

| | | | |
|---|---|---|---|
| 8. | $f^1 = \alpha u_1^{-2}u_2^3$, $f^2 = \nu u_1^{-3}u_2^4$ | | $\tilde{D}$, $\quad u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}$ |
| 9. | $f^1 = \alpha e^{3u_1}$, $f^2 = \nu e^{4u_1}$ | | $\tilde{D} - \frac{\partial}{\partial u_1}$, $\Psi(\tilde{x})\frac{\partial}{\partial u_2}$ |
| 10. | $f^1 = \alpha e^{-2u_2}$, $f^2 = \nu e^{-3u_2}$ | | $\tilde{D} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_2}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$ |
| 11. | $f^1 = \alpha u_1^{3\mu+1}$, $f^2 = \nu u_1^{4\mu+1}$ | | $\mu\tilde{D} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$, $\Psi(\tilde{x})\frac{\partial}{\partial u_2}$ |
| 12. | $f^1 = \alpha u_2^{2\nu+1}$, $f^2 = \nu u_2^{3\nu+1}$ | | $\nu\tilde{D} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$ |
| 13. | $f^1 = \alpha u_1^{\frac{1}{4}}$, $f^2 = \nu \ln u_1$ | | $\tilde{D} + 3u_2\frac{\partial}{\partial u_2} + 4u_1\frac{\partial}{\partial u_1} + 4\nu t\frac{\partial}{\partial u_2}$, $\Psi(\tilde{x})\frac{\partial}{\partial u_2}$ |
| 14. | $f^1 = \nu \ln u_2$, $f^2 = \frac{\alpha}{\sqrt{u_2}}$ | | $\tilde{D} + 3u_1\frac{\partial}{\partial u_1} + 2u_2\frac{\partial}{\partial u_2} + 2\nu t\frac{\partial}{\partial u_1}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$ |

Here $\Psi_\mu(x)$ and $\Psi_\mu(\tilde{x}, x_m + t)$ are arbitrary solutions of the Laplace equation $\Delta\Psi_\mu = \mu\Psi_\mu$, $\mu$, $\nu$ and $\lambda$ are arbitrary parameters satisfying $\nu\lambda \neq 0$.

Equations (1.5) with the non-linearities given in Item 8 of Table 2 admit additional equivalence transformation $u_\alpha \to e^{\sigma t}u_\alpha$. Besides, for Items 9,11,13 and 10,12, 14 we have transformations 3 and 2 from the list (2.10) respectively.

## 9. Group Classification of Equations (1.3) with Invertible Diffusion Matrices

In this Section we present the group classification of systems of coupled reaction-diffusion equations (1.3) with invertible matrix $A$. In accordance with the plane outlined in Section 4 we first describe the main symmetries generated by operators (5.2) and then indicate extensions of these symmetries.

Like in Sections 5, 7 the first step of our analysis consists in description of realizations of Lie algebras $\mathcal{A}$ generating basic symmetries of equation (1.3). However, the basis elements of $\mathcal{A}$ are now of the general form (5.2) while in Sections 5 and 7 we were restricted to the representations (6.1) and (4.11) respectively which are particular cases of (5.1).

Thus the first step of our analysis is to describe non-equivalent realizations of finite dimension algebras $\mathcal{A}$ whose basis elements have the form (5.2).

Let us specify all non-equivalent "tails" of operators (5.2), i.e., the terms

$$\pi = C^{ab}u_b\frac{\partial}{\partial u_a} + B^a\frac{\partial}{\partial u_a}. \tag{9.1}$$

These terms can either be a constituent part of a more general symmetry (5.2) or represent a particular case of (5.2) corresponding to $\mu = 0$.

If equation (1.3) admits a one-dimensional invariance algebra $\mathcal{A}$ then commutators of $\pi$ with the basic symmetries $P_0$ and $P_a$ should be equal to a linear combination of $\pi$ and operators (5.1). In other words, there are three possibilities:

$$1.\ C^{ab} = \mu^{ab}, \quad B^a = \mu^a, \tag{9.2}$$

$$2.\ C^{ab} = e^{\lambda t}\mu^{ab}, \quad B^a = e^{\lambda t}\mu^a, \tag{9.3}$$

$$3.\ C^{ab} = 0, \quad B^a = e^{\lambda t+\omega\cdot x}\mu^a \tag{9.4}$$

where $\mu^{ab}, \mu^a, \lambda$, and $\omega$ are constants.

In any case the problem of classification of one-dimension algebras $\mathcal{A}$ includes the subproblem of classification of non-equivalent linear combinations (9.1) with *constant* coefficients $\mu^{ab}$ and $\mu^a$. To describe such linear combinations we will use the isomorphism of (9.1) with $3\times 3$ matrices of the following form

$$g = \begin{pmatrix} 0 & 0 & 0 \\ B^1 & C^{11} & C^{12} \\ B^2 & C^{21} & C^{12} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^{11} & \mu^{12} \\ \mu^2 & \mu^{21} & \mu^{12} \end{pmatrix}. \tag{9.5}$$

Equations (1.3) admit equivalence transformations (2.4) which change the term $\pi$ (9.1) and can be used to simplify it. The corresponding transformation for matrix (9.5) can be represented as

$$g \to g' = UgU^{-1} \tag{9.6}$$

where $U$ is a $3\times 3$ matrix of the following special form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & K^{11} & K^{12} \\ b^2 & K^{21} & K^{22} \end{pmatrix}. \tag{9.7}$$

We will use relations (9.2)–(9.4) and equivalence transformations (9.6) to construct basis elements of basic symmetry algebras. For different forms of matrix $A$ specified in (2.2) the transformation matrix (9.7) needs further specification in accordance with (2.5)–(2.8).

The obtained non-equivalent realizations of low dimension algebras $\mathcal{A}$ are present in Appendix. Starting with these realizations one easily solves the related determining equations (4.6) for non-linearities $f^1$ and $f^2$ and specify all cases when the main symmetries can be extended (i.e., when relations (5.7)–(5.9) are satisfied). In addition we have to control all cases when basis elements of $\mathcal{A}$ depend on arbitrary solutions $\Psi$ of the linear heat equation. Such algebras (whose basis elements can be obtained from (A.1.10), (A.1.11), (A.1.15)–(A.1.18) changing $g_5$ and $g_3$ by $\Psi g_5$ and $\Psi g_3$) are infinite dimensional but generate the same number of determining equations as the low-dimension algebras.

## 10. Classification Results

We will not reproduce the related exact calculations but present the results of group classification in Tables 3-9. In addition to equations with invertible diffusion matrix we present here the results of classification which are related to the diffusion matrix of type $IV$ while the type $V$ is will be considered separately (see (2.2) for classification of diffusion matrices).

The Tables 3–9 present the classification results for different types of equations (1.3) corresponding to non-equivalent diffusion matrices enumerated in (2.2). The type of diffusion matrix is indicated in the fourth columns of Tables 3, 4 and third columns of Tables 5 and 6. In Tables 7-9 the results of symmetry classification of special equations are presented; these equations are indicated in the table titles. In the last columns of Tables 3, 5 and 6 the additional equivalence transformations (AET) are specified, which are possible for the related class of non-linearities. Finally, the symbols $D$, $G_\alpha$, $\widehat{G}_\alpha$ denote generators (4.5), $\psi_\mu$ denotes an arbitrary solution of the linear heat equation $\frac{\partial}{\partial t}\psi_\mu - \Delta\psi_\mu = \mu\psi_\mu$,

$$\tilde{\psi}_\nu = \begin{cases} \psi_\nu & \text{for Class } III \\ e^{\nu t}\Psi(x) & \text{for Class } IV \end{cases}$$

and $\Psi(x)$, $\Psi_\nu(x)$ have the same meaning as in Tables 1,2.

The results of group classification are briefly discussed in Section 12.

**Table 3. Non-linearities with arbitrary functions and extendible symmetries for equations (1.3), (2.2)**

| No | Nonlinear terms | Arguments of $F_1$ $F_2$ | Type of matrix $A$ | Main symmetries | Additional symmetries | AET (2.10) |
|---|---|---|---|---|---|---|
| 1. | $f^1=u_1^{\nu+1}F_1$, $f^2=u_1^{\nu+\mu}F_2$ | $\frac{u_2}{u_1^{\mu}}$ | $I,\ IV$, $\mu\neq 1$; $I-IV$, $\mu=1$ | $\frac{\nu}{2}D$ $-u_1\frac{\partial}{\partial u_1}$ $-\mu u_2\frac{\partial}{\partial u_2}$ | For $I$: $G_\alpha$, if $\nu=0$, $a\mu=1$ | $1,\rho=\mu\omega$ if $\nu=0$ |
| 2. | $f^1=u_1(F_1$ $+\varepsilon\ln u_1)$, $f^2=u_2(F_2$ $+\varepsilon\mu\ln u_1)$ | $\frac{u_2}{u_1^{\mu}}$ | $I,\ IV$, $\mu\neq 1$; $I-IV$, $\mu=1$ | $e^{\varepsilon t}\big(u_1\frac{\partial}{\partial u_1}$ $+\mu u_2\frac{\partial}{\partial u_2}\big)$ | For $I$: $\widehat{G}_\alpha$, if $a\mu=1$ | |
| 3. | $f^1=u_1F_1+\nu u_2$, $f^2=\nu\frac{u_2}{u_1}(u_1$ $+u_2)+u_1F_2$ $+u_2F_1$, $\nu\neq 0$ | $u_1e^{-\frac{u_2}{u_1}}$ | $I^*, III$ | $e^{\nu t}\big(u_1\frac{\partial}{\partial u_2}$ $+u_1\frac{\partial}{\partial u_1}$ $+u_2\frac{\partial}{\partial u_2}\big)$ | For $III$: $\widehat{G}_\alpha$, if $a=-1$ | |
| 4. | $f^1=u_1^{\nu+1}F_1$, $f^2=u_1^{\nu}(F_1u_2$ $+F_2u_1)$ | $u_1e^{-\frac{u_2}{u_1}}$ | $I^*, III$ | $\frac{\nu}{2}D$ $-u_1\frac{\partial}{\partial u_2}$ $-u_1\frac{\partial}{\partial u_1}$ $-u_2\frac{\partial}{\partial u_2}$ | For $III$: $G_\alpha$, if $\nu=0$, $a=-1$ | |
| 5. | $f^1=e^{\nu\frac{u_2}{u_1}}F_1u_1$, $f^2=e^{\nu\frac{u_2}{u_1}}(F_1u_2$ $+F_2)$ | $u_1$ | $I^*, III$ | $\frac{\nu}{2}D$ $-u_1\frac{\partial}{\partial u_2}$ | For $I^*$: $u_2\frac{\partial}{\partial u_2}$, if $\nu=0$, $F_2=0$ | 6 if $\nu=0$ |
| | | | | | For $I^*$: $\psi_0\frac{\partial}{\partial u_2}$, $D$ $+2u_2\frac{\partial}{\partial u_2}$, if $F_1=0$, $\nu=0$ | 3,6 |
| 6. | $f^1=u_1(F_1-\nu)$, $f^2=F_1u_2+F_2$, $\nu\neq 0$ | $u_1$ | $I^*, III$ | $e^{\nu t}u_1\frac{\partial}{\partial u_2}$ | $\psi_\mu\frac{\partial}{\partial u_2}$, if $F_1=\mu$ | 3 if $F_1=0$ |
| 7. | $f^1=u_1F_1+u_2F_2$ $-\nu z(\mu u_1+u_2)$, $f^2=u_2F_1-u_1F_2$ $+\nu z(u_1-\mu u_2)$; $R=(u_1^2+u_2^2)^{\frac{1}{2}}$, $z=\tan^{-1}(\frac{u_2}{u_1})$ | $Re^{\mu z}$ | $I^*, II$ | $e^{\nu t}\big(\mu R\frac{\partial}{\partial R}$ $-\frac{\partial}{\partial z}\big)$ | For $II$: $\widehat{G}_\alpha$, if $\mu=a$, $\nu\neq 0$; $G_\alpha$, if $\mu=a$, $\nu=0$ | 15 if $\mu=0$ |

