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Asymptotic Expansion of Markov Random Evolution

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Abstract. It is studied asymptotic expansion for solution of singularly perturbed equation for Markov random evolution in $\mathbb{R}^d$. The views of regular and singular parts of solution are found.

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1. Introduction

A Markov random evolution (MRE) is created by a solution of the evolutionary equation in Euclidean space $\mathbb{R}^d$, $d \geq 1$

$$du^\varepsilon(t)/dt = v(u^\varepsilon(t); æ(t/\varepsilon))$$

with the ergodic Markov switching process $æ(t)$, $t \geq 0$ on the standard (Polish) phase-space $(E, \mathcal{E})$ by the operator $Q(x, B)$, $x \in E$, $B \in \mathcal{E}$ that defines transition probabilities of a Markov chain $æ_n, n \geq 0$

$$Q(x, B) = P\{æ_{n+1} \in B | æ_n = x\}.$$

The operator of transition probabilities Q is defined by

$$Qf(x) = \int_E Q(x, dy) f(y), \qquad x \in E, \tag{1.1}$$

for any bounded measurable real valued f defined on E.

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We will see later that the equation for the regular and the singular parts of a random evolution are defined by the generator (1.1) of a uniformly argodic Markov switching process. The Banach space $\mathcal{B}(E)$ is splitted onto the two subspaces [7]:

$$\mathcal{B}(E) = N_Q \bigoplus R_Q,$$

where $N_Q := \{\varphi : Q\varphi = 0\}$ is the null-space of Q, and $R_Q := \{\psi : Q\varphi = \psi\}$ is the range of Q.

We define the projector $\Pi : N_Q := \Pi\mathcal{B}(E)$, $R_Q := (I - \Pi)\mathcal{B}(E)$; $\Pi\varphi(x) := \widehat{\varphi}\mathbf{1}$, $\widehat{\varphi} := \int_E \varphi(x)\pi(dx)$, where the stationary distribution $\pi(B), B \in \mathcal{E}$ of the Markov process $æ(t), t \geq 0$ satisfies the relations [4]

$$\pi(dx) = \rho(dx)m_1(x)/\widehat{m},$$

$$\widehat{m} = \int\limits_E m_1(x)\rho(dx).$$

$\rho(B), B \in \mathcal{E}$ is the stationary distribution of the Markov chain $æ_n, n \geq 0$, given by the equation

$$\rho(B) = \int\limits_E Q(x, B)\rho(dx), \qquad \rho(E) = 1.$$

Let us consider the Banach space $\mathcal{B}(\mathbb{R}^d)$ of real-valued test-functions $\varphi(u), u \in \mathbb{R}^d$ which are bounded with all their derivatives equipped with sup-norm

$$\|\varphi\| := \sup_{u\in\mathbb{R}^d} |\varphi(u)| < C_\varphi.$$

The random evolution in $\mathcal{B}(\mathbb{R}^d)$ is given by the relation

$$\Phi_t^\varepsilon(u, x) := E[\varphi(u^\varepsilon(t))|u^\varepsilon(0) = u, æ^\varepsilon(0) = x]. \tag{1.2}$$

The asymptotic behavior of MRE (1.2) as $\varepsilon \to 0$ is investigated under the assumption of uniformly ergodicity of the Markov switching process $æ(t)$ described above and under the assumption of the existence of a global solution of the deterministic equations

$$du_x(t)/dt = v(u_x(t); x), \qquad x \in E.$$

Let us consider the deterministic evolution

$$\Phi_x(t, u) = \varphi(u_x(t)), \qquad u_x(0) = u.$$

It generates a corresponding semigroup

$$\mathbb{V}_t(x)\varphi(u) := \varphi(u_x(t)), \qquad u_x(0) = u,$$

and its generator has the form:

$$\mathbb{V}(x)\varphi(u) = v(u;x)\varphi'(u) := \sum_{k=1}^{d} v_k(u;x)\varphi'_k(u),$$

$$\varphi'_k(u) := \partial\varphi(u)/\partial u_k, \qquad \varphi(u) \in C^\infty(\mathbb{R}^d).$$

By the average principle [10] the weak convergence

$$u^\varepsilon(t) \Rightarrow \widehat{u}(t), \qquad \varepsilon \to 0 \tag{1.3}$$

takes place. The average limit evolution $\widehat{u}(t),\ t \geq 0$ is defined by a solution of the average equation

$$d\widehat{u}(t)/dt = \widehat{v}(\widehat{u}(t)).$$

The average velocity $\widehat{v}(u), u \in \mathbb{R}^d$ is defined by

$$\widehat{v}(u) = \int_E v(u;x)\pi(dx)$$

(i.e. by the average of the initial velocity $v(u;x)$ over the stationary distribution $\pi(B),\ B \in \mathcal{E}$).