**Table 4. Non-linearities with arbitrary functions and non-extendible symmetries for equations (1.3), (2.2)**

| No | Nonlinear terms | Arguments of $F_a$ | Type of matrix $A$ | Symmetries and AET (2.10) [in square brackets] |
|---|---|---|---|---|
| 1. | $f^1 = u_2^\nu F_1,$ $f^2 = u_2^{\nu+1} F_2$ | $u_2 e^{u_1}$ | $I, IV$ | $\nu D - 2u_2 \frac{\partial}{\partial u_2} + 2\frac{\partial}{\partial u_1}$ [4 if $\nu = 0$] |
| 2. | $f^1 = F_1 + \varepsilon u_1,$ $f^2 = F_2 u_2 + \varepsilon u_1 u_2$ | $u_2 e^{u_1}$ | $I, IV$ | $e^{\varepsilon t}\left(u_2 \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right),$ |
| 3. | $f^1 = e^{\nu u_1} F_1,$ $f^2 = e^{\nu u_1}(F_2 + F_1 u_1)$ | $2u_2$ $-u_1^2$ | $I^*, III$ | $\nu D - 2u_1 \frac{\partial}{\partial u_2} - 2\frac{\partial}{\partial u_1}$ |
| 4. | $f^1 = \nu u_1 + F_1,$ $f_2 = \nu u_1^2 + F_1 u_1 + F_2$ | $2u_2$ $-u_1^2$ | $I^*, III$ | $\tilde{\psi}_\nu\left(u_1 \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_1}\right)$ |
| 5. | $f^1 = \nu u_1 + F_1,$ $f_2 = -\mu u_1 + F_2$ | $u_2$ | $II, III$ | For $II: e^{(\nu - a\mu)t}\Psi_\mu \frac{\partial}{\partial u_1},$ For $III: e^{(\nu+\sigma a)t}\Psi_\sigma,$ $\mu = \sigma a$ |
| 6. | $f^1 = e^{\nu z}(F_1 u_2 + F_2 u_1),$ $f^2 = e^{\nu z}(F_2 u_2 - F_1 u_1)$ | $R e^{-\mu z}$ | $I^*, II$ | $\nu D - 2\mu\left(u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}\right)$ $-2\left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1}\right)$ |
| 7. | $f^1 = 0, \quad f^2 = F$ | $u_2$ | $I, IV$ | $\psi_0 \frac{\partial}{\partial u_1}, \quad u_1 \frac{\partial}{\partial u_1},$ [2; 1, $\rho = 0$] |
| 8. | $f^1 = 0, \quad f^2 = F$ | $u_1$ | $I, a \neq 1,$ $IV$ | $D + 2u_2 \frac{\partial}{\partial u_2}, \ \tilde{\psi}_0 \frac{\partial}{\partial u_2},$ [3, 6] |
| 9. | $f^1 = F_1,$ $f^2 = F_2 + \nu u_2$ | $u_1$ | $I, III, IV$ | $\tilde{\psi}_\nu \frac{\partial}{\partial u_2}$ |
| 10. | $f^1 = F_1 + (\nu - \mu)u_1,$ $f^2 = F_2 + (\nu - a\mu)u_2$ | $u_2 - u_1$ | $I, a \neq 1$ $IV$ | $e^{\nu t}\Psi_\mu(x)\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right)$ |
| 11. | $f^1 = \alpha u_1 + \mu,$ $f_2 = \nu u_2 + F,$ $\alpha\mu = 0$ | $u_1$ | $I^*, III$ | $\tilde{\psi}_\nu \frac{\partial}{\partial u_2},$ $e^{(\nu-\alpha)t}(u_1 - \mu t)\frac{\partial}{\partial u_2}$ |
| 12. | $f^1 = u_1^2,$ $f_2 = u_1 u_2 + \nu u_2 + F,$ | $u_1$ | $I^*, III$ | $e^{\nu t} u_1 \frac{\partial}{\partial u_2},$ $e^{\nu t}\left(\frac{\partial}{\partial u_2} + t u_1 \frac{\partial}{\partial u_2}\right)$ |
| 13. | $f^1 = \left(u_1^2 - 1\right),$ $f_2 = (u_1 + \nu) u_2 + F$ | $u_1$ | $I^*, III$ | $e^{(\nu+1)t}\left(u_1 \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_2}\right),$ $e^{(\nu-1)t}\left(u_1 \frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_2}\right)$ |
| 14. | $f^1 = \left(u_1^2 + 1\right),$ $f_2 = (u_1 + \nu) u_2 + F$ | $u_1$ | $I^*, III$ | $e^{\nu t}(\cos t u_1 \frac{\partial}{\partial u_2} - \sin t \frac{\partial}{\partial u_2}),$ $e^{\nu t}(\sin t u_1 \frac{\partial}{\partial u_2} + \cos t \frac{\partial}{\partial u_2})$ |
| 15. | $f^1 = e^{\nu u_2} F_1,$ $f^2 = e^{\nu u_2} F_2$ | $\mu u_2$ $-u_1$ | $I, IV$ $\mu \neq 0;$ $II, III$ $\mu = 0$ | $\nu D - 2\left(\mu \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right)$ |
| 16. | $f^1 = e^{\nu u_1} F_1,$ $f^2 = e^{\nu u_1} F_2$ | $u_2$ | $III$ | $\nu D - 2\frac{\partial}{\partial u_1}$ |

**Table 5. Non-linearities with arbitrary parameters and extendible symmetries for equations (1.3), (2.2)**

| No | Nonlinear terms | Type of matrix $A$ | Main symmetries | Additional symmetries | AET (2.10) |
|---|---|---|---|---|---|
| 1. | $f^1=\lambda u_1^{\nu+1}u_2^{\mu}$, $f^2=\sigma u_1^{\nu}u_2^{\mu+1}$ | $I, IV$ | $\mu D-2u_2\frac{\partial}{\partial u_2}$, $\nu D-2u_1\frac{\partial}{\partial u_1}$ | $G_\alpha$ if $a\nu=-\mu$ $\neq 0$ & $K$ if $\nu=\frac{4}{m(1-a)}, a\neq 1$; | $1, \nu\omega$ $+\mu\rho$ $=0$ |
| | | | | $\psi_0\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=-1$ & $G_\alpha$ if $\mu=a\neq 0$ & $K$ if $a=1+\frac{m}{4}$; | $2; 1,$ $\nu\omega$ $+\mu\rho$ $=0$ |
| | | | | $\psi_0\frac{\partial}{\partial u_1}, u_2\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=-1, a=1$; | $2; 1,$ $\nu\omega$ $+\mu\rho$ $=0$ |
| | | | | $G_\alpha, \mathcal{K}$ if $\sigma=1, \lambda\neq 1$ $\mu=-\nu=a=1$; | $1, \nu\omega$ $+\mu\rho$ $=0$ |
| | | | | $u_1\frac{\partial}{\partial u_2}$ if $\mu=0$, $\lambda=\sigma, a=1$ & $\frac{\partial}{\partial u_2}+tu_1\frac{\partial}{\partial u_2}$ if $\nu=1$ | $1, \nu\omega$ $+\mu\rho$ $=0$ |
| 2. | $f^1=\lambda u_1^{\nu+1}$, $f^2=\sigma u_1^{\nu+\mu}$, $\lambda\sigma\neq 0$ | $I, IV$ | $\nu D-2u_1\frac{\partial}{\partial u_1}$ $-2\mu u_2\frac{\partial}{\partial u_2}$, $\tilde{\psi}_0\frac{\partial}{\partial u_2}$ | $(u_1-\lambda t)\frac{\partial}{\partial u_2}$ if $-\nu=a=1$; | 3 |
| | | | | $G_\alpha$ if $\nu=0,\ a\mu=1$; | 3 |
| | | | | $e^{-\lambda t}u_1\frac{\partial}{\partial u_2}$ if $\nu=0,\ a=1$, & $e^{\lambda t}\left(\sigma u_1\frac{\partial}{\partial u_2}+\lambda\frac{\partial}{\partial u_1}\right)$ if $\mu=2$ | 3; $1,\ \rho$ $=\mu\omega$ |
| 3. | $f^1=\lambda u_1$, $f^2=\sigma u_1^{\mu}$ | $III$ | $\psi_0\frac{\partial}{\partial u_2}$, $e^{-\lambda t}u_1\frac{\partial}{\partial u_2}$ | $e^{\lambda t}\left(u_1\frac{\partial}{\partial u_2}+\lambda\frac{\partial}{\partial u_1}\right)$ if $\mu=2$ | 3; 6 if $\lambda=0$ |
| 4. | $f^1=\lambda e^{\nu u_1}$, $f^2=\sigma e^{(\nu+1)u_1}$, $\lambda\neq 0$ | $I, IV$ | $\nu D-2u_2\frac{\partial}{\partial u_2}$ $-2\frac{\partial}{\partial u_1}$, $\tilde{\psi}_0\frac{\partial}{\partial u_2}$ | $u_2\frac{\partial}{\partial u_2}$ if $\sigma=0$ & $u_2\frac{\partial}{\partial u_1}$ if $a=1$; | 3; 1, $\omega=0$ |
| | | | | $(u_1-\lambda t)\frac{\partial}{\partial u_2}$ if $\nu=0, a=1$ | 3,4 |
| 5. | $f^1=\lambda e^{u_1}$, $f^2=\sigma e^{u_1}$ | $III$ | $D-2\frac{\partial}{\partial u_1}$, $\tilde{\psi}_0\frac{\partial}{\partial u_2}$ | $u_1\frac{\partial}{\partial u_2}$ if $\lambda=0$ | 3; 6 if $\lambda=0$ |
| 6. | $f^1=\lambda e^{u_2}$, $f^2=\sigma e^{u_2}$, $\lambda\neq 0$ | $I, IV$ | $D-2\frac{\partial}{\partial u_2}$, $\psi_0\frac{\partial}{\partial u_1}$ | $u_1\frac{\partial}{\partial u_1}+\frac{\partial}{\partial u_2}$ if $\sigma=0$ & for $I$ $u_2\frac{\partial}{\partial u_1}$ if $a=1$ | $2; 5$ if $\sigma=0$ |

| | | | | | |
|---|---|---|---|---|---|
| 7. | $f^1 = \lambda u_1^{\nu+1} e^{\mu \frac{u_2}{u_1}}$, | $I^*, III$ | $\mu D - 2u_1 \frac{\partial}{\partial u_2}$, | For $I^*$ : $G_\alpha$<br>if $\nu = 0$; | 1,<br>$\rho = \omega$;<br>6 if<br>$\mu = 0$ |
| | $f^2 = e^{\mu \frac{u_2}{u_1}} (\lambda u_2$<br>$+ \sigma u_1) u_1^\nu$ | | $\nu D - 2u_1 \frac{\partial}{\partial u_1}$<br>$-2u_2 \frac{\partial}{\partial u_2}$ | For $III$ : $G_\alpha$<br>if $\mu = a\nu$<br>& $K$ if $\nu = \frac{4}{m}$ | 1,<br>$\rho = \omega$<br>if<br>$\nu = 0$;<br>6 if<br>$\mu = 0$ |
| 8. | $f^1 = e^{\mu z} R^\nu (\lambda u_1$<br>$- \sigma u_2)$, | $I^*, II$ | $\nu D - 2u_1 \frac{\partial}{\partial u_1}$<br>$-2u_2 \frac{\partial}{\partial u_2}$, | For $I^*$ :<br>$G_\alpha$ if $\nu = 0$; | 1,<br>$\rho = \omega$ |
| | $f^2 = e^{\mu z} R^\nu (\lambda u_2$<br>$+ \sigma u_1)$ | | $\mu D - 2u_1 \frac{\partial}{\partial u_2}$<br>$+2u_2 \frac{\partial}{\partial u_1}$ | For $II$ : $G_\alpha$<br>if $\mu = a\nu$<br>& $K$ if $\nu = \frac{4}{m}$ | 1,<br>$\rho = \omega$<br>if<br>$\nu = 0$ |
| 9. | $f^1 = \varepsilon u_1^{\mu+1}$,<br>$f^2 = \varepsilon u_1^\mu (u_2$<br>$- \ln u_1)$, $\mu \neq 0$, | $I^*$ | $\mu D - 2u_1 \frac{\partial}{\partial u_1}$<br>$-2 \frac{\partial}{\partial u_2}$,<br>$u_1 \frac{\partial}{\partial u_2}$ | $\frac{\partial}{\partial u_2} + t u_1 \frac{\partial}{\partial u_2}$<br>if $\mu = 1$ | 6 |
| 10. | $f^1 = \lambda$,<br>$f^2 = \varepsilon \ln u_1$ | $I - IV$ | $\frac{1}{2} D + u_1 \frac{\partial}{\partial u_1}$<br>$+u_2 \frac{\partial}{\partial u_2}$<br>$+\varepsilon t \frac{\partial}{\partial u_2}$, | For $I, a \neq 1, IV$:<br>$u_1 \frac{\partial}{\partial u_1} + \varepsilon t \frac{\partial}{\partial u_2}$<br>if $\lambda = 0$; | 3, 7, 9<br>(for $II$ :<br>3, 7) |
| | | | $\tilde{\psi}_0 \frac{\partial}{\partial u_2}$ | For $I^*, III$<br>$(u_1 - \lambda t) \frac{\partial}{\partial u_2}$;<br>& (for $I^*$)<br>$u_1 \frac{\partial}{\partial u_1} + \varepsilon t \frac{\partial}{\partial u_2}$<br>if $\lambda = 0$ | 3, 9;<br>& 6, 7<br>if<br>$\lambda = 0$<br>(7 for $I^*$<br>only) |
| 11. | $f^1 = 0$,<br>$f^2 = \varepsilon u_2 + \ln u_1$ | $I, IV$ | $\mu u_1 \frac{\partial}{\partial u_1}$<br>$-\varepsilon \frac{\partial}{\partial u_2}$,<br>$\tilde{\psi}_\varepsilon \frac{\partial}{\partial u_2}$ | $e^{\varepsilon t} u_1 \frac{\partial}{\partial u_2}$<br>if $a = 1$ | 10,<br>$\kappa = \varepsilon$ |
| 12. | $f^1 = \lambda u_1 \ln u_1$,<br>$f^2 = \nu u_2 + \ln u_1$ | $I, IV$ | $\tilde{\psi}_\nu \frac{\partial}{\partial u_2}$ | $e^{\nu t} \Big( u_1 \frac{\partial}{\partial u_1} + t \frac{\partial}{\partial u_2} \Big)$<br>if $\nu = \lambda$; | 10,<br>$\kappa = \nu$ |
| | | | | $e^{\lambda t} \Big( (\lambda - \nu) u_1 \frac{\partial}{\partial u_1}$<br>$+ \frac{\partial}{\partial u_2} \Big)$ if $\nu \neq \lambda$ | 10,<br>$\kappa = \nu$ |
| 13. | $f^1 = \lambda u_1^{\mu+1}$,<br>$f^2 = \sigma u_1^{\mu+1}$,<br>$\lambda \sigma = 0$ | $III$ | $\mu D - 2u_1 \frac{\partial}{\partial u_1}$<br>$-2u_2 \frac{\partial}{\partial u_2}$,<br>$\tilde{\psi}_0 \frac{\partial}{\partial u_2}$ | $u_1 \frac{\partial}{\partial u_2}$<br>if $\lambda = 0$ | 3;<br>6 if<br>$\lambda = 0$ |

Here and in the following $\varepsilon = \pm 1$, $K$ is generator defined in (4.6), $\mathcal{K} = K + \frac{2}{\lambda - 1} \left[ t \left( \lambda u_1 \frac{\partial}{\partial u_1} + (2 - \lambda) u_2 \frac{\partial}{\partial u_2} \right) + u_1 \frac{\partial}{\partial u_2} \right]$. In the following table $Q = 2 \left( (\mu - a\nu) t - \frac{\nu}{2m} x^2 \right)$ for version $II$ and $Q = 2 \left( (\mu - \nu) t - \frac{\nu}{2am} x^2 \right)$, $a \neq 0$ for version $III$.

**Table 6. Non-linearities with arbitrary parameters and non-extendible symmetries for equations (1.3), (2.2)**

| No | Nonlinear terms | Type of matrix $A$ | Symmetries | AET (2.10) |
|---|---|---|---|---|
| 1. | $f^1 = \lambda u_2^{\nu+1}$, $f^2 = \mu u_2^{\nu+1}$ | $II, III$ | $\nu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2u_2 \frac{\partial}{\partial u_2}$, $\Psi_0(x) \frac{\partial}{\partial u_1}$ | 2 |
| 2. | $f^1 = \lambda \left(u_1 + u_2\right)^{\nu+1}$, $f^1 = \mu \left(u_1 + u_2\right)^{\nu+1}$ | $I, a \neq 1$ $IV$ | $\nu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2u_2 \frac{\partial}{\partial u_2}$, $\Psi_0(x) \left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right)$ | 12 |
| 3. | $f^1 = \lambda u_1^{\nu+1}$, $f^2 = u_1^{\nu} \left(\lambda u_2 + \mu u_1^{\sigma}\right)$, $\nu + \sigma \neq 0, 1, \mu \neq 0$ | $I^*$ | $\nu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2\sigma u_2 \frac{\partial}{\partial u_2}$, $u_1 \frac{\partial}{\partial u_2}$ | 6 |
| 4. | $f^1 = \lambda e^{u_2}$, $f^2 = \sigma e^{u_2}$ | $II, III$ | $D - 2\frac{\partial}{\partial u_2}$, $\Psi_0(x) \frac{\partial}{\partial u_1}$ | 2 |
| 5. | $f^1 = \lambda e^{(u_1+u_2)}$, $f^2 = \sigma e^{(u_1+u_2)}$ | $I, a \neq 1$, $IV$ | $D - 2\frac{\partial}{\partial u_2}$, $\Psi_0(x) \left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right)$ | 12 |
| 6. | $f^1 = \lambda u_2^{\nu} e^{u_1}$, $f^2 = \sigma u_2^{\nu+1} e^{u_1}$, $\nu^2 + (a-1)^2 \neq 0$ | $I, IV$ | $D - 2\frac{\partial}{\partial u_1}$, $u_2 \frac{\partial}{\partial u_2} - \nu \frac{\partial}{\partial u_1}$ | 13 if $\sigma = 0$ |
| 7. | $f^1 = \lambda e^{u_1}$, $f^2 = \sigma u_1 e^{u_1}$ | $I^*, III$ | $D - 2\frac{\partial}{\partial u_1} - 2u_1 \frac{\partial}{\partial u_2}$, $\psi_0 \frac{\partial}{\partial u_2}$ (& $u_2 \frac{\partial}{\partial u_2}$ for $I^*$) | 3; 6 if $\lambda = 0$ |
| 8. | $f^1 = \varepsilon e^{u_1}$, $\varepsilon = \pm 1$, $f^2 = \lambda u_1$ | $I, IV$ | $D + 2u_2 \frac{\partial}{\partial u_2} - 2\frac{\partial}{\partial u_1}$ $-2\lambda t \frac{\partial}{\partial u_2}$, $\tilde{\psi}_0 \frac{\partial}{\partial u_2}$ | 3 |
| 9. | $f^1 = \nu e^{\lambda\left(2u_2 - u_1^2\right)}$, $f^2 = \left(\nu u_1 + \mu\right) e^{\lambda\left(2u_2 - u_1^2\right)}$ | $I^*, III$ | $\lambda D - \frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}$ | 14 |
| 10. | $f^1 = \lambda \ln(2u_2 - u_1^2)$, $f^2 = \sigma(2u_2 - u_1^2)$ $+\lambda u_1 \ln(2u_2 - u_1^2)$ | $I^*$ | $D + 2u_1 \frac{\partial}{\partial u_1} + 4u_2 \frac{\partial}{\partial u_2}$ $+4\lambda t \left(\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}\right)$, $\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}$ | 14 |
| 11. | $f^1 = \mu \ln u_2$, $f^2 = \nu \ln u_2$ | $II, III$ | $\Psi_0(x) \frac{\partial}{\partial u_1}$, $D + 2u_1 \frac{\partial}{\partial u_1}$ $+2u_2 \frac{\partial}{\partial u_2} + Q \frac{\partial}{\partial u_1}$ | 2 |
| 12. | $f^1 = \varepsilon \ln\left(u_1 + u_2\right)$, $f^2 = \nu \ln\left(u_1 + u_2\right)$ | $I, a \neq 1$, $IV$, $a = 0$ | $\Psi_0(x) \left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right)$, $(a-1)\Big(D + 2u_1 \frac{\partial}{\partial u_1}$ $+2u_2 \frac{\partial}{\partial u_2}\Big) + (2(a\varepsilon + \nu)t$ $+\frac{\varepsilon+\nu}{m} x^2) \left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right)$ | 12 |

| | | | | |
|---|---|---|---|---|
| 13. | $f^1 = \lambda u_1^{\nu+1}$, $f^2 = \ln u_1$, $\lambda(\nu+1) \neq 0$ | $I, IV$ | $\nu\left(D + 2u_2 \frac{\partial}{\partial u_2}\right)$ $-2u_1 \frac{\partial}{\partial u_1} - 2t \frac{\partial}{\partial u_2}$, $\tilde{\psi}_0 \frac{\partial}{\partial u_2}$ | 3 |
| 14. | $f^1 = \lambda u_1^{\nu+1}$, $f^2 = \lambda u_1^{\nu+1} \ln u_1$ | $I^*, III$ | $\nu D - 2\Big(u_1 \frac{\partial}{\partial u_1}$ $+u_2 \frac{\partial}{\partial u_2} + u_1 \frac{\partial}{\partial u_2}\Big)$, $\psi_0 \frac{\partial}{\partial u_2}$ | 3 |
| 15. | $f^1 = \lambda u_1^{\nu+1}$, $f^2 = \lambda u_1^{\nu} u_2 + u_1 \ln u_1$, $\lambda(\nu-1) \neq 0$ | $I^*$ | $\nu D - 2u_1 \frac{\partial}{\partial u_1}$ $-2tu_1 \frac{\partial}{\partial u_2}$ $-2(1-\nu)u_2 \frac{\partial}{\partial u_2}, \quad u_1 \frac{\partial}{\partial u_2}$ | 6 |
| 16. | $f^1 = \lambda\left(2u_2 - u_1^2\right)^{\nu+\frac{1}{2}}$, $f^2 = \lambda u_1\big(2u_2 - u_1^2\big)^{\nu+\frac{1}{2}}$ $+\mu\left(2u_2 - u_1^2\right)^{\nu+1}$ | $I^*$ | $\nu D - u_1 \frac{\partial}{\partial u_1} - 2u_2 \frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}$ $\Big(\& \ 2\lambda t \left(\frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}\right)$ $+ \frac{\partial}{\partial u_2}$ if $\mu = 0, \ \nu = \frac{1}{2}\Big)$ | $14; 1$, $\rho = 2\omega$ if $\nu = 0$ |
| 17. | $f^1 = 2\nu u_1 \ln u_1 + u_1 u_2$, $f^2 = -(\nu-\mu)^2 \ln u_1$ $+2\mu u_2$ | $I, IV$ | $X = e^{(\mu+\nu)t}\Big(u_1 \frac{\partial}{\partial u_1}$ $+(\mu-\nu)\frac{\partial}{\partial u_2}\Big)$, $tX + e^{(\mu+\nu)t} \frac{\partial}{\partial u_2}$ | $10$, $\kappa = 2\nu$ if $\mu + \nu$ $= 0$ |
| 18. | $f^1 = 2\nu u_1 \ln u_1 + u_1 u_2$, $f^2 = 2\mu u_2$ $+\left(1 - (\nu-\mu)^2\right) \ln u_1$ | $I, IV$ | $X^{\pm} = e^{\lambda_{\pm} t}\Big(u_1 \frac{\partial}{\partial u_1}$ $+(\lambda_{\pm} - 2\nu)\frac{\partial}{\partial u_2}\Big)$, $\lambda_{\pm} = \mu + \nu \pm 1$ | $10$, $\kappa = 2\nu$ if $\mu + \nu$ $= \pm 1$ |
| 19. | $f^1 = 2\nu u_1 \ln u_1 + u_1 u_2$, $f^2 = 2\mu u_2$ $-\left(1 + (\nu-\mu)^2\right) \ln u_1$ | $I, IV$ | $e^{(\mu+\nu)t}\Big[\cos t u_1 \frac{\partial}{\partial u_1}$ $-(\sin t + (\nu$ $-\mu)\cos t)\frac{\partial}{\partial u_2}\Big]$, $e^{(\mu+\nu)t}\Big[\sin t u_1 \frac{\partial}{\partial u_1}$ $+(\cos t + (\mu$ $-\nu)\sin t)\frac{\partial}{\partial u_2}\Big]$ | |
| 20. | $f^1 = \varepsilon(2u_2 - u_1^2)$, $f^2 = (\mu + \varepsilon u_1)\left(2u_2 - u_1^2\right)$ $-\frac{\mu^2}{2}\varepsilon u_1, \quad \mu \neq 0$ | $I^*, III$ | $X_1 = e^{\mu t}\Big(2\frac{\partial}{\partial u_1}$ $+2u_1 \frac{\partial}{\partial u_2} + \varepsilon\mu \frac{\partial}{\partial u_2}\Big)$, $tX_1 + \varepsilon e^{\mu t} \frac{\partial}{\partial u_2}$ | |
| 21. | $f^1 = \varepsilon(2u_2 - u_1^2)$, $f^2 = (\mu + \varepsilon u_1)\left(2u_2 - u_1^2\right)$ $+\frac{1-\mu^2}{2}\varepsilon u_1$ | $I^*, III$ | $X^{\pm} = e^{\mu \pm 1}\Big(2\frac{\partial}{\partial u_1}$ $+2u_1 \frac{\partial}{\partial u_2}$ $+\varepsilon(\mu \pm 1)\frac{\partial}{\partial u_2}\Big)$ | 14 if $\mu^2 = 1$ |
| 22. | $f^1 = \varepsilon(2u_2 - u_1^2)$, $f^2 = -\frac{1+\mu^2}{2}\varepsilon u_1$ $+(\mu + \varepsilon u_1)\left(2u_2 - u_1^2\right)$ | $I^*, III$ | $e^{\mu t}\Big(2\varepsilon \cos t\Big(\frac{\partial}{\partial u_1}$ $+u_1 \frac{\partial}{\partial u_2}\Big) + (\mu \cos t$ $-\sin t)\frac{\partial}{\partial u_2}\Big)$, $e^{\mu t}\Big(2\varepsilon \sin t\Big(\frac{\partial}{\partial u_1}$ $+u_1 \frac{\partial}{\partial u_2}\Big) + (\mu \sin t$ $+\cos t)\frac{\partial}{\partial u_2}\Big)$ | |