The rate of convergence in (1.3) can be investigated in two directions:
i) asymptotic analysis of the fluctuations

$$\zeta^\varepsilon(t) = u^\varepsilon(t) - \widehat{u}(t); \tag{1.4}$$

ii) asymptotic analysis of the average deterministic evolution (1.2).

The asymptotic analysis of fluctuations (1.4) leads to the diffusion approximation of the random evolution [5, 10].

The asymptotic analysis of evolution (1.2) is realized in what follows by constructing the asymptotic expansion in power of the small parameter series $\varepsilon \to 0 (\varepsilon > 0)$ in the following form ($\tau = t/\varepsilon$):

$$\Phi_t^\varepsilon(u, x) = u^{(0)}(t) + \sum_{k=1}^{\infty} \varepsilon^k [u^{(k)}(t) + w^{(k)}(\tau)]. \tag{1.5}$$

The asymptotic expansion (1.5) contains two parts:
i) the regular term $u^\varepsilon(t) := u^{(0)}(t) + \sum_{k\geq 1}^{\infty} \varepsilon^k u^{(k)}(t)$,
ii) the singular term (boundary layer) $w^\varepsilon(\tau) := \sum_{k\geq 1}^{\infty} \varepsilon^k w^{(k)}(\tau),\ \tau = t/\varepsilon$.

In addition the initial condition:

$$u^{(0)}(0) = \varphi(u)\mathbf{1}$$

has to be valid for any $x \in E, u \in \mathbb{R}^d$.

It's well-known (see, e.g. [8]), that the evolution, determined by a test-function $\varphi(u) \in C^\infty(R^d)$ (here $\varphi(u)$ is integrable on $\mathbb{R}^d$): satisfy the system of Kolmogorov backward differential equations:

$$\begin{gathered}\frac{\partial}{\partial t}\Phi_t^\varepsilon(u,x) = [\varepsilon^{-1}Q + \mathbb{V}]\Phi_t^\varepsilon(u,x),\\ \Phi_0^\varepsilon(u,x) = \varphi(u).\end{gathered} \tag{1.6}$$

Asymptotic expansions with "boundary layers" were studied by many authors (see [2, 3, 12]). In particular, functionals of Markov and semi-Markov processes are investigated from this point of view in [6, 9, 11].

In this work we study system (1.6) with the first order singularity. To find asymptotic expansion of the solution of (1.2) we use the method proposed in [3, 12]. The solution consists of two parts, regular terms and singular terms, which are determined by different equations. Asymptotic expansion lets not only determine the terms of asymptotic, but to see the velocity of convergence in hydrodynamic limit.

Besides, when studying this problem, we improved the algorithm of asymptotic expansion. Partially, the initial conditions for the regular terms of asymptotic are determined without the use of singular terms, i.e. the regular part of the solution may be found by a separate recursive algorithm; scalar part of the regular term is found and without the use of singular terms. These and other improves of the algorithm are pointed later.

2. Asymptotic Expansion of the Solution

Let $P(t) = e^{Qt} = \{p_{ij}(t); i,j \in E\}$. Put $\pi_j = \lim_{t\to\infty} p_{ij}(t)$ and $-R_0 = \{\int_0^\infty (p_{ij}(t) - \pi_j)\,dt; i,j \in E\} = \{r_{ij}; i,j \in E\}$.

Let Π be a projecting operator on the null-space N_Q of the operator Q. For any vector g we have $\Pi g = \widehat{g}\mathbf{1}$, where $\widehat{g} = (g,\pi)$, $\mathbf{1} = (1,\dots,1)$. Then for the operator Q the following correlations are true (see [7], chapter 3)

$$\Pi Q \Pi = 0,$$

$$QR_0 = R_0 Q = \Pi - I.$$

We put:

$$exp_0(Qt) := e^{Qt} - \Pi.$$

Theorem 2.1. *The solution of equation* (1.6) *with initial condition* $\Phi_0^\varepsilon(u,x) = \varphi(u)$, *where* $\varphi(u) \in C^\infty(R^d)$ *and integrable on* $\mathbb{R}^d$ *has asymptotic expansion*

$$\Phi_t^\varepsilon(u,x) = u^{(0)}(t) + \sum_{n=1}^{\infty} \varepsilon^n \big(u^{(n)}(t) + w^{(n)}\,(t/\varepsilon)\big). \tag{2.1}$$

Regular terms of the expansion are: $u^{(0)}(x,t)$, *the solution of equation*

$$\frac{\partial}{\partial t} u^{(0)}(t) - \Pi\mathbb{V}\Pi u^{(0)}(t) = 0 \tag{2.2}$$

with initial condition $u^{(0)}(0) = \varphi(u)$,

$$u^{(1)}(t) = R_0\Big[\,\frac{d}{dt}\,u^{(0)}(t) - \mathbb{V}u^{(0)}(t)\Big] + c^{(1)}(t) := R_0\mathbb{L}u^{(0)}(t) + c^{(1)}(t),$$

for $k \geq 2:$

$$u^{(k)}(t) = R_0\mathbb{L}u^{(k-1)}(t) + c^{(k)}(t)$$

where $c^{(k)}(t) \in N_Q$,

$$c^{(k)}(u,t) = c^{(k)}(V^{-1}(t+V(u)),0) + \int \frac{\mathcal{L}_k(V^{-1}(t+V(u)),0)}{V^{-1}(t+V(u))}\,du - \int \frac{\mathcal{L}_k(u,t)}{v(u)}\,du, \qquad k > 0,$$

here $V(u) = \int \frac{du}{v(u)}$, $V^{-1}(w)$ *is the backward function for* $V(u)$,

$$\mathcal{L}_k(u,t) = \sum_{i=0}^{k-1}\sum_{n=1}^{k-i}(-1)^k(k-i-n+1)\Pi\mathbb{V}\mathbb{R}_0\mathbb{V}^n\Pi\frac{d^{k-i-n}}{dt^{k-i-n}}c^{(i)}(t).$$

The singular terms of the expansion have the view:

$$w^{(1)}(\tau) = exp_0(Q\tau)\mathbb{V}\varphi(u),$$

for $k > 1:$

$$w^{(k)}(\tau) = exp_0(Q\tau)w^{(k)}(0) + \int_0^{\tau} exp_0(Q(\tau-s))\mathbb{V}w^{(k-1)}(s)\,ds - \Pi\int_{\tau}^{\infty}\mathbb{V}w^{(k-1)}(s)\,ds.$$

Initial conditions:

$$c^{(0)}(0) = \varphi(u),$$

$$w^{(1)}(0) = -R_0\mathbb{L}\varphi(u),$$

$$c^{(1)}(0) = 0,$$

for $k > 1$:

$$w^{(k)}(0) = -R_0\mathbb{L}u^{(k-1)}(0),$$

$$c^{(k)}(0) = \mathbb{V}\widetilde{w}^{(k-1)}(0),$$

where $\widetilde{w}^{(1)}(0) = -R_0\mathbb{V}\varphi(u)$,

$$\widetilde{w}^{(k)}(0) = R_0\mathbb{L}u^{(k-1)}(0) + R_0\mathbb{V}\widetilde{w}^{(k-1)}(0) + \Pi\mathbb{V}(\widetilde{w}^{(k-1)}(\lambda))'_\lambda\big|_{\lambda=0},$$

$$(\widetilde{w}^{(k)}(\lambda))'_\lambda\big|_{\lambda=0} = R_0^2\mathbb{L}u^{(k-1)}(0) + R_0^2Q_1\widetilde{w}^{(k-1)}(0) + R_0\mathbb{V}(\widetilde{w}^{(k-1)}(\lambda))'_\lambda\big|_{\lambda=0}.$$

Remark 2.1. The initial conditions for the regular terms of asymptotic are determined without the use of singular terms, i.e. the regular part of the solution may be found by a separate recursive algorithm (comp. with [6]).