**Table 7. Symmetries of equations (1.3) with diagonal matrix $A$ and non-linearities**

$$f^1 = u_1\left(\mu \ln u_1 + \lambda \ln u_2\right), \;\; f^2 = u_2\left(\nu \ln u_2 + \sigma \ln u_1\right)$$

| No | Conditions for coefficients and notations | Main symmetries | Additional symmetries |
|---|---|---|---|
| 1. | $\lambda = 0$, $\mu = \nu$ | $e^{\mu t} u_2 \frac{\partial}{\partial u_2}$, $e^{\mu t}\left(u_1 \frac{\partial}{\partial u_1} + \sigma t u_2 \frac{\partial}{\partial u_2}\right)$ | $\widehat{G}_\alpha$, if $a \neq 0$, $\sigma = 0$, $\mu \neq 0$ |
| 2. | $\lambda = 0$, $\mu \neq \nu$ | $e^{\nu t} u_2 \frac{\partial}{\partial u_2}$ , $e^{\mu t}\left((\mu - \nu) u_1 \frac{\partial}{\partial u_1} + \sigma u_2 \frac{\partial}{\partial u_2}\right)$ | $\widehat{G}_\alpha$ if $\mu \neq 0$, $\mu - \nu = a\sigma$ |
| | | | $G_\alpha$ if $a\sigma = -\nu$, $\mu = 0$; |
| | | | $\psi_0 \frac{\partial}{\partial u_2}$ if $\sigma = \nu = 0$; |
| | | | $\psi_0 \frac{\partial}{\partial u_1}$ if $\sigma = \mu = 0$; |
| | | | $u_1 \frac{\partial}{\partial u_2}$, $\widehat{G}_\alpha$ if $a = 1$, $\nu = 0$ $\mu = \sigma \neq 0$ |
| 3. | $\delta = \frac{1}{4}(\mu - \nu)^2 + \lambda\sigma = 0$, $\mu + \nu = 2\omega_0$ | $X_2 = e^{\omega_0 t}\left(2\lambda u_1 \frac{\partial}{\partial u_1} + (\nu - \mu) u_2 \frac{\partial}{\partial u_2}\right)$, | $\widehat{G}_\alpha$ if $\nu \neq -\mu$, $2\lambda = a(\nu - \mu)$ |
| | $\lambda\sigma \neq 0$ | $e^{\omega_0 t} 2u_2 \frac{\partial}{\partial u_2} + tX_2$ | $G_\alpha$ if $\lambda = a\nu$ $\mu = -\nu \neq 0$ |
| 4. | $\lambda\sigma \neq 0$, $\delta = 1$, $\omega_\pm = \omega_0 \pm 1$ | $e^{\omega_+ t}\left(\lambda u_1 \frac{\partial}{\partial u_1} + (\omega_+ - \mu) u_2 \frac{\partial}{\partial u_2}\right)$, $e^{\omega_- t}\left(\lambda u_1 \frac{\partial}{\partial u_1} + (\omega_- - \mu) u_2 \frac{\partial}{\partial u_2}\right)$ | $\widehat{G}_\alpha$ if $\mu\nu \neq \lambda\sigma$, $\lambda = a(\nu - \mu + a\sigma)$ |
| | | | $G_\alpha$ if $\nu\mu = \lambda\sigma$, $\lambda = -a\mu$ |
| 5. | $\delta = -1$ | $e^{\omega_0 t}\left(2\lambda \cos t u_1 \frac{\partial}{\partial u_1} + ((\nu - \mu)\cos t - 2\sin t) u_2 \frac{\partial}{\partial u_2}\right)$, $e^{\omega_0 t}\left(2\lambda \sin t u_1 \frac{\partial}{\partial u_1} + ((\nu - \mu)\sin t + 2\cos t) u_2 \frac{\partial}{\partial u_2}\right)$ | none |

Equations (1.3) with the nonlinearities present in Table 7 admit equivalence transformation 1 from the list (2.10) provided $\mu\nu = \lambda\sigma$. The related parameters $\rho$ and $\omega$ should satisfy $\mu\omega + \lambda\rho = 0$. In addition, the equations corresponding to the last version enumerated in Item 2 admit additional equivalence transformation 6 given by formula (2.10).

**Table 8. Symmetries of equations (1.3) with matrix $A$ of type $I^*, II$ and non-linearities $f^1 = (\mu u_1 - \sigma u_2)\ln R + z(\lambda u_1 - \nu u_2)$, $f^2 = (\mu u_2 + \sigma u_1)\ln R + z(\lambda u_2 + \nu u_1)$**

| No | Conditions for coefficients | Main symmetries | Additional symmetries |
|---|---|---|---|
| 1. | $\lambda = 0,$ $\mu = \nu$ | $e^{\mu t}\frac{\partial}{\partial z},$ $e^{\mu t}\left(R\frac{\partial}{\partial R} + \sigma t\frac{\partial}{\partial z}\right)$ | For $II:\ \hat{G}_\alpha$ if $a\sigma = 0,\ \mu \neq 0$ |
| | | | For $II:\ G_\alpha$ if $a = \nu = 0,\ \sigma \neq 0$ |
| | | | For $I^*:\ \widehat{G}_\alpha$ if $\sigma = 0,\ \mu \neq 0$ |
| 2. | $\lambda = 0,$ $\mu \neq \nu,$ | $e^{\nu t}\frac{\partial}{\partial z},$ $e^{\mu t}\left(\sigma\frac{\partial}{\partial z} + (\mu - \nu) R\frac{\partial}{\partial R}\right)$ | For $II:\ \hat{G}_\alpha$ if $a\sigma = \nu - \mu,$ $\mu \neq 0$ or $a = 0,\ \mu \neq 0$ |
| | | | For $II:\ G_\alpha$ if $a\sigma = \nu,\ \mu = 0$ |
| | | | For $I^*:\ \widehat{G}_\alpha$ if $\mu \neq 0,\ \sigma = 0$ |
| | | | For $I^*:\ G_\alpha$ if $\mu = 0,\ \sigma = 0$ |
| 3. | $\delta = 0,$ $\lambda \neq 0$ | $X_3 = e^{\omega_0 t}\left(2\lambda R\frac{\partial}{\partial R} + (\nu - \mu)\frac{\partial}{\partial z}\right),$ | For $II:\ \widehat{G}_\alpha$ if $\mu \neq -\nu$ $a(\mu - \nu) = 2\lambda$ |
| | | | For $II:\ G_\alpha$ if $a\nu = -\lambda,\ \omega_0 = 0$ |
| | | | For $I^*:\ \widehat{G}_\alpha$ if $\mu = \nu \neq 0$ |
| | | | For $I^*:\ G_\alpha$ if $\mu = \nu = 0$ |
| 4. | $\lambda \neq 0,$ $\delta = 1$ | $e^{\omega_+ t}\left(\lambda R\frac{\partial}{\partial R} + (\omega_+ - \mu)\frac{\partial}{\partial z}\right),$ $e^{\omega_- t}\left(\lambda R\frac{\partial}{\partial R} + (\omega_- - \mu)\frac{\partial}{\partial z}\right)$ | $\widehat{G}_\alpha$ if $\mu\nu \neq \lambda\sigma,$ $\lambda = a(\nu - \mu + a\sigma)$ |
| | | | $G_\alpha$ if $\nu\mu = \lambda\sigma,\ \lambda = -a\mu$ |
| | | | For $I^*:\ \widehat{G}_\alpha$ if $\sigma = 0,\ \mu \neq 0$ |
| | | | For $I^*:\ G_\alpha$ if $\sigma = \mu = 0$ |
| 5. | $\delta = -1$ | $\exp(\omega_0 t)\left[2\lambda \cos t R\frac{\partial}{\partial R} + ((\nu - \mu)\cos t - 2\sin t)\frac{\partial}{\partial z}\right],$ $\exp(\omega_0 t)\left[2\lambda \sin t R\frac{\partial}{\partial R} + ((\nu - \mu)\sin t + 2\cos t)\frac{\partial}{\partial z}\right]$ | none |

All equations enumerated in Table 8 admit additional equivalence transformations 15 from the list (2.10).

**Table 9. Symmetries of equations (1.3) with non-linearities $f^1 = \lambda u_2 + \mu u_1 \ln u_1$, $f^2 = \lambda \frac{u_2^2}{u_1} + (\sigma u_1 + \mu u_2) \ln u_1 + \nu u_2$ and matrices $A$ of type $III$ (and $I^*$ if $a = 0$)**

| No | Conditions for coefficients | Main symmetries | Additional symmetries |
|---|---|---|---|
| 1. | $\lambda = 0,$ $\mu \neq \nu$ | $e^{\nu t} u_1 \frac{\partial}{\partial u_2},$ | $\psi_\nu \frac{\partial}{\partial u_2}$ if $\mu = 0,$ & $G_\alpha$ if $a\nu = \sigma \neq 0$ |
| | | $e^{\mu t}\left((\mu-\nu) R \frac{\partial}{\partial R} + \sigma u_1 \frac{\partial}{\partial u_2}\right)$ | $\widehat{G}_a,$ if $\mu \neq 0,$ $\sigma = a(\nu - \mu) \neq 0$ |
| 2. | $\lambda = 0, \mu = \nu$ | $e^{\mu t} u_1 \frac{\partial}{\partial u_2},$ $e^{\mu t}\left(R \frac{\partial}{\partial R} + \sigma t u_1 \frac{\partial}{\partial u_2}\right)$ | $\psi_0 \frac{\partial}{\partial u_2}$ if $\mu = 0,$ $\sigma \neq 0$ & $D + u_2 \frac{\partial}{\partial u_2}$ if $a = 0$ |
| | | | $\widehat{G}_a$ if $\sigma = 0,\ \mu \neq 0$ |
| 3. | $\sigma = 0,$ $\mu\lambda \neq 0,$ | $e^{\nu t}\left(\lambda R \partial_R + (\mu - \nu) u \partial_v\right)$ | $G_a$ if $\nu = 0,$ $\mu = -\lambda$ |
| | $\mu \neq \nu, a = 1$ | $e^{\mu t} R \partial_R$ | $\hat{G}_a$ if $\nu - \mu = \lambda$ |
| 4. | $\delta = 0$ $\mu + \nu = 2\omega_0,$ $\lambda \neq 0$ | $X_4 = e^{\omega_0 t}\big(2\lambda R \frac{\partial}{\partial R}$ $+(\nu - \mu) u_1 \frac{\partial}{\partial u_2}\big),$ | $G_a$ if $\omega_0 = 0,$ $\nu = -a\lambda$ & $D + 2u_1 \frac{\partial}{\partial u_1}$ if $a = 0$ |
| | | $2e^{\omega_0 t} u_1 \frac{\partial}{\partial u_2} + t X_4$ | $\widehat{G}_a,$ if $\omega_0 \neq 0,$ $2a\lambda = \mu - \nu$ |
| 5. | $\lambda \neq 0,$ $\delta = 1,$ | $e^{\omega_+ t}\left(\lambda R \frac{\partial}{\partial R} + (\omega_+ - \mu) u_1 \frac{\partial}{\partial u_2}\right),$ | $G_a,$ if $\mu = a\lambda,$ $\mu\nu = \lambda\sigma$ |
| | $\omega_\pm = \omega_0 \pm 1$ | $e^{\omega_- t}\left(\lambda R \frac{\partial}{\partial R} + (\omega_- - \mu) u_1 \frac{\partial}{\partial u_2}\right)$ | $\widehat{G}_\alpha,$ if $\mu\nu \neq \lambda\sigma,$ $\mu - \nu = \lambda - \sigma,$ $a = 1$ or $\sigma = a = 0,\ \mu \neq 0$ |
| 6. | $\delta = -1,$ | $e^{\omega_0 t}[2\lambda \cos t R \frac{\partial}{\partial R}$ $+((\nu - \mu)\cos t - 2\sin t) u_1 \frac{\partial}{\partial u_2}],$ $e^{\omega_0 t}[2\lambda \sin t R \frac{\partial}{\partial R}$ $+((\nu - \mu)\sin t + 2\cos t) u_1 \frac{\partial}{\partial u_2}]$ | none |

If $\lambda = \mu = 0$ or $\lambda = \nu = 0$ then the related equation (1.3) admits additional equivalence transformations 16 or 6 from the list (2.10) correspondingly.

Tables 3–9 present results of group classification of equations (1.3) with invertible diffusion matrix $A$. The results present in Tables 3–7 are valid for equations with the singular matrix $A$ of type $IV$ also but do not exhaust all non-equivalent non-linearities for such equations. Moreover, the equations with singular diffusion matrix admit strong equivalence transformations $u_1 \to u_1$, $u_2 \to \varepsilon(u_2)$ where $\varepsilon(u_2)$ is an arbitrary function of $u_2$ which reduce the number of non-equivalent symmetries in Tables 3–9 for $a = 0$.

The completed group classification of equations (1.3) with matrix $A$ of type $IV$ is given in paper [21]

## 11. Classification of Reaction-Diffusion Equations with Nilpotent Diffusion Matrix

To complete the classification of systems (1.3) we need to consider the remaining class of these equations when matrix $A$ belongs to type $V$, i.e., is nilpotent. The procedure of classification of such equations appears to be more complicated then in the case of invertible or diagonalizable diffusion matrix. The general form of symmetry admitted by this equation is given by equation (4.13) while the classifying equations take the form (4.14).

A specific feature of symmetries (4.13) is that the coefficient $B^3$ can be a function of $u_1$. One more specific point in the classification of equations with matrix $A$ of type $V$ is that they admit powerful equivalence relations

$$u_1 \to u_1, \; u_2 \to u_2 + \Phi(u_1) \tag{11.1}$$

and

$$u_1 \to u_1, \; u_2 \to u_2 + \hat{\Phi}(u_1, t, x) \tag{11.2}$$

which did not appear in our analysis presented in the previous sections.

Transformation (11.1) (where $\Phi(u_1)$ is an arbitrary function of $u_1$) are admitted by any equation (1.3) with matrix $A$ of type $V$. Transformations (11.2) are valid for the cases when $f^1$ does not depend on $u_2$ and at the same time $f^2$ is linear in $u_2$. Moreover, the related functions $\hat{\Phi}(u_1, t, x)$ should satisfy the following system of equations

$$\begin{aligned} &f^2_{u_2}\hat{\Phi}_t - \hat{\Phi}_{tt} - f^1\hat{\Phi}_{tu_1} = 0, \\ &f^2_{u_2}\hat{\Phi}_{x_\nu} - \hat{\Phi}_{tx_\nu} - f^1\hat{\Phi}_{u_1x_\nu} = 0 \end{aligned} \tag{11.3}$$

Thus the group classification of equation (1.3) with the nilpotent diffusion matrix is reduced to solving the classifying equations (4.14) with applying the equivalence transformations discussed in Section 2 and transformations (11.1), (11.2) as well. To do this we again use the analysis of low dimension algebras $\mathcal{A}$ whose results are given in the Appendix. We will not reproduce the related routine calculations but present the classification results in Tables 8–10.