Proof of Theorem 2.1. Let us substitute the solution $\Phi_t^\varepsilon(u,x)$ in the form (2.1) to the equation (1.6) and equal the terms at ε degrees. We'll have the system for the regular terms of asymptotic:

$$\begin{cases} Qu^{(0)} = 0 \\ Qu^{(k)} = \frac{d}{dt}u^{(k-1)} - \mathbb{V}u^{(k-1)} := \mathbb{L}u^{(k-1)}, \ k \geq 1 \end{cases} \tag{2.3}$$

and for the singular terms

$$\frac{dw^\varepsilon}{dt} = \frac{dw^\varepsilon}{d\tau}\frac{d\tau}{dt} = \varepsilon^{-1}\frac{dw^\varepsilon}{d\tau} = (\varepsilon^{-1}Q + \mathbb{V})w^\varepsilon.$$

Thus, from $\frac{dw^\varepsilon}{d\tau} = (Q + \varepsilon\mathbb{V})w^\varepsilon$ we obtain:

$$\begin{cases} \frac{d}{d\tau}w^{(1)} = Qw^{(1)} \\ \frac{d}{d\tau}w^{(k)} - Qw^{(k)} = \mathbb{V}w^{(k-1)}, \ k > 1. \end{cases} \tag{2.4}$$

From (2.3) we have: $u^{(0)}(t) \in N_Q$.
The solvability condition for $u^{(1)}(t)$ has the view:

$$\Pi Q\Pi u^{(1)}(t) = 0 = \frac{\partial}{\partial t}u^{(0)}(t) - \Pi\mathbb{V}\Pi u^{(0)}(t).$$

So, we have equation (2.2) for $u^{(0)}(t)$.

For $u^{(1)}(t)$ we have:

$$u^{(1)}(t) = R_0 \mathbb{L} u^{(0)}(t) + c^{(1)}(t).$$

Using the second equation from (2.3) we obtain:

$$u^{(k)}(t) = R_0 \mathbb{L} u^{(k-1)}(t) + c^{(k)}(t),$$

where $c^{(k)}(t) \in N_Q$.

To find $c^{(k)}(t)$ we'll use the fact that $u^{(0)}(t) \in N_Q$. Let us put $c^{(0)}(t) = u^{(0)}(t)$.

For the equation

$$\begin{aligned} Qu^{(2)}(t) = \frac{\partial}{\partial t} u^{(1)}(t) - \mathbb{V} u^{(1)}(t) = \frac{d}{dt} R_0 \Big[\frac{d}{dt}\, c^{(0)}(t) - \mathbb{V} c^{(0)}(t) \Big] \\ + \frac{d}{dt} c^{(1)}(t) - \mathbb{V} R_0 \Big[\frac{d}{dt}\, c^{(0)}(t) - \mathbb{V} c^{(0)}(t) \Big] - \mathbb{V} c^{(1)}(t) \end{aligned}$$

we use the solvability condition

$$\begin{aligned} \Pi Q \Pi u^{(2)}(t) = 0 = \frac{d}{dt}\, c^{(1)}(t) - \mathbb{V} c^{(1)}(t) + \Pi R_0 \Pi \frac{d^2}{dt^2}\, c^{(0)}(t) \\ - \Pi R_0 \mathbb{V} \Pi \frac{d}{dt}\, c^{(0)}(t) - \Pi \mathbb{V} R_0 \Pi \frac{d}{dt}\, c^{(0)}(t) + \Pi \mathbb{V} R_0 \mathbb{V} \Pi c^{(0)}(t). \end{aligned}$$

We find:

$$\frac{d}{dt}\, c^{(1)}(t) - \mathbb{V} c^{(1)}(t) = -\Pi \mathbb{V} R_0 \mathbb{V} \Pi c^{(0)}(t).$$