In Tables 8–10 we use without explanations all the notations applied in Tables 1–9. In addition, a number of classified equations appear a specific symmetry $W\partial_{u_2}$ where $W$ is a function of $t, x$ and $u_1$ which solve the following equation:

$$f^2_{u_2} - W_t - W_{u_1} f^1 = 0.$$

**Table 10. Non-linearities with arbitrary functions for equations (1.3) with nilpotent diffusion matrix**

| No | Nonlinear terms | Arguments of $F_\alpha$ | Symmetries |
|---|---|---|---|
| 1. | $f^1 = F_1 u_1^{\mu-\nu}$, $f^2 = F_2 u_1^\mu$ | $\frac{u_1^{\nu+1}}{u_2}$ | $Q_1 = (\mu-1)D - \nu t\frac{\partial}{\partial t}$ $-u_1\frac{\partial}{\partial u_1} - (\nu+1)u_2\frac{\partial}{\partial u_2}$ & $(m-2)x^2\frac{\partial}{\partial x_a} - x_a Q_1$ if $\nu(m-2)=4$, $\mu(m-2) = m+2,\ m \neq 2$ |
| 2. | $f^1 = F_1 u_1 u_2^{\mu-1}$, $f^2 = F_2 u_2^\mu$, $F_2 \neq 0$ | $u_1$ | $\mu D - t\frac{\partial}{\partial t} - u_2\frac{\partial}{\partial u_2}$ & $e^W\frac{\partial}{\partial u_2}$ if $\mu = 1$ & $H^a\frac{\partial}{\partial x_a} - H^b_{x_b} u_2\frac{\partial}{\partial u_2}$ if $m = 2$ |
| 3. | $f^1 = F_1 u_2^{-1}$, $f^2 = F_2 + \nu u_2$ | $u_1$ | $e^{\nu t}\left(\frac{\partial}{\partial t} + \nu u_2\frac{\partial}{\partial u_2}\right)$ & $e^W\frac{\partial}{\partial u_2}$ if $F_1 = 0$ |
| 4. | $f^1 = F_1 u_2^{\mu-1}$, $f^2 = F_2 u_2^\mu$ | $u_2 e^{u_1}$ | $\mu D - t\frac{\partial}{\partial t} - u_2\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_1}$ |
| 5. | $f^1 = \frac{F_1}{u_2} + \nu$, $f^2 = F_2 + \nu u_2$ | $u_2 e^{u_1}$ | $e^{\nu t}\left(\frac{\partial}{\partial t} + \nu u_2\frac{\partial}{\partial u_2} - \nu\frac{\partial}{\partial u_1}\right)$ |
| 6. | $f^1 = 0,\ f^2 = F_2$ | $u_2$ | $\Psi_0(x)\frac{\partial}{\partial u_1}$, $x_a\frac{\partial}{\partial x_a} + 2u_1\frac{\partial}{\partial u_1}$ |
| 7. | $f^1 = F_1,\ f^2 = 0$ | $u_1$ | $e^W\frac{\partial}{\partial u_2}$, $x_a\frac{\partial}{\partial x_a} - 2u_2\frac{\partial}{\partial u_2}$ |
| 8. | $f^1 = \frac{\nu}{\mu-1}u_1 + F_1 u_1^{2-\mu}$, $f^2 = \frac{\mu\nu}{\mu-1}u_2 + F_2 u_1$, $\mu \neq 1$ | $u_2 u_1^{-\mu}$ | $e^{\nu t}\Big((1-\mu)t\frac{\partial}{\partial t} - \nu u_1\frac{\partial}{\partial u_1}$ $-\nu\mu u_2\frac{\partial}{\partial u_2}\Big)$ |

| | | | |
|---|---|---|---|
| 9. | $f^1 = u_1F_1,\ m = 1$<br>$f^2 = u_2F_2 + u_1$ | $u_2u^3$ | $Q_2 = \cos(2x)\left(u_1\frac{\partial}{\partial u_1} - 3u_2\frac{\partial}{\partial u_2}\right)$<br>$+\sin(2x)x\frac{\partial}{\partial x},\quad Q_3 = (Q_2)_x$ |
| 10. | $f^1 = u_1F_1,\ m = 1$<br>$f^2 = u_2F_2 - u_1$ | $u_2u_1^3$ | $Q_4 = e^{2x}\left(\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_1} - 3u_2\frac{\partial}{\partial u_2}\right),$<br>$Q_5 = e^{-2x}\left(\frac{\partial}{\partial x} - u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2}\right)$ |
| 11. | $f^1 = F_1,$<br>$f^2 = u_2F_2,$<br>$m = 2$ | $u_2e^{u_1}$ | $H^a\frac{\partial}{\partial x_a} - H^b_{x_b}\left(u_2\frac{\partial}{\partial u_2} - \frac{\partial}{\partial u_1}\right)$ |
| 12. | $f^1 = \nu e^{\frac{u_2}{u_1}},$<br>$f^2 = e^{\frac{u_2}{u_1}}F$ | $u_1$ | $D - u_1\frac{\partial}{\partial u_2}$ |
| 13. | $f^1 = F_1,$<br>$F_2 = u_2F_2 + F_3$ | $u_1$ | $e^W\frac{\partial}{\partial u_2}$ |
| 14. | $f^1 = e^{\nu u_2}F_1,$<br>$f^2 = e^{\nu u_2}F_2$ | $u_1$ | $\nu D - \frac{\partial}{\partial u_2}$ |
| 15. | $f^1 = e^{\nu u_1}F_1,$<br>$f^2 = e^{\nu u_1}F_2$ | $u_2$ | $\nu D - \frac{\partial}{\partial u_1}$ |
| 16. | $f^1 = \nu u_1 + F_1,$<br>$f_2 = \mu u_1 + F_2$ | $u_2$ | $e^{(\nu-a\mu)t}\Psi_\mu(x)\frac{\partial}{\partial u_1}$ |
| 17. | $f^1 = u_1(F_1 + \nu\ln u_1),$<br>$f^2 = u_2(F_2 + \nu\ln u_1),$<br>$\nu \neq 0$ | $\frac{u_1}{u_2}$ | $e^{\nu t}(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2})$ |
| 18. | $f^1 = u_1F_1 - \nu u_2,$<br>$f^2 = \nu\frac{u_2}{u_1}(u_2 - u_1)$<br>$+u_1F_2 - u_2F_1$ | $u_1e^{\frac{u_2}{u_1}}$ | $e^{\nu t}\left(u_1\frac{\partial}{\partial u_2} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}\right)$<br>$\&\ \widehat{G}_\alpha$ if $a = 1$ |
| 19. | $f^1 = u_1^{\nu+1}F_1,$<br>$f^2 = u_1^\nu\,(F_2u_1 - F_1u_2)$ | $u_1e^{\frac{u_2}{u_1}}$ | $\nu D + u_1\frac{\partial}{\partial u_2} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$<br>$\&\ G_\alpha$ if $\nu = 0,\ a = 1$ |
| 20. | $f^1 = u_1^{\mu+1}F_1,$<br>$f^2 = u_1^{\mu+1}F_2$ | $\frac{u_2}{u_1}$ | $\mu D - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}$ |

For the non-linearities enumerated in Items 2 (when $\mu = 1$), 3 (when $F_1 = 0$), 4 and 8 of Table 8 the related equation (1.3) admits additional equivalence transformations (11.2). In addition, transformations (2.4) and (11.1) and some equivalence transformations from the list (2.10) are admissible, namely, transformations 9 for the non-linearities given in Item 1 (when $\nu = -1, \mu = 0$) and Item 6, Item 6,Item 18, Item 19 and Item 20 transformations 1 with $\rho = \omega$ for the non-linearities from Item I (when $\nu = 1, \mu = 0$), Item 20 (when $\mu = 0$) and Items 18, 19. Finally for $f^1$ and $f^2$ present in Item 7 transformation 3 of (2.10) is admissible.

**Table 11. Non-linearities with arbitrary parameters and extendible symmetries for equations (1.3) with nilpotent diffusion matrix**

| No | Non-linearities | Main symmetries | Additional symmetries | AET (2.10) |
|---|---|---|---|---|
| 1. | $f^1=\lambda u_1^{\nu+1}u_2^{\mu}$, $f^2=\sigma u_1^{\nu}u_2^{\mu+1}$ | $(\mu+\nu)t\frac{\partial}{\partial t}$ $-(\mu+1)u_1\frac{\partial}{\partial u_1}$ $+(\nu-1)u_2\frac{\partial}{\partial u_2}$ | $x_aQ_6-2\kappa x^2\frac{\partial}{\partial x_a}$ if $\kappa(m+2)=\nu$, $\kappa(2-m)=\mu$ | |
| | | $Q_6=2\mu u_1\frac{\partial}{\partial u_1}$ $+(\mu+\nu)x_a\frac{\partial}{\partial x_a}$ $-2\nu u_2\frac{\partial}{\partial u_2}$ | $e^W\frac{\partial}{\partial u_2}$ if $\lambda=0$, $\mu=-1$ & $2x^2\frac{\partial}{\partial x_a}$ $-(m-2)x_aQ_6$ if $\nu=\frac{m+2}{m-2}$, $m\neq 2$ | 17, $3,6$ |
| | | | $\Psi_0(x)\frac{\partial}{\partial u_1}$, if $\sigma=0,\nu=-1$, & $x_aQ_6+\frac{2}{m+2}x^2\frac{\partial}{\partial x_a}$ if $\mu=\frac{m-2}{m+2}$, $m\neq 2$ | 17, 9 |
| | | | $\Psi_0(x)\frac{\partial}{\partial u_1}$ if $\lambda=\nu=0$ | 17, 9 |
| | | | $e^W\frac{\partial}{\partial u_2}$ if $\mu=0$ & $H^a\frac{\partial}{\partial x_a}-\frac{\partial H^a}{\partial x_a}u_2\frac{\partial}{\partial u_2}$ if $m=2$ | 17 & 6 if $\lambda=\sigma$ |
| 2. | $f^1=\lambda u_1^{\nu+1}u_2^{-1}$, $f^2=\sigma u_1^{\nu}+\varepsilon u_2$, $\lambda\neq 0$ | $e^{\varepsilon t}\left(\frac{\partial}{\partial t}+\varepsilon u_2\frac{\partial}{\partial u_2}\right)$, $Q'_6=Q_6\vert_{\mu=-1}$ | $x_aQ'_6-\frac{2}{m-2}x^2\frac{\partial}{\partial x_a}$ if $\nu=\frac{m+2}{m-2}$, $m\neq 2$ | |
| | | | $\Psi_0(x)\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=-1$ | 9 |
| | | | $u_2\frac{\partial}{\partial u_2}+u_1\frac{\partial}{\partial u_1}$ if $\sigma=0$, $\nu=1$ | 1 |
| 3. | $f^1=\lambda e^{\nu u_1}$, $f^2=\sigma e^{(\nu+1)u_1}$, $\lambda\sigma=0$ | $(\nu+1)D-u_2\frac{\partial}{\partial u_2}$ $-t\frac{\partial}{\partial t}-\frac{\partial}{\partial u_1}$, $e^W\frac{\partial}{\partial u_2}$ | $\nu\left(u_2\frac{\partial}{\partial u_2}+t\frac{\partial}{\partial t}\right)$ $-\frac{\partial}{\partial u_1}$ if $\sigma=0$ | 17, 3 |
| | | | $u_2\frac{\partial}{\partial u_2}+t\frac{\partial}{\partial t}$ if $\lambda=0$ | 10, $3,6$ |
| 4. | $f^1=\lambda e^{(\nu+1)u_2}$, $f^2=\sigma e^{\nu u_2}$ | $(\nu-1)D-u_1\frac{\partial}{\partial u_1}$ $+t\partial_t-\frac{\partial}{\partial u_2}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$ | $\nu\left(u_1\frac{\partial}{\partial u_1}-t\frac{\partial}{\partial t}\right)$ $+\frac{\partial}{\partial u_2}$ if $\lambda=0$ | 9 for any $\lambda$ |
| 5. | $f^1=\lambda u_2^{\mu-1}e^{u_1}$, $f^2=\sigma u_2^{\mu}e_1^{u}$ | $D-\frac{\partial}{\partial u_1}$, $t\partial_t+u_2\frac{\partial}{\partial u_2}-\mu\frac{\partial}{\partial u_1}$ | $e^W$ if $\mu=1$ & $H^a\frac{\partial}{\partial x_a}-\frac{\partial H^a}{\partial x_a}u_2\frac{\partial}{\partial u_2}$ if $m=2$ | 17 if $\mu=1$ |
| 6. | $f^1=\lambda\ln u_2$, $f^2=\sigma u_2^{\frac{\mu+1}{2}}$ | $\mu D-\frac{\mu+1}{2}t\frac{\partial}{\partial t}$ $-\frac{1-\mu}{2}u_1\frac{\partial}{\partial u_1}$ $-u_2\frac{\partial}{\partial u_2}-\lambda t\frac{\partial}{\partial u_1}$, $\Psi_0(x)\frac{\partial}{\partial u_1}$ | $x_a\frac{\partial}{\partial x_a}+2u_1\frac{\partial}{\partial u_1}$, $u_1\frac{\partial}{\partial u_1}+2u_2\frac{\partial}{\partial u_2}$ $+t\partial_t+2\lambda t\frac{\partial}{\partial u_1}$ if $\sigma=0$ | 9 |

**Table 12. Non-linearities with arbitrary parameters and non extendible symmetries for equations (1.3) with $a = 0$**

| No | Non-linearities | Condi-tions | Symmetries | AET (2.10) |
|---|---|---|---|---|
| 1. | $f^1 = \lambda u_1^{3\mu+1} u_2^{\mu}$, $f^2 = \sigma u_1^{3\mu} u_2^{\mu+1} - \alpha u_1$, | $\mu \neq 0$, $m = 1$, $\alpha = -1$ | $Q_7 = 4\mu t \frac{\partial}{\partial t}$ $-(\mu+1) u_1 \frac{\partial}{\partial u_1}$ $+(3\mu - 1) u_2 \frac{\partial}{\partial u_2}$, $Q_2, Q_3$ | |
| | | $\mu \neq 0$, $m = 1$, $\alpha = 1$ | $Q_4, Q_5, Q_7$ | |
| 2. | $f^1 = \lambda u_1^{-2} u_2^{-1}$, | $m = 1$, $\alpha = -1$ | $e^{\varepsilon t} \left( \frac{\partial}{\partial t} + \varepsilon u_2 \frac{\partial}{\partial u_2} \right)$, $Q_2,\ Q_3$ | 17 if |
| | $f^2 = \sigma u_1^{-3} + \varepsilon u_2 - \alpha u_1$ | $m = 1$, $\alpha = 1$ | $e^{\varepsilon t} \left( \frac{\partial}{\partial t} + \varepsilon u_2 \frac{\partial}{\partial u_2} \right)$, $Q_4,\ Q_5$ | $\lambda = 0$ |
| 3. | $f^1 = \lambda u_2^{\mu+1}$, $f^2 = \sigma u_2^{\mu-\nu+1}$ | $\mu \neq -1$ | $(\mu - 2\nu) D + \nu t \frac{\partial}{\partial t}$ $-u_2 \frac{\partial}{\partial u_2} - (\nu+1) u_1 \frac{\partial}{\partial u_1}$, $\Psi_0(x) \frac{\partial}{\partial u_1}$ | 9 |
| 4. | $f^1 = \lambda u_2,\ f^2 = e^{-u_2}$ | $\lambda \neq 0$ | $2D - t\partial_t + u_1 \frac{\partial}{\partial u_1}$ $+ \frac{\partial}{\partial u_2} + \lambda t \frac{\partial}{\partial u_1}$, $\Psi_0(x) \frac{\partial}{\partial u_1}$ | 9 |
| 5. | $f^1 = \lambda e^{u_2}$, $f^2 = \sigma e^{u_2}$ | $\lambda\sigma \neq 0$ | $D - \frac{\partial}{\partial u_2},\ \Psi_0(x) \frac{\partial}{\partial u_1}$ | 9 |
| 6. | $f^1 = \lambda u_1^{\nu+1} e^{\mu \frac{u_2}{u_1}}$, $f^2 = e^{\mu \frac{u_2}{u_1}} (\lambda u_2 + \sigma u_1) u_1^{\nu}$ | $\mu\lambda \neq 0$ | $\mu D - u_1 \frac{\partial}{\partial u_2}$, $\nu D - u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2}$ | |
| 7. | $f^1 = \mu \ln u_2$, $f^2 = \nu \ln u_2$ | $\nu \neq 0$ | $\Psi_0(x) \frac{\partial}{\partial u_1}$, $D + u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2}$ $+ \left( \mu t - \frac{\nu}{2m} x^2 \right) \frac{\partial}{\partial u_1}$ | 9 |
| 8. | $f^1 = 0,\ f^2 = \varepsilon \ln u_1$ | $\varepsilon = \pm 1$ | $D - t\partial_t + u_1 \frac{\partial}{\partial u_1}$ $+\varepsilon t \frac{\partial}{\partial u_2},\ t\partial_t + u_2 \frac{\partial}{\partial u_2}$, $\Phi(u_1, x) \frac{\partial}{\partial u_2}$ | $3, 6,$ $17$ |
| 9. | $f^1 = \varepsilon (\ln u_2 - \kappa \ln u_1) u_1$, $f^2 = \varepsilon (\ln u_2 - \kappa \ln u_1) u_2$ | $m \neq 2$, $\kappa \neq \frac{m+2}{m-2}$ | $(1-\kappa) x_a \frac{\partial}{\partial x_a}$ $+2\kappa u_2 \frac{\partial}{\partial u_2} + 2u_1 \frac{\partial}{\partial u_1}$, $e^{(1-\kappa)\varepsilon t} \left( u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right)$ | 1, $\rho = \omega$ if $\kappa = 1$ |
| 10. | $f^1 = \varepsilon u_1 \left( (m+2) \ln u_1 \right.$ $\left. +(2-m) \ln u_2 \right)$, | $m \neq 1, 2$ $\alpha = 0$ | $Q_1,\ x_a Q_1 - x^2 \partial x_a$, $e^{4\varepsilon t} \left( u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right)$ | |
| | $f^2 = \varepsilon u_2 \left( (m+2) \ln u_1 \right.$ $\left. +(2-m) \ln u_2 \right) - \alpha u_1$ | $m = 2$, $\alpha = 0$ | $H^a \frac{\partial}{\partial x_a} - H^a_{x_a} u_2 \frac{\partial}{\partial u_2}$, $e^{4\varepsilon t} \left( u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \right)$ | |

| | | | | |
|---|---|---|---|---|
| | | $m=1$, $\alpha=1$, $\varepsilon=1$ | $Q_2,\ Q_3$, $e^{4t}\left(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}\right)$ | |
| | | $m=1$, $\alpha=1$, $\varepsilon=-1$ | $Q_4,\ Q_5$, $e^{-4t}\left(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}\right)$ | |
| 11. | $f^1=\mu u_1\ln u_1$, $f^2=\mu u_2\ln u_1+\nu u_2$ | $\mu\neq 0$ | $e^W\frac{\partial}{\partial u_2}$, $e^{\mu t}(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$ | 17 |
| 12. | $f^1=\varepsilon u_2$, $f^2=\lambda\frac{u_2^2}{u_1}+2\nu u_2$ $+\sigma u_1\ln u_1$ | $\lambda=\pm1$, $\sigma=\mp\nu^2$ | $Q_8=e^{\nu t}(\lambda(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$ $+\nu u_1\frac{\partial}{\partial u_2})$, $e^{\nu t}u_1\frac{\partial}{\partial u_2}+tQ_8$ | |
| | | $\lambda\neq 0$, $\nu^2+\lambda\sigma=1$ | $X_\pm=e^{\nu\pm1}(\lambda(u_1\frac{\partial}{\partial u_1}$ $+u_2\frac{\partial}{\partial u_2})+(\nu\pm1)u_1\frac{\partial}{\partial u_2})$ | 1, $\rho=\omega$ if $\sigma=0$ |
| | | $\lambda\neq 0$, $\nu^2+\lambda\sigma$ $=-1$ | $e^{\nu t}(\lambda\cos t(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$ $+(\nu\cos t-\sin t)u_1\frac{\partial}{\partial u_2})$, $e^{\nu t}(\lambda\sin t(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2})$ $+(\nu\sin t+\cos t)u_1\frac{\partial}{\partial u_2})$ | |

## 12. Discussion

In this paper we present the completed group classification of systems of two coupled reaction-diffusion equations with general diffusion matrix. In other words we specify essentially different equations of this type defined up to equivalence transformations and describe their symmetries.

We consider only nonlinear equations, i.e., exclude the cases when $f^1$ and $f^2$ in the right hand side of (1.3) are linear in $u_1$, $u_2$. Such cases are presented in paper [23].

The analyzed class of equations includes six non-equivalent subclasses corresponding to different canonical forms of diffusion matrix $A$ enumerated in (2.2). In the particular case when matrix $A$ has the forms $I$ and $I^*$ from (2.2) our results can be compared with those of [7] and also [3]–[5].

Paper [7] was apparently the first work were the problem of group classification of equations (1.3) with a diagonal diffusion matrix was formulated and partially solved. Unfortunately, the classification results presented in [7] are incomplete and in many points incorrect. Thus, all cases enumerated above in Table 7, Items 1,2 of Table 3, Items 1,2, 7-10, 15 of Table 4, Items 2, 12, 16 and 17 of Table 6, were overlooked, symmetries of equations with non-linearities given in Items 1 and 2 of Table 5 were presented incompletely, etc.

In papers [3]–[5] Lie symmetries of the same equations and also of systems of diffusion equations with the unit diffusion matrix were classified.

The results obtained in [3]–[5] are much more advanced then the pioneer Davidov ones, nevertheless they are still incomplete. In particular, the cases indicated above in Items 5 and 6 of Table 3; Items 12–14 of Table 4; the last line of Item 1, Item 9 and Item 11 for a=1 of Table 5; Items 15 and 22 of Table 6 and Item I for $\sigma = 0$, $\mu \neq 0$ of Table 7 were not indicated in [5], which is in conflict with the statement of Theorem 1 formulated here. Moreover, many of equations presented in [5] as non-equivalent ones, in fact are equivalent one to another even in frames of equivalence relations (7) of [3]. The related examples are not enumerated here in as much as we believe that all non-equivalent equations (1.3) with different symmetries are present in Tables 1–9.

Except the points mentioned in the previous paragraph our results concerning equations with a diagonal diffusion matrix are in accordance with ones obtained in [3]–[5].

Consider examples of well known reaction diffusion equations which appear to be particular subjects of our analysis.

- The Jackiw-Teitelboim model of two-dimension gravity with the non-relativistic gauge [19]

$$\begin{aligned} &\frac{\partial}{\partial t}u_1 - \frac{\partial^2 u_1}{\partial x^2} - 2ku_1 + 2u_1^2 u_2 = 0, \\ &\frac{\partial}{\partial t}u_2 + \frac{\partial^2 u_2}{\partial x^2} + 2ku_2 - 2u_1 u_2^2 = 0 \end{aligned} \tag{12.1}$$

  admits the equivalence transformation 1 (2.10) for $\rho = -\omega$. Choosing $\rho = 2k$ we transform equation (12.1) to the form (1.2) where $a = -1$, $f^1 = -2u_1^2 u_2$ and $f^2 = 2u_2^2 u_1$. The symmetries corresponding to these non-linearities are given in the first line of Table 5. Symmetries of equations (12.1) were investigated in paper [16]. In accordance with our analysis, generalized equation (12.1) with two spatial variables admits additional conformal symmetry generated by operator $K$ (4.5).

- The primitive predator-prey system can be defined by [20]

$$\dot{u}_1 - D\frac{\partial^2 u_1}{\partial x_2} = -u_1 u_2, \qquad \dot{u}_2 - \lambda D\frac{\partial^2 u_2}{\partial x^2} = u_1 u_2.$$

  and this is again a particular case of equation (1.2) with the non-linearities given in the first line of Table 3 where however $\mu=\nu=1$, $F_1 = -F_2 = \frac{u_1}{u_2}$ . In addition to the basic symmetries $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \rangle$ this equation admits the (main) symmetry:

$$X = \left( D - 2u_1 \frac{\partial}{\partial u_1} - 2u_2 \frac{\partial}{\partial u_2} \right).$$

- The $\lambda - \omega$ reaction-diffusion system

$$\dot{u}_1 = D\Delta u_1 + \lambda(R)u_1 - \omega(R)u_2, \quad \dot{u}_2 = D\Delta u_2 + \omega(R)u_1 + \lambda(R)u_2, \tag{12.2}$$

where $R^2 = u_1^2 + u_2^2$, has symmetries that were analyzed in paper [1]. Again we recognize that this system is a particular case of (1.2) with non-linearities given in Item 6 of Table 4 with $\mu = \nu = 0$. Hence it admits the five dimensional Lie algebra generated by main symmetries (2.2) with $\mu, \nu = 1, 2$ and:

$$X = \left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1}\right) \tag{12.3}$$

which is in accordance with results of paper [1] for arbitrary functions $\lambda$ and $\omega$. Moreover, using Table 5, Item 8 we find that for the cases when

$$\lambda(R) = \tilde{\lambda} R^\nu, \quad \omega = \sigma R^\nu \tag{12.4}$$

equation (12.2) admits additional symmetry with respect to scaling transformations generated by the operator:

$$X = \left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1}\right) + \nu D. \tag{12.5}$$

The other extensions of the basic symmetries correspond to the case when $\lambda(R) = \mu \ln(R), \omega(R) = \sigma \ln(R)$, the related additional symmetries are given in Table 8 where $\nu = \lambda = 0$.

- The nonlinear Schrödinger equation (NSE) in $m$-dimensional space:

$$\left(i\frac{\partial}{\partial t} - \Delta\right)\psi = F(\psi, \psi^*) \tag{12.6}$$

also is a particular case of (1.2). If we denote $\psi = u_1 + iu_2$, $F = f_1 + if_2$ then (12.6) reduces to the form (1.3) with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In other words, any solution given in Tables 3–6, 8 with matrices $A$ belonging to Class $II$ gives rise for the NSE (9.4) that admits a main or extended symmetry. Thus our analysis makes it possible to present the completed group classification of the NSE as a particular case of general study of systems of reaction-diffusion equations with arbitrary diffusion matrix. Our results are in complete accordance with ones obtained in paper [22] where symmetries of the general NSE were described.