By induction:

$$\begin{aligned} &\frac{d}{dt}\, c^{(k)}(t) - \mathbb{V} c^{(k)}(t) \\ &= \sum_{i=0}^{k-1} \sum_{n=1}^{k-i} (-1)^k (k-i-n+1) \Pi \mathbb{V} R_0 \mathbb{V}^n \Pi \frac{d^{k-i-n}}{dt^{k-i-n}} c^{(i)}(t), \qquad k > 0. \end{aligned}$$

So, we have the following equation for $c^{(k)}(u,t)$:

$$\frac{d}{dt}\, c^{(k)}(u,t) - v(u) \frac{d}{du}\, c^{(k)}(t) = \mathcal{L}_k(u,t),$$

here

$$\mathcal{L}_k(u,t) = \sum_{i=0}^{k-1} \sum_{n=1}^{k-i} (-1)^k (k-i-n+1) \Pi \mathbb{V} \mathbb{R}_0 \mathbb{V}^n \Pi \frac{d^{k-i-n}}{dt^{k-i-n}}\, c^{(i)}(t).$$

To find a solution we should write down a system

$$\frac{dt}{1} = -\frac{du}{v(u)} = \frac{dc^{(k)}}{\mathcal{L}_k(u,t)}.$$

The independent integrals of this system are:

$$t + \int \frac{du}{v(u)} = C_1,$$

$$c^{(k)}(u,t) + \int \frac{\mathcal{L}_k(u,t)}{v(u)}\, du = C_2.$$

As soon as $c^{(k)}(u,t)$ is only in one of the first integrals, we may present the solution in the form:

$$c^{(k)}(u,t) = f_k\left(t + \int \frac{du}{v(u)}\right) - \int \frac{\mathcal{L}_k(u,t)}{v(u)}\, du, k > 0,$$

where f_k is any differentiable function. Using initial condition for $c^{(k)}(u,t)$ we find a condition for f_k:

$$f_k\left(\int \frac{du}{v(u)}\right) = c^{(k)}(u,0) + \int \frac{\mathcal{L}_k(u,0)}{v(u)}\, du.$$

We may put now $V(u) = \int \frac{du}{v(u)}$ and make a change of variables $w = V(u)$. So, $u = V^{-1}(w)$ and we have:

$$f_k(w) = c^{(k)}(V^{-1}(w),0) + \int \frac{\mathcal{L}_k(V^{-1}(w),0)}{v(V^{-1}(w))} \frac{dw}{w}.$$

Thus, we obtain

$$\begin{aligned} c^{(k)}(u,t) &= c^{(k)}(V^{-1}(t+V(u)),0) \\ &\quad + \int \frac{\mathcal{L}_k(V^{-1}(t+V(u)),0)}{V^{-1}(t+V(u))}\, du - \int \frac{\mathcal{L}_k(u,t)}{v(u)}\, du, \qquad k > 0. \end{aligned}$$

Initial conditions for $c^{(k)}(u,0)$ are found later through Laplace transform for the singular terms of asymptotic.

For the singular terms we have from (10):

$$w^{(1)}(\tau) = exp_0(Q\tau)w^{(1)}(0).$$

Here we should note that the ordinary solution $w^{(1)}(\tau) = exp(Q\tau) \times w^{(1)}(0)$ is corrected by the term $-\Pi w^{(1)}(0)$ in order to receive the following $\lim_{\tau\to\infty} w^{(1)}(\tau) = 0$. We choose this limit to be equal 0 for all singular terms, that may done due to uniform ergodicity of switching Markovian process.

The following statements are made using a method proposed in [2]. For the second equation of the system the corresponding solution should be

$$w^{(k)}(\tau) = exp_0(Q\tau)w^{(k)}(0) + \int\limits_0^{\tau} exp_0(Q(\tau - s))\mathbb{V}w^{(k-1)}(s)\, ds,$$

where the homogenous part has the following solution

$$w^{(k)}(\tau) = exp_0(Q\tau)w^{(k)}(0).$$

But here we should again correct the solution, in order to receive the limit $\lim\limits_{\tau\to\infty} w^{(k)}(\tau) = 0$, by the term $-\Pi\int_\tau^\infty \mathbb{V}w^{(k-1)}(s)\, ds$.

And so the solution is:

$$w^{(k)}(\tau) = exp_0(Q\tau)w^{(k)}(0) + \int_0^{\tau} exp_0(Q(\tau - s))\mathbb{V}w^{(k-1)}(s)ds \\ - \Pi\int\limits_{\tau}^{\infty} \mathbb{V}w^{(k-1)}(s)\, ds.$$

We should finally find the initial conditions for the regular and singular terms.

We put $c^{(0)}(t) = u^{(0)}(t)$, so $c^{(0)}(0) = u^{(0)}(0) = \varphi(u)$.

From the initial condition for the solution $u^\varepsilon(0) = u^{(0)}(0) = \varphi(u)$, we have to determine $u^{(k)}(0) + w^{(k)}(0) = 0,\ k \geq 1$. Let us rewrite this equation for the null-space N_Q of matrix Q:

$$\Pi u^{(k)}(0) + \Pi w^{(k)}(0) = 0, \qquad k \geq 1, \tag{2.5}$$

and the space of values R_Q:

$$(I - \Pi)u^{(k)}(0) + (I - \Pi)w^{(k)}(0) = 0, \qquad k \geq 1. \tag{2.6}$$

So, for $k = 1$ we obtain:

$$u^{(1)}(0) = R_0\mathbb{L}u^{(0)}(0) + c^{(1)}(0) = (I - \Pi)R_0\mathbb{L}\varphi(u) + \Pi c^{(1)}(0),$$

$$w^{(1)}(0) = (I - \Pi)w^{(1)}(0).$$

Thus, $c^{(1)}(0) = 0,\ w^{(1)}(0) = -R_0\mathbb{L}\varphi(u)$.

By analogy, for $k > 1$:

$$u^{(k)}(0) = R_0\mathbb{L}u^{(k-1)}(0) + c^{(k)}(0) = (I - \Pi)R_0\mathbb{L}u^{(k-1)}(0) + \Pi c^{(k)}(0),$$

$$w^{(k)}(0) = (I - \Pi)w^{(k)}(0) - \Pi\int_0^\infty \mathbb{V}w^{(k-1)}(s)\,ds.$$

Functions $w^{(k-1)}(s), u^{(k-1)}(0)$ are known from the previous steps of induction. So, we've found $\Pi w^{(k)}(0)$ in (2.5) and $(I - \Pi)u^{(k)}(0)$ in (2.6).

Now we may use the correlations (2.5), (2.6) to find the unknown initial conditions:

$$c^{(k)}(0) = \int_0^\infty \mathbb{V}w^{(k-1)}(s)\,ds,$$

$$w^{(k)}(0) = -R_0\mathbb{L}u^{(k-1)}(0).$$

In [6] an analogical correlation was found for $c^{(k)}(0)$. To find $c^{(k)}(0)$ explicitly and without the use of singular terms we'll find Laplace transform for the singular term. The following lemma is true.

Lemma 2.1. *Laplace transform for the singular term of asymptotic expansion*

$$\widetilde{w}^{(k)}(\lambda) = \int_0^\infty e^{-\lambda s}w^{(k)}(s)\,ds$$

has the view:

$$\widetilde{w}^{(1)}(\lambda) = (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}[-R_0\mathbb{V}\varphi(u)],$$

$$\begin{aligned}\widetilde{w}^{(k)}(\lambda) = (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}\mathbb{L}u^{(k-1)}(0)\\ + (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}\mathbb{V}\widetilde{w}^{(k-1)}(\lambda)\\ + \frac{1}{\lambda}\Pi\mathbb{V}[\widetilde{w}^{(k-1)}(\lambda) - \widetilde{w}^{(k-1)}(0)],\end{aligned}$$

where

$$\widetilde{w}^{(1)}(0) = -R_0\mathbb{V}\varphi(u),$$

$$(\widetilde{w}^{(1)}(\lambda))'_\lambda\big|_{\lambda=0} = -R_0^2\mathbb{V}\Pi\varphi(u),$$

$$\widetilde{w}^{(k)}(0) = R_0\mathbb{L}u^{(k-1)}(0) + R_0\mathbb{V}\widetilde{w}^{(k-1)}(0) + \Pi\mathbb{V}(\widetilde{w}^{(k-1)}(\lambda))'_\lambda\big|_{\lambda=0},$$

$$(\widetilde{w}^{(k)}(\lambda))'_\lambda\big|_{\lambda=0} = R_0^2\mathbb{L}u^{(k-1)}(0) + R_0^2Q_1\widetilde{w}^{(k-1)}(0) + R_0\mathbb{V}(\widetilde{w}^{(k-1)}(\lambda))'_\lambda\big|_{\lambda=0}.$$