Among the solutions present in Tables 3–6, 8 we recognize ones which correspond to the well-known non-linearities [11]

$$F = F(\psi^*\psi)\psi, \quad F = (\psi^*\psi)^k\psi, \quad F = (\psi^*\psi)^{\frac{2}{m}}\psi, \quad F = \ln(\psi^*\psi)\psi$$

One more interesting particular case of the NSE with extended symmetry can be found using Table 6 Item 1 for $\nu = 2, m = 1$:

$$\left(i\frac{\partial}{\partial t} - \Delta\right)\psi = (\psi - \psi^*)^2$$

which is a potential equation for the Boussinesq equation for function $V = \frac{\partial}{\partial t}(\psi - \psi^*)$.

- Generalized complex Ginzburg-Landau (CGL) equation

$$\frac{\partial W}{\partial \tau} - (1 + i\beta)\Delta W = F(W, W^*) \qquad (12.7)$$

is a particular case of system (1.3) with matrix $A$ belonging to Class $II$ with $a \neq 0$, refer to (2.2). Indeed, representing $W$ and $F$ as $W = (u_1 + iu_2), F = \beta(f^1 + if^2)$ and changing independent variable $\tau \to t = \beta\tau$ we transform (12.7) to the form (1.3) with $A = \begin{pmatrix} \beta^{-1} & -1 \\ 1 & \beta^{-1} \end{pmatrix}$. All non-equivalent non-linearities $f^1$, $f^2$ and the corresponding symmetries are given in Table 3, Items 1, 3, Table 4, Items 5, 6, 15, Table 5, Items 8, 10, Table 6, Items 1, 4, 11 and Table 8. The ordinary CGL equation corresponds to the case $F = W - (1 + i\alpha)W|W|^2, m = 2$ and admits basic symmetries (5.1) only.

- Non-autonomous dynamical systems in phase space [8]

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} - A(u_1, u_2) = h_1(t, x), \\ &\frac{\partial u_2}{\partial t} + \alpha\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} - \nu u_1 = h_2(t, x) \end{aligned} \qquad (12.8)$$

also are equivalent to a system of type (1.3) at least in the case of constant $h_1$ and $h_2$. The related matrix $A$ belongs to Type $III$. Using the results present in Tables 3–6 and 9 we can specify all cases when the considered system admits main or extended symmetries.

We see that the class of equations which is classified in present paper includes a number of important particular systems. Moreover, we present *a priori* description of symmetries of all possible systems of two reaction-diffusion equations with general diffusion matrix.

## Appendix

### A.1. Algebras $\mathcal{A}$ for Equations (1.3) with Diagonal Diffusion Matrix

Let us consider equation (1.3) with a diagonal matrix $A$ (version I of (2.2) where $a \neq 0$) and find the related low-dimension algebras $\mathcal{A}$. In this case matrix (9.5) and the equivalence transformation matrix (9.7) reduce to the forms

$$g = \begin{pmatrix} 0 & 0 & 0 \\ B^1 & C^{11} & 0 \\ B^2 & 0 & C^{22} \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^{11} & 0 \\ \mu^2 & 0 & \mu^{22} \end{pmatrix} \tag{A.1.1}$$

and

$$U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & K^1 & 0 \\ b^2 & 0 & K_2 \end{pmatrix}. \tag{A.1.2}$$

Up to equivalence transformations (9.6), (A.1.2) there exist three non-equivalent matrices (A.1.1), namely

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A.1.3}$$

In accordance with (9.1)–(9.4) the related symmetry operator can be represented in one of the following forms

$$X_1 = \mu D - 2(g_a)_{bc}\tilde{u}_c \frac{\partial}{\partial u_b}, \quad X_2 = e^{\lambda t}(g_a)_{bc}\tilde{u}_c \frac{\partial}{\partial u_b} \tag{A.1.4}$$

or

$$X_3 = e^{\lambda t + \omega \cdot x}\left(\frac{\partial}{\partial u_2} + \mu \frac{\partial}{\partial u_1}\right). \tag{A.1.5}$$

Here $(g_a)_{bc}$ are elements of matrices (A.1.3), $b, c = 0, 1, 2$, $\tilde{u} =$ column $(u_0, u_1, u_2)$, $u_0 = 1$.

Formulae (A.1.4) and (A.1.5) give the principal description of one-dimension algebras $\mathcal{A}$ for equation (1.3), with matrix $A$ of type I.

To describe two-dimension algebras $\mathcal{A}$ we classify matrices $g$ (A.1.1) forming two-dimension Lie algebras. Choosing one of the basis elements in the forms given in (A.1.3) and the other element in the general form (A.1.1) we find that up to equivalence transformations (9.6) there exist six algebras $\langle e_1, e_2 \rangle$:

$$A_{2,1} = \{\tilde{g}_1, g_4\}, \quad A_{2,2} = \{\tilde{g}_1, \tilde{g}_3\}, \quad A_{2,3} = \{g_5, \tilde{g}_3\}, \tag{A.1.6}$$

$$A_{2,4} = \{g_1, g_5\}, \quad A_{2,5} = \{g_1', g_3\}, \quad A_{2,6} = \{g_2, \tilde{g}_3\} \tag{A.1.7}$$

where $\tilde{g}_1 = g_1|_{\lambda=0}$, $g_1' = g_1|_{\lambda=1}$, $\tilde{g}_3 = g_3|_{\lambda=0}$, and

$$g_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.1.8}$$

Algebras (A.1.6) are Abelian while algebras (A.1.7) are characterized by the following commutation relations:

$$[e_1, e_2] = e_2 \tag{A.1.9}$$

where $e_1$ is the first element given in the brackets (A.1.7), i.e., for $A_{2,4}$ $e_1 = g_1$, etc.

Using (A.1.6), (A.1.7) and applying arguments analogous to those which follow equations (6.2) we easily find pairs of operators (5.2) forming Lie algebras. Denoting

$$\hat{e}_\alpha = (e_\alpha)_{ab}\tilde{u}_b \frac{\partial}{\partial u_a}, \quad \alpha = 1, 2$$

we represent them as follows:

$$\begin{gathered} \langle \mu D + \hat{e}_1 + \nu t \hat{e}_2, \hat{e}_2 \rangle, \quad \langle \mu D + \hat{e}_2 + \nu t \hat{e}_1, \hat{e}_1 \rangle, \\ \langle \mu D - \hat{e}_1, \nu D - \hat{e}_2 \rangle, \quad \langle F_1 \hat{e}_1 + G_1 \hat{e}_2,\ F_2 \hat{e}_1 + G_2 \hat{e}_2 \rangle \end{gathered} \tag{A.1.10}$$

for $e_1, e_2$ belonging to algebras (A.1.6), and

$$\langle \mu D - \hat{e}_1, \hat{e}_2 \rangle, \quad \langle \mu D + \hat{e}_1 + \nu t \hat{e}_2, \hat{e}_2 \rangle \tag{A.1.11}$$

for $e_1, e_2$ belonging to algebras (A.1.7).

Here $\mu$ and $\nu$ are parameters which can take on any (including zero) finite values, $\{F_1, G_1\}$ and $\{F_2, G_2\}$ are fundamental solutions of the following system

$$F_t = \lambda F + \nu G, \quad G_t = \sigma F + \gamma G \tag{A.1.12}$$

with arbitrary parameters $\lambda, \nu, \sigma, \gamma$.

The list (A.1.10)–(A.1.11) does not includes algebras spanned on the vectors $\langle F\hat{e},\ G\hat{e} \rangle$ (with $F, G$ satisfying (A.1.12)) and $\langle \mu D + \lambda e^{\nu t + \omega \cdot x} \hat{e},$ $e^{\nu t + \omega \cdot x} \hat{e} \rangle$ which are either incompatible with classifying equations (4.6) or reduce to one-dimension algebras. In the following we ignore algebras $\mathcal{A}$ which include such subalgebras.

All the other two-dimension algebras $\mathcal{A}$ can be reduced to one of the form given in (A.1.10), (A.1.11) using equivalence transformations (2.4), (6.7).

There exist one more type of (m+2)-dimensional algebras $\mathcal{A}$ generated by two-dimension algebras (A.1.6), namely:

$$\langle \mu D + \hat{e}_1 + (\alpha t + \lambda_{\sigma\rho} x_\sigma x_\rho)\hat{e}_2, x_\nu \hat{e}_2, \hat{e}_2 \rangle$$

where $\nu, \sigma, \rho$ run from 1 to $m$. The related classifying equations generated by all symmetries $x_1\hat{e}_2, x_2\hat{e}_2, \cdots, x_m\hat{e}_2$ and $\hat{e}_2$ coincides and we have the same number of constrains for $f^1, f^2$ as in the case of two-dimension algebras $\mathcal{A}$.

Up to equivalence there exist three realizations of three-dimension algebras in terms of matrices (A.1.3), (A.1.8):

$$\begin{array}{ll} A_{3,1}: & e_1 = \tilde{g}_1,\ e_2 = g_4,\ e_3 = \tilde{g}_3, \\ A_{3,2}: & e_1 = g_5,\ e_2 = g_4,\ e_3 = \tilde{g}_3, \end{array} \tag{A.1.13}$$

$$A_{3,3}: \quad e_1 = g_1',\ e_2 = g_5,\ e_3 = \tilde{g}_3. \tag{A.1.14}$$

Non-zero commutators for matrices (A.1.13) and (A.1.14) are $[e_2, e_3] = e_3$ and $[e_1, e_\alpha] = e_\alpha (\alpha = 2, 3)$. The algebras of operators (5.2) corresponding to realizations (A.1.13) and (A.1.14) are of the following general forms:

$$\langle \mu D - 2\hat{e}_1,\ \nu D - 2\hat{e}_2 - 2\lambda t \hat{e}_3, \hat{e}_3 \rangle \tag{A.1.15}$$

and

$$\begin{array}{l} \langle \mu D - 2\hat{e}_1 - 2\nu t \hat{e}_2 - 2\sigma t \hat{e}_3,\ \hat{e}_2,\ \hat{e}_3 \rangle, \\ \langle \hat{e}_1,\quad F_1 \hat{e}_2 + G_1 \hat{e}_3,\ F_2 \hat{e}_2 + G_2 \hat{e}_3 \rangle \end{array} \tag{A.1.16}$$

respectively.

In addition, we have the only four-dimension algebra

$$\hat{A}_{4,1}: \quad e_1 = \tilde{g}_1,\ e_2 = g_5,\ e_3 = \tilde{g}_3,\ e_4 = g_4 \tag{A.1.17}$$

which generates the following algebras of operators (5.2):

$$\begin{array}{l} \langle \mu D - 2\hat{e}_1 - 2\nu t \hat{e}_2,\ \hat{e}_2, \hat{e}_3,\ \hat{e}_4 \rangle, \\ \langle \mu D - 2\hat{e}_1 - 2\nu t \hat{e}_3,\ \hat{e}_2,\ \hat{e}_3,\ \hat{e}_4 \rangle, \\ \langle \mu D - 2\hat{e}_1,\ \nu D - \hat{e}_4,\ \hat{e}_2,\ \hat{e}_3 \rangle. \end{array} \tag{A.1.18}$$

Thus we have specified all low dimension algebras $\mathcal{A}$ which can be admitted by equations (1.3) with a diagonal (but not unit) matrix $A$.

## A.2. Algebras $\mathcal{A}$ for Equations (1.3) with $A^{12} \neq 0$

Consider equation (1.3) with matrix $A$ of type $II$ (refer to (2.3)) and find the corresponding algebras $\mathcal{A}$. The related matrices (9.5) and (9.7) are

$$g = \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^2 & \mu^3 \\ \mu^4 & -\mu^3 & \mu^5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & k^1 & k^2 \\ b^2 & -k^2 & k^3 \end{pmatrix}. \tag{A.2.1}$$

Up to equivalence transformations (9.6), (A.2.1) there exist three matrices $g$, namely

$$g_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; g_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; g_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & -1 \\ 0 & 1 & \mu \end{pmatrix} \tag{A.2.2}$$

and three two-dimension algebras of matrices $g$ (A.2.1):

$$A_{2,7} = \{g_1', g_6\}, \;\; A_{2,8} = \{g_5, \tilde{g}_3\}, \tag{A.2.3}$$

$$A_{2,9} = \{g_1', g_5\} \tag{A.2.4}$$

where $\tilde{g}_3$ is matrix (A.1.3) with $\lambda = 0$.

Algebras (A.2.3) are Abelian while the basis elements of $A_{2,9}$ satisfy commutation relations (A.1.9).

Like in previous subsection we easily find the related basis elements of one-dimension algebras $\mathcal{A}$ in the form (A.1.4) and (A.1.5) for $\mu = 0$.

The two-dimension algebras $\mathcal{A}$ generated by (A.2.3) and (A.2.4) again are given by relations (A.1.10) and (A.1.11) respectively, where $e_1$ and $e_2$ are the first and second elements of algebras $A_{2,7} - A_{2.9}$.

In addition, we have two three-dimension algebras

$$\begin{aligned} &A_{3,3}: \quad e_1 = g_1', \; e_2 = g_5, \; e_3 = \tilde{g}_3; \\ &A_{3,4}: \quad e_1 = g_5, \; e_2 = g_6, \; e_3 = \tilde{g}_3 \end{aligned} \tag{A.2.5}$$

and the only four-dimension algebra:

$$A_{4,2}: \quad e_1 = g_1', \; e_2 = g_6, \; e_3 = \tilde{g}_3, \; e_4 = g_5. \tag{A.2.6}$$

Algebra $A_{3,4}$ generates algebras (A.1.16) while $A_{3,5}$ corresponds to (A.1.15) with $\nu = 0$. Finally, $A_{4,2}$ generates the following algebras $\mathcal{A}$

$$\begin{aligned} &\langle \mu D - 2\hat{e}_1, \; \nu D - 2\hat{e}_2, \; \hat{e}_3, \; \hat{e}_4 \rangle, \\ &\langle \hat{e}_1, \; \hat{e}_2, \; e^{\mu t + \nu \cdot x} \hat{e}_3, \; e^{\mu t + \nu \cdot x} \hat{e}_4 \rangle. \end{aligned} \tag{A.2.7}$$

### A.3. Algebras $\mathcal{A}$ for Equations (1.3) with Triangular Matrix $A$

If matrix $A$ belongs to type $III$ given in (2.3) the related matrices (9.5) and (9.7) take the form

$$g = \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^2 & 0 \\ \mu^3 & \mu^4 & \mu^5 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & k^1 & 0 \\ b^2 & k^2 & k^3 \end{pmatrix}. \tag{A.3.1}$$

There exist six non-equivalent matrices $g$, i.e., matrices $g_1'$, $g_3$, $g_5$ (A.1.3), (A.2.2), and the following ones

$$g_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_9 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{A.3.2}$$

In addition, we have six two-dimension algebras,

$$\begin{aligned} &A_{2,3} = \{g_5, \tilde{g}_3\}, \ A_{2,10} = \{g_1', g_8\}, \\ &A_{2,11} = \{g_8, \tilde{g}_3\}, \ A_{2,12} = \{g_9, \tilde{g}_3\}, \end{aligned} \tag{A.3.3}$$

$$A_{2,5} = \{g_1', g_3\}, \ A_{2,13} = \{g_1', g_5\}, \tag{A.3.4}$$

four three-dimension algebras:

$$\begin{aligned} A_{3,3}: &\quad e_1 = g_1', \ e_2 = g_5, \ e_3 = \tilde{g}_3, \\ A_{3,5}: &\quad e_1 = g_8, \ e_2 = g_1', \ e_3 = \tilde{g}_3, \\ A_{3,6}: &\quad e_1 = \tilde{g}_3, \ e_2 = g_8, \ e_3 = g_9, \\ A_{3,7}: &\quad e_1 = \tilde{g}_3, \ e_2 = g_5, \ e_3 = g_7 \end{aligned} \tag{A.3.5}$$

and the only four-dimension algebra:

$$A_{4,3}: \quad e_1 = \tilde{g}_3, \ e_2 = g_5, \ e_3 = g_1', \ e_4 = g_8. \tag{A.3.6}$$

Algebras (A.3.3) are Abelian while (A.3.4) are characterized by commutation relations (A.1.9). The related two-dimension algebras $\mathcal{A}$ are given by formulae (A.1.10) and (A.1.11) respectively.

Algebra $A_{3,3}$ generates three-dimension algebras $\mathcal{A}$ enumerated in (A.1.16). Algebra $A_{3,5}$ is isomorphic to $A_{3,1}$ and so we come to the related algebras $\mathcal{A}$ given in (A.1.15). Algebras $A_{3,6}$ and $A_{3,7}$ are characterized by the following non-zero commutators

$$[e_2, e_3] = e_1 \tag{A.3.7}$$

and

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_2 + e_3 \tag{A.3.8}$$

respectively.

Using (A.3.7) and (A.3.8) we come to the following related three-dimension algebras $\mathcal{A}$ :

$$\begin{gathered}\langle \mu D - 2\hat{e}_2,\ \nu D - 2\hat{e}_3,\ \hat{e}_1\rangle,\ \langle \hat{e}_1,\ D + 2e_\alpha + 2\nu t\hat{e}_1,\ \hat{e}_{\alpha'}\rangle,\\ \langle e^{\nu t+\omega\cdot x}\hat{e}_1,\ e^{\nu t+\omega\cdot x}\hat{e}_\alpha,\ \hat{e}_{\alpha'}\rangle\end{gathered} \tag{A.3.9}$$

where $\alpha,\ \alpha' = 2, 3, \alpha' \neq \alpha$, and

$$\langle \mu D - 2\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3\rangle,\ \langle \hat{e}_1,\ e^{\nu t+\omega\cdot x}\hat{e}_2,\ \ e^{\nu t+\omega\cdot x}\hat{e}_3\rangle. \tag{A.3.10}$$

Finally, four-dimension algebras $\mathcal{A}$ corresponding to $A_{4,3}$ have the following general form

$$\langle \mu D - 2\hat{e}_1,\ \nu D - 2\hat{e}_2,\ \hat{e}_3,\ \hat{e}_4\rangle,\ \ \langle e^{\nu t+\omega\cdot x}\hat{e}_1,\ e^{\nu t+\omega\cdot x}\hat{e}_2,\ \hat{e}_3,\ \hat{e}_4\rangle \tag{A.3.11}$$

### A.4. Algebras $\mathcal{A}$ for Equations (1.3) with the Unit Matrix $A$

Group classification of these equations appears to be the most complicated. The related matrices $g$ are of the most general form (9.5) and defined up to the general equivalence transformation (9.6), (9.7). In other words there are seven non-equivalent matrices (9.5), namely, $g_1, g_2$ (A.1.3), $g_5$ , $g_6$ (A.2.2) and $g_7 - g_9$ (A.3.2). In addition, we have fifteen two-dimension algebras of matrices (9.5),

$$\begin{gathered}A_{2,1} = \{\tilde{g}_1, g_4\},\ A_{2,2} = \{\tilde{g}_1, \tilde{g}_3\},\ A_{2,3} = \{\tilde{g}_3, g_5\},\\ A_{2,10} = \{g_7, g_8\},\ A_{2,11} = \{\tilde{g}_3, g_8\},\\ A_{2,12} = \{\tilde{g}_3, g_9\},\ A_{2,13} = \{g'_1, g_6\},\end{gathered} \tag{A.4.1}$$

$$\begin{gathered}A_{2,4} = \{g_1, g_5\},\ A_{2,5} = \{g'_1, g_3\},\ A_{2,6} = \{g_2, \tilde{g}_3\},\\ A_{2,14} = \{g_1|_{\lambda\neq 1}, g_8\},\ A_{2,15} = \{g_{11}, -g_8\},\ A_{2,16} = \{g_9, {g''}_1\},\\ A_{2,17} = \{g_4, g_8\},\ A_{2,18} = \{g_7, \tilde{g}_3\}\end{gathered} \tag{A.4.2}$$

where

$$g_{10} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},\ g''_1 = g_1|_{\lambda=2} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}.$$

Algebras (A.4.1) are Abelian while algebras (A.4.2) are characterized by relations (A.1.9).

Three-dimension algebras are $A_{3,1} - A_{3,7}$ given by relations (A.1.13), (A.2.5) and (A.3.3) (where tildes should be omitted) and also $A_{3,8} - A_{3,11}$ given below:

$$\begin{aligned}&A_{3,8}:\ e_1 = g_1,\ e_2 = g_8,\ e_3 = \tilde{g}_3,\\ &A_{3,9}:\ e_1 = g_4,\ e_2 = g_8,\ e_3 = \tilde{g}_3,\\ &A_{3,10}:\ e_1 = g_2,\ e_2 = g_8,\ e_3 = -\tilde{g}_3,\\ &A_{3,11}:\ e_1 = \tilde{g}_1,\ e_2 = -g_8,\ e_3 = \tilde{g}_4.\end{aligned}$$

Algebras ($A_{3,8}, A_{3,11}$) and $A_{3,9}$ and $A_{3,10}$ are isomorphic to $A_{3,1}$ and $A_{3,3}$ and $A_{3,6}$ respectively. The related algebras $\mathcal{A}$ are given by relations (A.1.15), (A.1.16) and (A.3.9) correspondingly.

Finally, four-dimension algebras of matrices (9.6) are $A_{4,1}$ , $A_{4,2}$ and $A_{4,3}$ given by equations (A.1.17), (A.2.6) and (A.3.6), and also $A_{4,4}$, $A_{4,5}$ given below:

$$\begin{array}{l} A_{4,4}: \ e_1 = g_1, \ e_2 = g_4, \ e_3 = g_8, \ e_4 = g_3; \\ A_{4,5}: \ e_1 = g_4, \ e_2 = g_8, \ e_3 = g_5, \ e_4 = g_3. \end{array}$$

Using found algebras and solving the related equations (4.6) we easily make the group classification of equations (1.3).

## References


[1] J. F. R. Archilla et al, *Symmetry analysis of an integrable reaction-diffusion equation* // J. Phys. A **39** (1997), 185–191.

[2] G. Bluman, *Simplifying the form of Lie groups admitted by a given differential equation* // J. Math. Anal. Appl. **145** (1990), 52–62.

[3] R. M. Cherniha and J. R. King, *Lie symmetries of nonlinear multidimensional reaction-diffusion systems: I* // J. Phys. A **33** (2000), 267–282.

[4] R. M. Cherniha and J. R. King, *Lie symmetries of nonlinear multidimensional reaction-diffusion systems: I. Addendum* // J. Phys. A **33** (2000), 7839–7841.

[5] R. M. Cherniha and J. R. King, *Lie symmetries of nonlinear multidimensional reaction-diffusion systems: II* // J. Phys. A **36** (2002), 405–425.

[6] P. A. Clarkson and E. I. Mansfield, *Symmetry reductions and exact solutions of a class ofnonlinear heat equations* // Physica D **70** (1993), 250.

[7] Yu. A. Danilov, *Group analysis of the Turing systems and of its analogues*, Preprint of Kurchatov Institute for Atomic Energy IAE-3287/1, 1980.

[8] R. L Devaney, *An Introduction to Chaotic Dynamical Systems.* Benjamin-Cummings, 1989.

[9] R. Fitzhugh, *Impulses and physiological states in models of nerve membrane* // Biophys. J. **1** (1961), 445–466.

[10] W. I. Fushchich and M. I. Serov, *Conditional invariance and reduction of the nonlinear heat equation* // Dokl. Acad. Nauk Ukrainy **7** (1990) 24–27.

[11] W. I. Fushchych, W. M. Shtelen and N. I. Serov, *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics.* Kluwer, Dordrecht, 1993.

[12] W. I. Fushchich and A. G. Nikitin, *Symmetries of Maxwell Equations.* Reidel, Dordrecht, 1987.

[13] L. Gangon and P. Winternitz, *Symmetry classes of variable coefficient nonlinear Schrödinger equations* // J. Phys. A, **26** (1993), 7061–7076.

[14] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations, in 3 Volumes*, 1994.

[15] I. V. Knyazeva and M. D. Popov, *Group classification of diffusion equations.* Preprint 6, Keldysh Inst. of Math.,Acad. Sci. of USSR, Moskow, 1986.

[16] R. A. Kraenkel and M. Senthilvelan, *Symmetry analysis of an integrable reaction-diffusion equation* // Ch. Sol. Fract. **12** (2001), 463–474.

[17] S. Lie and F. Engel, *Theorie der Transformationsgruppen: In 3 Bd. Teubner*, Leipzig, 1888, 1890, 1893.

[18] S. Lie, *Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen*, Arch. for Math. **6** (1881), 328–368. (Translation by N.H. Ibragimov: Lie, S. *On integration of a Class of Linear Partial Differential Equations by Means of Definite Integrals*, see [14], Vol. 2, 473-508).

[19] I. Martina, O. K. Pashaev and G. Soliani, *Integrable dissipative structures in the gauge theory of gravity* // Class. Quantum Grav. **14** (1997), No 12, 3179–3186.

[20] J. D. Murray. *Mathematical Biology.* Springer, 1991.

[21] A. G. Nikitin, *Group classification of systems of non-linear reaction-diffusion equations with general diffusion matrix. II. Diadonal diffusion matrix*, math-ph/0411028, 2004.

[22] A. G. Nikitin and R. O. Popovych, *Group classification of nonlinear Schrödinger equations* // Ukr. Mat. Zhurn. **53** (2001), 1053–1060.

[23] A. G. Nikitin and R. Wiltshire, *Systems of Reaction Diffusion Equations and their symmetry properties* // J. Math. Phys. **42** (2001) 1667–1688.

[24] A. G. Nikitin and R. Wiltshire,*Symmetries of Systems of Nonlinear Reaction-Diffusion Equations*, in: *Symmetries in Nonlinear Mathematical Physics, Proc. of the Third Int. Conf. , Kiev, July 12–18, 1999, Ed. A.M. Samoilenko* (Inst. of Mathematics of Nat. Acad. Sci. of Ukraine, Kiev, pp. 47–59, 2000).

[25] P. Olver, *Application of Lie Groups to Differential Equations.* Springer, N.Y., 1986.

[26] V. A. Dorodnitsyn, *Group properties and invariant solutions of a linear heat equation with a source.* Preprint **57**, Keldysh Institute of Applied Mathematics, Academy of Sci. of U.S.S.R (1979).

[27] J. Rinzel and J. B. Keller, *Traveling wave solutions of a nerve conduction equation* // Biophys. J. **13** (1973), 1313–1337.

[28] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdiumov and A. P. Mikhailov, *Blow-up Rates In Problems for Quasilinear Parabolic Equations.* Nauka, Moscow, 1987.

[29] R. Z. Zhdanov and V. I. Lagno, *Conditional symmetry of a porous medium equation* // Physica D, **122** (1998), 178–186.

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