Proof.

$$\widetilde{w}^{(1)}(\lambda) = \int_0^\infty e^{-\lambda s} w^{(1)}(s)\,ds = \int_0^\infty e^{-\lambda s}[e^{Qs} - \Pi]\,ds \cdot w^{(1)}(0)$$
$$= (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}[-\mathbb{V}\varphi(u)],$$

where the correlation for the resolvent was found in [7].

$$\widetilde{w}^{(1)}(0) = -R_0\mathbb{V}\varphi(u),$$
$$(\widetilde{w}^{(1)}(\lambda))'_\lambda\big|_{\lambda=0} = \lim_{\lambda\to 0}\frac{R(\lambda) - R_0}{\lambda}[-\mathbb{V}\varphi(u)] = -R_0^2\mathbb{V}\varphi(u).$$

For the next terms we have:

$$\widetilde{w}^{(k)}(\lambda) = (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}\mathbb{L}u^{(k-1)}(0)$$
$$+ (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}\mathbb{V}\widetilde{w}^{(k-1)}(\lambda)$$
$$+ \frac{1}{\lambda}\Pi\mathbb{V}[\widetilde{w}^{(k-1)}(\lambda) - \widetilde{w}^{(k-1)}(0)],$$

here the last term was found using the following correlation:

$$\int_0^\infty e^{-\lambda s}\int_s^\infty \mathbb{V}w^{(k-1)}(\theta)\,d\theta\,ds = \int_0^\infty\int_0^\theta e^{-\lambda s}\mathbb{V}w^{(k-1)}(\theta)\,ds\,d\theta$$
$$= \int_0^\infty \Big(-\frac{1}{\lambda}\Big)(e^{-\lambda\theta} - 1)\mathbb{V}w^{(k-1)}(\theta)\,d\theta = \frac{1}{\lambda}\mathbb{V}[\widetilde{w}^{(k-1)}(\lambda) - \widetilde{w}^{(k-1)}(0)].$$

So,

$$\widetilde{w}^{(k)}(0) = R_0\mathbb{L}u^{(k-1)}(0) + R_0\mathbb{V}\widetilde{w}^{(k-1)}(0) + \Pi\mathbb{V}(\widetilde{w}^{(k-1)}(\lambda))'_\lambda\big|_{\lambda=0},$$

$$(\widetilde{w}^{(k)}(\lambda))'_\lambda\big|_{\lambda=0} = R_0^2\mathbb{L}u^{(k-1)}(0) + R_0^2 Q_1\widetilde{w}^{(k-1)}(0) + R_0\mathbb{V}(\widetilde{w}^{(k-1)}(\lambda))'_\lambda\big|_{\lambda=0}$$
$$- \lim_{\lambda\to 0}\Big\{\frac{1}{\lambda^2}\Pi\mathbb{V}[\widetilde{w}^{(k-1)}(\lambda) - \widetilde{w}^{(k-1)}(0)] - \frac{1}{\lambda}\Pi\mathbb{V}(\widetilde{w}^{(k-1)}(\lambda))'_\lambda\Big\},$$

where the last limit tends to 0.

Lemma is proved. □

So, the obvious view of the initial condition for the $c^{(k)}(t)$ is:

$$c^{(k)}(0) = \mathbb{V}\widetilde{w}^{(k-1)}(0).$$

Theorem is proved. □

3. Estimate of the Remainder

Let function $\varphi(u)$ in the definition of the functional Φ_t^ε belongs to Banach space of twice continuously differentiable by u functions $C^2(\mathbb{R}^d)$.

Let us write (1.6) in the view

$$\tilde{\Phi}^\varepsilon(t) = \Phi^\varepsilon(t) - \Phi_2^\varepsilon(t) \tag{3.1}$$

where $\Phi_2^\varepsilon(t) = u^{(0)}(t) + \varepsilon(u^{(1)}(t) + w^{(1)}(t)) + \varepsilon^2(u^{(2)}(t) + w^{(2)}(t))$, and the explicit view of the functions $u^{(i)}(t), w^{(j)}(t), i = \overline{0,2}, j = 1,2$ is given in Theorem 2.1.

By Theorem 3.2.1 from [7] in Banach space $C^2(R^d \times E)$ for the generator of Markovian evolution $L^\varepsilon = \varepsilon^{-1}Q + \mathbb{V}$, exists bounded inverse operator $(L^\varepsilon)^{-1}$.

Let us substitute the function (3.1) into equation (1.6):

$$\frac{d}{dt}\tilde{\Phi}^\varepsilon - L^\varepsilon\tilde{\Phi}^\varepsilon = \frac{d}{dt}\Phi_2^\varepsilon - L^\varepsilon\Phi_2^\varepsilon := \varepsilon\theta^\varepsilon. \tag{3.2}$$

Here $\varepsilon\theta^\varepsilon = \varepsilon[\frac{d}{dt}u^{(1)} - \varepsilon\mathbb{V}(u^{(2)} + w^{(2)})]$.

The initial condition has the order ε, so we may write it in the view:

$$\tilde{\Phi}^\varepsilon(0) = \varepsilon\tilde{\Phi}^\varepsilon(0).$$

Let $L_t^\varepsilon\varphi(u) = E[\varphi(u^\varepsilon(t))|u^\varepsilon(0) = u, æ^\varepsilon(0) = x]$ be the semigroup corresponding to the operator L^ε.

Theorem 3.1. *The following estimate is true for the remainder* (3.1) *of the solution of equation* (1.6)*:*

$$\|\tilde{\Phi}^\varepsilon(t)\| \le \varepsilon\|\tilde{\Phi}^\varepsilon(0)\|\exp\{\varepsilon L\|\theta^\varepsilon\|\},$$

where $L \ge 2||(L^\varepsilon)^{-1}||$.

Proof. The solution of equation (3.2) is:

$$\tilde{\Phi}^\varepsilon(t) = \varepsilon\left[L_t^\varepsilon\tilde{\Phi}^\varepsilon(0) + \int_0^t L_{t-s}^\varepsilon\theta^\varepsilon(s)\,ds\right].$$

For the semigroup we have $L_t^\varepsilon = I + L^\varepsilon\int_0^t L_s^\varepsilon ds$, so $\int_0^t L_s^\varepsilon ds = (L^\varepsilon)^{-1} \times (L_t^\varepsilon - I)$.

Using Gronwell–Bellman inequality [1], we receive

$$\|\tilde{\Phi}^\varepsilon(t)\| \le \varepsilon L_t^\varepsilon\|\tilde{\Phi}^\varepsilon(0)\|\exp\left\{\varepsilon\int_0^t L_s^\varepsilon\theta^\varepsilon(t-s)\,ds\right\}$$
$$\le \varepsilon L_t^\varepsilon\|\tilde{\Phi}^\varepsilon(0)\|\exp\{\varepsilon L\|\theta^\varepsilon\|\},$$

where $L \geq 2\|(L^\varepsilon)^{-1}\|$.

Theorem is proved. $\square$

Remark 3.1. For the remainder of asymptotic expansion (1.5) of the view

$$\tilde{\Phi}^\varepsilon_{N+1}(t) := \Phi^\varepsilon(t) - \Phi^\varepsilon_{N+1}(t),$$

where $\Phi^\varepsilon_{N+1}(t) = u^{(0)}(t) + \sum_{k=1}^{N+1} \varepsilon^k (u^{(k)}(t) + w^{(k)}(t))$ we have analogical estimate:

$$\|\tilde{\Phi}^\varepsilon_{N+1}(t)\| \leq \varepsilon^N \|\tilde{\Phi}^\varepsilon(0)\| \exp\{\varepsilon^N L \|\theta^\varepsilon_N\|\},$$

where $\frac{d}{dt}\Phi^\varepsilon_{N+1} - L^\varepsilon \Phi^\varepsilon_{N+1} := \varepsilon^N \theta^\varepsilon_N$.

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